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Research Article

On the Boundary Value Condition of an Isotropic Parabolic Equation

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The well-posedness problem of anisotropic parabolic equation with variable exponents is studied in this paper. The weak solutions and the strong solutions are introduced, respectively. By a generalized Gronwall inequality, the stability of strong solutions to this equation is established, and the uniqueness of weak solutions is proved. Compared with the related works, a new boundary value condition, $\prod_{i=1}^{N} a_i(x,t)=0$, $(x,t)\in\partial\Omega\times[0,T]$, is introduced the first time and has been proved that it can take place of the Dirichlet boundary value condition in some way.

1. Introduction

In this paper, we mainly pay attention on the stability of solutions to the following anisotropic parabolic equation with variable exponents:

$$u_t=\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(a_i(x,t)\left|u_{x_i}\right|^{p_i(x,t)-2}u_{x_i}\right)+\sum_{i=1}^{N}\frac{\partial b_i(u,x,t)}{\partial x_i},\quad (x,t)\in Q_T,\tag{1}$$

with the initial value

$$u(x,0)=u_0(x),\quad x\in\Omega\tag{2}$$

and the boundary value condition

$$u(x,t)=0,\quad (x,t)\in\partial\Omega\times(0,T).\tag{3}$$

Here, $a_i(x,t)\geq 0, p_i(x,t)>1$, $a_i(x,t)\in C(\bar{Q_T})$, $p_i(x,t)\in C(\bar{Q_T})$, and $\Omega\subset\mathbb{R}^N$ is a bounded domain with the smooth boundary $\partial\Omega$, $Q_T=\Omega\times(0,T)$.

When $p_i(x,t)=p$ is a constant, $i=1,2,\cdots,N$, equation (1) arises in the mathematical modelling of various physical processes such as flows of incompressible turbulent fluids, gases in pipes, and processes of filtration in glaciology. The equation in this case has been studied widely [1–5]. When $p_i(x,t)=p(x,t)$ is a measurable function, $i=1,2,\cdots,N$, equation (1) is similar with the equation with the type

$$u_t=\operatorname{div}\left(a(x,t)|\nabla u|^{p(x,t)}\nabla u\right)+f(x,t,u,\nabla u),\tag{4}$$

which arises in the phenomena of electrorheological fluids [6, 7]. The existence of solutions of the initial-boundary value problem to this equation can be found in [8–11]. Also, one can refer to [12–18] for some other related works.

If $a_i(x,t)=a_i(x)$ and satisfies

$$a_i(x)|_{x\in\Omega}>0, a_i(x)|_{x\in\partial\Omega}=0,\quad i=1,2,\cdots,N,\tag{5}$$

the well-posedness problem of the following equations

$$u_t=\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(a_i(x)\left|u_{x_i}\right|^{p_i(x)-2}u_{x_i}\right)+f(x,t,u,\nabla u)\tag{6}$$

has been studied by the second author in recent years [19–21]. Instead of boundary value condition (3), only a partial boundary value condition

$$u(x,t)=0, \quad (x,t)\in \sum_p \times (0,T) \tag{7}$$

is imposed, where $\sum_p \in \partial\Omega$ is a relatively open subset which has different expression according to different kinds of $f(x,t,u,\nabla u)$ and sometimes is just an empty set [19–21].

Compared with [19, 20] and [21], since the diffusion coefficient $a_i(x,t)$ and the variable exponent $p_i(x,t)$ both depend on the time variable t, equation (1) has a wider applications than equation (6), and in mathematical theory, there are some essential difficulties to be overcome. More than that, instead of (5), we only assume that

$$a_i(x,t)>0, \quad (x,t)\in\Omega\times[0,T], i=1,2,\cdots,N, \tag{8}$$

$$\prod_{i=1}^{N} a_i(x,t)=0, \quad (x,t)\in\partial\Omega\times[0,T] \tag{9}$$

and do not require that

$$a_i(x,t)=0, \quad (x,t)\in\partial\Omega\times[0,T], i=1,2,\cdots,N, \tag{10}$$

which is similar as (5) in [19–21].

To see the essential difference between (9) and (10), let us give a special case of equation when $N=2$, $\Omega\subset\mathbb{R}^2$, $\partial\Omega=\bar{\Gamma}_1\cup\bar{\Gamma}_2$, and Γ_1 and Γ_2 are relatively open subset of $\partial\Omega$, $\Gamma_1\cap\Gamma_2=\varnothing$. Consider the equation

$$\begin{aligned} u_t=&\frac{\partial}{\partial x_1}\left(a_1(x)\left|u_{x_1}\right|^{p_1(x)-2}u_{x_i}\right)\\ &+\frac{\partial}{\partial x_2}\left(a_2(x)\left|u_{x_1}\right|^{p_2(x)-2}u_{x_i}\right), \quad (x,t)\in Q_T, \end{aligned} \tag{11}$$

where

$$a_1(x)=0, \quad x\in\Gamma_1, a_1(x)>0, x\in\Gamma_2, \tag{12}$$

$$a_2(x)>0, \quad x\in\Gamma_2, a_2(x)=0, x\in\Gamma_2, \tag{13}$$

then (9) is true, i.e.,

$$\prod_{i=1}^{2} a_i(x)=0, \quad (x,t)\in\partial\Omega\times[0,T]. \tag{14}$$

More precisely, for example,

$$\begin{aligned} &\Omega=\left\{x=(x_1,x_2): 1<x_1^2+x_2^2<4\right\},\\ &a_1(x)=x_1^2+x_2^2-1,\\ &a_2(x)=4-\left(x_1^2+x_2^2\right). \end{aligned} \tag{15}$$

However, in (12) and (13),

$$a_1(x)+a_2(x)>0, \quad x\in\partial\Omega. \tag{16}$$

This fact makes us feel that only under the boundary value condition (3), the uniqueness (or the stability) of weak solutions to equation (11) can be true. The following works seem to supply more evidences. One is [22] in which the equation

$$\begin{aligned} v_t=&\operatorname{div}\left(\left|\nabla v^m\right|^{p(x,t)-2}\nabla v^m+b(x,t)\nabla v^m\right)\\ &+v^{q(x,t)}, \quad (x,t)\in Q_T \end{aligned} \tag{17}$$

is studied. The others are the equations arising from the double phase obstacle problems

$$\begin{aligned} v_t=&\operatorname{div}\left(a(x)|\nabla v|^{p-2}\nabla v+b(x)|\nabla v|^{q-2}\nabla v\right)\\ &+f(x,t,v,\nabla v), \quad (x,t)\in Q_T, \end{aligned} \tag{18}$$

where $a(x)+b(x)>0$, which have gained a wide attention in recent years, one can refer to [23, 24] and the references therein. In these papers, the boundary value condition (3) is imposed without exception.

The main dedication of this paper is that the stability of weak solutions to equation (11)(in general, (1)) can be established independent of boundary value condition (3). Such a conclusion totally overthrows our imagination. In theory, condition (9) can take place of boundary value condition (3) is found the first time. In applications, condition (9) reflects a synthesized effect of an anisotropic diffusion process.

This paper is arranged as follows. In Section 1, we have given a simple introduction. In Section 2, we will introduce the definitions of weak solution and strong weak solution, respectively, quote some basic lemmas, and give the main results. In Section 3, we will study the stability of weak solutions to equation (1) with the new boundary value condition (9). In Section 4, we will study the uniqueness of weak solution to equation (1) independent of the boundary value condition (7). In Section 5, we will give the outline of the proof on the existence of strong solutions.

2. Definitions and Main Results

We denote

$$\begin{aligned} &p_i^+=\max_{(x,t)\in\bar{Q}_T} p_i(x,t), p_{it}^+=\max_{x\in\bar{\Omega}} p_i(x,t), \quad i=1,2,\cdots,N,\\ &p_i^-=\min_{(x,t)\in\bar{Q}_T} p_i(x,t), p_{it}^-=\min_{x\in\bar{\Omega}} p_i(x,t), \quad i=1,2,\cdots,N,\\ &p_-=\min\left\{p_1^-,p_2^-,\cdots,p_{N-1}^-,p_N^-\right\}, \quad p_->1,\\ &p_+=\max\left\{p_1^+,p_2^+,\cdots,p_{N-1}^+,p_N^+\right\}. \end{aligned} \tag{19}$$

First of all, let us introduce the definition of solutions.

Definition 1. A function $u(x,t)$ is said to be a weak solution of equation (1), if

$$u \in L^{\infty}(Q_T),\, u_t \in L^{p_+'}\left(0,T;W^{-1,p_+'}(\Omega)\right) a_i(x,t) \cdot \left|u_{x_i}\right|^{p_i(x,t)} \in L^1(Q_T), \quad i=1,2,\cdots,N, \tag{20}$$

and for any function $\varphi \in C_0^1(Q_T)$,

$$\iint_{Q_T} u_t \varphi dxdt + \sum_{i=1}^{N} \iint_{Q_T} \left[a_i(x,t)\left|u_{x_i}\right|^{p_i(x,t)-2} u_{x_i} \cdot \varphi_{x_i} + b_i(u,x,t) \cdot \varphi_{x_i}\right] dxdt = 0. \tag{21}$$

This definition of weak solution is similar as that defined in [20], where $a_i(x,t) = a_i(x)$, $p_i(x,t) = p_i$ is a constant. Also, we can prove the existence of weak solutions similar as that defined in [20], so we do not repeat the details in this paper. As an improvement from the existing result in [20], we introduce the following definition.

Definition 2. A function $u(x,t)$ is said to be a strong solution of equation (1) with the initial value (2), if

$$u \in L^{\infty}(Q_T),\, u_t \in L^2(Q_T),\, a_i(x,t)\left|u_{x_i}\right|^{p_i(x,t)} \in L^1(Q_T), \quad i=1,2,\cdots,N, \tag{22}$$

and for any function $\varphi \in C_0^1(Q_T)$, u satisfies (21) and the initial value is satisfied in the sense

$$\lim_{t\to 0}\int_{\Omega} (u(x,t) - u_0(x))\phi(x)dx = 0, \quad \phi(x) \in C_0^{\infty}(\Omega). \tag{23}$$

The proof of the existence of strong solution will be given at Section 5 of this paper. Since $a_i(x,t)$ is positive when $x \in \Omega$, (22) means that u_t and ∇u exist almost everywhere in Q_T. This is the reason that we call $u(x,t)$ as a strong solution of equation (1). Moreover, from Definition 2, for all $\varphi(x,t) \in L^{p_+}(0,T;W_0^{1,p_+}(\Omega))$, we still have the integral equality (21), which implies that

$$\|u_t\|_{L^{p_+'}\left(0,T;W^{-1,p_+'}(\Omega)\right)} \le c. \tag{24}$$

Thus, if $u(x,t)$ is a strong solution of equation (1), then it is a weak solution.

Secondly, in order to make the paper sufficiently self-contained and present our discussions in a straightforward manner, let us briefly recall some preliminary results on properties of variable exponent Lebesgue spaces $L^{p(x)}(\Omega)$ and variable exponent Sobolev spaces $W^{1,p(x)}(\Omega)$ [24–26].

Set

$$C_+(\bar{\Omega}) = \left\{h \in C(\bar{\Omega}) \colon \min_{x\in\bar{\Omega}} h(x) > 1\right\}. \tag{25}$$

For any $h \in C_+(\bar{\Omega})$, we define

$$h^+ = \sup_{x\in\Omega} h(x), \qquad h^- = \inf_{x\in\Omega} h(x). \tag{26}$$

For any $p \in C_+(\bar{\Omega})$, let $L^{p(x)}(\Omega)$ consist of all measurable real-valued functions $u(x)$ satisfying

$$\int_{\Omega} |u(x)|^{p(x)} dx < \infty, \tag{27}$$

endowed with the Luxemburg norm

$$\|u\|_{L^{p(x)}(\Omega)} = \inf\left\{\lambda > 0 : \int_{\Omega}\left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \le 1\right\}. \tag{28}$$

Define

$$W^{1,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega) \colon |\nabla u| \in L^{p(x)}(\Omega)\right\}, \tag{29}$$

endowed with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{L^{p(x)}(\Omega)} + \|\nabla u\|_{L^{p(x)}(\Omega)}. \tag{30}$$

Let $W_0^{1,p(x)}(\Omega)$ be the closure space of $C_0^{\infty}(\Omega)$ in $W^{1,p(x)}(\Omega)$.

From [25–27], we have

Lemma 3. *The following three statements are true.*

(i) *The space* $(L^{p(x)}(\Omega), \|\cdot\|_{L^{p(x)}(\Omega)})$, $(W^{1,p(x)}(\Omega), \|\cdot\|_{W^{1,p(x)}(\Omega)})$, *and* $W_0^{1,p(x)}(\Omega)$ *are reflexive Banach spaces*

(ii) *($p(x)$-Hölder's inequality), let $q_1(x)$ and $q_2(x)$ be real functions satisfying $1/q_1(x) + 1/q_2(x) = 1$ with $q_1(x) > 1$. Then, the conjugate space of $L^{q_1(x)}(\Omega)$ is $L^{q_2(x)}(\Omega)$. For any $u \in L^{q_1(x)}(\Omega)$ and $v \in L^{q_2(x)}(\Omega)$, we have*

$$\left|\int_{\Omega} uv dx\right| \le 2\|u\|_{L^{q_1(x)}(\Omega)}\|v\|_{L^{q_2(x)}(\Omega)} \tag{31}$$

(iii) *There are the following properties:*

(1) *if $\|u\|_{L^{p(x)}(\Omega)} = 1$, then $\int_{\Omega}\|u\|^{p(x)}dx = 1$*

(2) *if $\|u\|_{L^{p(x)}(\Omega)} > 1$, then $\|u\|_{L^{p(x)}(\Omega)}^{p^-} \le \int_{\Omega}|u|^{p(x)}dx \le \|u\|_{L^{p(x)}(\Omega)}^{p^+}$*

(3) *if $\|u\|_{L^{p(x)}(\Omega)} < 1$, then $\|u\|_{L^{p(x)}(\Omega)}^{p^+} \le \int_{\Omega}|u|^{p(x)}dx \le \|u\|_{L^{p(x)}(\Omega)}^{p^-}$*

Basing on Lemma 3, by generalizing the Gronwall inequality, we will prove the following stability theorems, in which the initial values satisfy

$$\begin{aligned} &u_0 \in L^{\infty}(\Omega), \left|u_{0x_i}\right| \in L^{p_i}(\Omega), \quad i = 1, 2, \cdots, N, \\ &v_0 \in L^{\infty}(\Omega), \left|v_{0x_i}\right| \in L^{p_i}(\Omega), \quad i = 1, 2, \cdots, N. \end{aligned} \tag{32}$$

Theorem 4. *Let $u(x,t)$ and $v(x,t)$ be two strong solutions of (1) with the initial values $u_0(x)$ and $v_0(x)$, respectively; $p_- > 1$ and $a_i(x,t) \in C(\bar{Q}_T)$ satisfy (8) and (9) and*

$$\int_{\Omega} a_i(x,t)^{-1/(p_i(x,t)-1)} dx \le c(T), \quad i = 1, 2, \cdots, N, \tag{33}$$

$b_i(s,x,t)$ satisfies

$$\left|b_i(u,x,t) - b_i(v,x,t)\right| \le c a_i(x,t)^{1/p_i(x,t)} |u - v|, \quad i = 1, 2, \cdots, N. \tag{34}$$

If for η small enough,

$$\begin{aligned} &\frac{1}{\eta}\left(\int_{\Omega\setminus\Omega_{\eta t}} a_i(x,t)\left|\left(\prod_{j=1}^{N} a_j(x)\right)_{x_i}\right|^{p_i(x,t)} dx\right)^{1/p_{it}^{+}} \\ &\quad \le c(T), \quad i = 1, 2, \cdots, N, \end{aligned} \tag{35}$$

then

$$\int_{\Omega} |u(x,t) - v(x,t)| dx \le \int_{\Omega} |u_0(x) - v_0(x)| dx, \quad t \in [0, T), \tag{36}$$

where for any $t \in [0,T)$, $\Omega_{\eta t} = \{x \in \Omega : (\prod_{j=1}^{N} a_j(x,t)) > \eta\}$.

In this paper, the constant $c(T)$ represents that c depends on T. If we only want to prove the uniqueness of weak solutions, condition (34) is not necessary; we have the following result.

Theorem 5. *Let $p_- > 1, a_i(x,t) \in C^1(\bar{Q}_T)$ satisfy (8) and (9) and $(\partial a_i(x,t))/\partial t \le 0, b_i(s,x,t)$ be a Lipschitz function on $\mathbb{R} \times \bar{\Omega} \times [0,T]$. If $u(x,t)$ and $v(x,t)$ are two strong solutions of equation (1) with the initial values $u_0(x)$ and $v_0(x)$, respectively, then for any $\Omega_1 \subset\subset \Omega$,*

$$\int_{\Omega_1} |u(x,t) - v(x,t)|^2 dx \le c(\Omega_1) \int_{\Omega} |u_0(x) - v_0(x)|^2 dx, \tag{37}$$

which implies that the uniqueness of weak solution is true.

One can see that both Theorem 4 and Theorem 5 imply the uniqueness of solution is true. However, in Theorem 5, the convection function $b_i(\cdot,x,t)$ is independent of the diffusion coefficient $a_i(x,t)$, so as a uniqueness theorem, it is a better than Theorem 4.

3. The Stability of Strong Solutions Independent of the Boundary Value Condition

For small $\eta > 0$, let

$$\begin{aligned} h_{\eta}(s) &= \frac{2}{\eta}\left(1 - \frac{|s|}{\eta}\right)_{+}, \\ S_{\eta}(s) &= \int_0^s h_{\eta}(\tau) d\tau. \end{aligned} \tag{38}$$

Obviously, $h_{\eta}(s) \in C(\mathbb{R})$, and

$$\left|S_{\eta}(s)\right| \le 1\ ; \lim_{\eta\to 0} S_{\eta}(s) = \operatorname{sgn} s, \lim_{\eta\to 0} s S_{\eta}'(s) = 0. \tag{39}$$

In addition, if we denote $H_{\eta}(s) = \int_0^s S_{\eta}(\tau) d\tau$, then

$$\lim_{\eta\to 0} H_{\eta}(s) = |s|, s \in (-\infty, +\infty). \tag{40}$$

At first, we give a generalization of the Gronwall inequality.

Lemma 6. *If $a(x,t)|u(x,t)|^{p(x,t)}, a(x,t)|v(x,t)|^{p(x,t)} \in L^1(Q_T)$ and*

$$\begin{aligned} &\int_{\Omega} a(x,s)|u(x,s) - v(x,s)|^{p(x,s)} dx \\ &\quad \le \int_{\Omega} a(x,\tau)|u(x,\tau) - v(x,\tau)|^{p(x,\tau)} dx \\ &\quad + c\left(\int_{\tau}^{s}\int_{\Omega} a(x,t)|u - v|^{p(x,t)} dx dt\right)^{l}, \end{aligned} \tag{41}$$

where $0 < l \le 1$, then,

$$\begin{aligned} &\int_{\Omega} a(x,t)|u(x,s) - v(x,s)|^{p(x,t)} dx \\ &\quad \le c \int_{\Omega} a(x,\tau)|u(x,\tau) - v(x,\tau)|^{p(x,t)} dx. \end{aligned} \tag{42}$$

Proof. Define that $\kappa(t) = \int_{\Omega} a(x,t)|u(x,t) - v(x,t)|^{p(x,t)} dx$. Without loss of the generality, we may assume that there exists $\tau \in [0,T)$, $\kappa(\tau) > 0$. Then, for any $s > \tau$, $\int_{\tau}^{s} k(t) dt > 0$. Denoting

$$\begin{aligned} &\tau_0 = \max\{t \in [\tau, s], \kappa(t) > 0\}, \\ &\int_{\tau}^{\tau_0} k(t) dt = c_1, \end{aligned} \tag{43}$$

then, $\tau < \tau_0 \le s$, and

$$\int_{\tau}^{s} k(t) dt \ge \int_{\tau}^{\tau_0} k(t) dt = c_1. \tag{44}$$

Since $a(x,t)|u(x,t)|^{p(x,t)}, a(x,t)|v(x,t)|^{p(x,t)} \in L^1(Q_T)$, there exists a constant $C > 0$ such that

$$\frac{c\left(\int_\tau^s k(t)dt\right)^l}{\int_\tau^s k(t)dt} \le \frac{c\left(\int_\tau^s k(t)dt\right)^l}{c_1} \le C = C(c, c_1, T). \tag{45}$$

Combing (41) with (45), we obtain

$$\kappa(s) - \kappa(\tau) \le (C + c)\int_\tau^s k(t)dt. \tag{46}$$

Using the Gronwall inequality, we have

$$\begin{aligned}&\int_\Omega a(x,s)|u(x,s) - v(x,s)|^{p(x,s)}dx \\ &\quad \le c\int_\Omega a(x,\tau)|u(x,\tau) - v(x,\tau)|^{p(x,\tau)}dx.\end{aligned} \tag{47}$$

If $u(x,t), v(x,t) \in L^\infty(Q_T)$, this lemma has been proved in [19] recently.

Secondly, we give the proof of Theorem 4.

Proof of Theorem 4. Let $u(x,t)$ and $v(x,t)$ be two strong solutions of equation (1) with the initial values $u_0(x)$ and $v_0(x)$, respectively.

For any $t \in [0,T]$, set $\Omega_{\eta t} = \{x \in \Omega : \prod_{i=1}^N a_i(x,t) > \eta\}$, and

$$\phi_{\eta t}(x) = \begin{cases} 1, & \text{if } x \in \Omega_{\eta t}, \\ \dfrac{1}{\eta}\displaystyle\prod_{i=1}^N a_i(x,t), & \text{if } x \in \Omega \setminus \Omega_{\eta t}. \end{cases} \tag{48}$$

By a process of limit, we can choose $\chi_{[\tau,s]}\phi_{\eta t}(x)S_\eta(u-v)$ as the test function. Here, $\chi_{[\tau,s]}$ is the characteristic function of $[\tau,s] \subset (0,T)$. Then,

$$\begin{aligned}&\int_\tau^s\int_\Omega \phi_{\eta t}S_\eta(u-v)\frac{\partial(u-v)}{\partial t}dxdt + \sum_{i=1}^N\int_\tau^s\int_\Omega a_i(x,t) \\ &\quad \cdot \left(|u_{x_i}|^{p_i(x,t)-2}u_{x_i} - |v_{x_i}|^{p_i(x,t)-2}\nabla v\right)(u_{x_i} - v_{x_i})h_\eta \\ &\quad \cdot (u-v)\phi_{\eta t}(x)dxdt + \sum_{i=1}^N\int_\tau^s\int_\Omega a_i(x,t)\left(|u_{x_i}|^{p_i(x,t)-2}u_{x_i}\right. \\ &\qquad \left. - |v_{x_i}|^{p_i(x,t)-2}v_{x_i}\right)(u-v)S_\eta(u-v)\frac{\partial\phi_{\eta t}}{\partial x_i}dxdt \\ &\qquad + \sum_{i=1}^N\int_\tau^s\int_\Omega [b_i(u,x,t) - b_i(v,x,t)]\frac{\partial\phi_{\eta t}}{\partial x_i}S_\eta(u-v)dxdt \\ &\qquad + \sum_{i=1}^N\int_\tau^s\int_\Omega [b_i(u,x,t) - b_i(v,x,t)] \\ &\quad \cdot (u-v)_{x_i}\phi_{\eta t}h_\eta(u-v)dxdt = 0.\end{aligned} \tag{49}$$

In the first place, using the dominated convergence theorem, we have

$$\begin{aligned}&\lim_{\eta\to 0}\int_\tau^s\int_\Omega \phi_{\eta t}(x)S_\eta(u-v)\frac{\partial(u-v)}{\partial t}dxdt \\ &\quad = \int_\tau^s\int_\Omega \operatorname{sgn}(u-v)\frac{\partial(u-v)}{\partial t}dxdt \\ &\quad = \lim_{\eta\to 0}\int_\tau^s\int_\Omega \frac{\partial H_\eta(u-v)}{\partial t}dxdt \\ &\quad = \lim_{\eta\to 0}\int_\Omega \left[H_\eta(u-v)(x,s) - H_\eta(u-v)(x,\tau)\right]dx \\ &\quad = \int_\Omega |u-v|(x,s)dx - \int_\Omega |u-v|(x,\tau)dx,\end{aligned} \tag{50}$$

and for any $1 \le i \le N$,

$$\begin{aligned}&\int_\Omega a_i(x,t)\left(|u_{x_i}|^{p_i(x,t)-2}u_{x_i} - |v_{x_i}|^{p_i(x,t)-2}v_{x_i}\right) \\ &\quad \cdot (u_{x_i} - v_{x_i})h_\eta(u-v)\phi_{\eta t}(x)dx \ge 0.\end{aligned} \tag{51}$$

In the second place, we notice $\partial\phi_{\eta t}/\partial x_i = (1/\eta)(\prod_{j=1}^N a_j(x,t))_{x_i}$ when $x \in \Omega \setminus \Omega_{\eta t}$; in the other places, it is identical to zero. Since we assume that

$$\prod_{j=1}^N a_j(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T], \tag{52}$$

by Lemma 3, we have

$$\begin{aligned}&\left|\int_\Omega a_i(x,t)\left(|u_{x_i}|^{p_i(x,t)-2}u_{x_i} - |v_{x_i}|^{p_i(x,t)-2}v_{x_i}\right)\frac{\partial\phi_{\eta t}}{\partial x_i}S_\eta(u-v)dx\right| \\ &\quad = \left|\int_{\Omega\setminus\Omega_{\eta t}} a_i(x,t)\left(|u_{x_i}|^{p_i(x,t)-2}u_{x_i} - |v_{x_i}|^{p_i(x,t)-2}v_{x_i}\right)\frac{\partial\phi_{\eta t}}{\partial x_i}S_\eta(u-v)dx\right| \\ &\quad \le \frac{1}{\eta}\int_{\Omega\setminus\Omega_{\eta t}} a_i(x,t)\left(|u_{x_i}|^{p_i(x,t)-1} + |v_{x_i}|^{p_i(x,t)-1}\right)\left(\prod_{j=1}^N a_j(x,t)\right)_{x_i} S_\eta(u-v)\Big|dx \\ &\quad \le c\frac{1}{\eta}\left(\int_{\Omega\setminus\Omega_{\eta t}}\left(a_i(x,t)|u_{x_i}|^{p_i(x,t)} + |v_{x_i}|^{p_i(x,t)}\right)dx\right)^{1/p'_{it}+} \\ &\qquad \cdot \left(\int_{\Omega\setminus\Omega_{\eta t}} a_i(x,t)\left|\left(\prod_{j=1}^N a_j(x,t)\right)_{x_i}\right|^{p_i(x,t)}dx\right)^{1/p_{it}^+} \\ &\quad \le c\left[\left(\int_{\Omega\setminus\Omega_{\eta t}} a_i(x,t)|u_{x_i}|^{p_i(x,t)}dx\right)^{(p_i-1)/p_i} + \left(\int_{\Omega\setminus\Omega_{\eta t}} a_i(x)|v_{x_i}|^{p_i(x,t)}dx\right)^{1/p'_{it}+}\right] \\ &\qquad \cdot \left[\frac{1}{\eta}\left(\int_{\Omega\setminus\Omega_{\eta t}} a_i(x,t)\left|\left(\prod_{j=1}^N a_j(x,t)\right)_{x_i}\right|^{p_{it}}dx\right)^{1/p_{it}^+}\right] \\ &\quad \le c\left[\left(\int_{\Omega\setminus\Omega_{\eta t}} a_i(x,t)|u_{x_i}|^{p_i(x,t)}dx\right)^{1/p'_{it}+} + \left(\int_{\Omega\setminus\Omega_{\eta t}} a_i(x,t)|v_{x_i}|^{p_{it}}dx\right)^{1/p'_{it}+}\right],\end{aligned} \tag{53}$$

where $p_{it}^+ = \max_{x\in\bar{\Omega}}p(x,t)$ and $p_i{}'(x,t) = p(x,t)/(p(x,t)-1)$.

(53) implies

$$\lim_{\eta\to 0}\left|\int_\tau^s\int_\Omega a_i(x,t)\left(\left|u_{x_i}\right|^{p_i(x,t)-2}u_{x_i}-\left|v_{x_i}\right|^{p_i(x,t)-2}v_{x_i}\right)\frac{\partial\phi_{\eta t}}{\partial x_i}S_\eta(u-v)\,dx\,dt\right|$$
$$\le c\lim_{\eta\to 0}\int_\tau^s\left[\left(\left(\int_{\Omega\setminus\Omega_{\eta t}}a_i(x,t)\left|u_{x_i}\right|^{p_i(x,t)}dx\right)^{1/p'^{+}_{it}}+\left(\int_{\Omega\setminus\Omega_{\eta t}}a_i(x,t)\left|v_{x_i}\right|^{p_i(x,t)}dx\right)^{1/p'^{+}_{it}}\right]dt=0. \tag{54}$$

In the third place, since $a_i(x,t)$ satisfies condition (34), we can deduce that

$$\lim_{\eta\to 0}\sum_{i=1}^N\int_\tau^s\int_\Omega\left[b_i(u,x,t)-b_i(v,x,t)\right](u-v)_{x_i}\phi_\eta h_\eta(u-v)\,dx\,dt=0. \tag{55}$$

In details,

$$\left|\int_\Omega\left[b_i(u,x,t)-b_i(v,x,t)\right](u-v)_{x_i}h_\eta(u-v)\,dx\right|$$
$$=\left|\int_{\{\Omega:|u-v|<\eta\}}\left[b_i(u,x,t)-b_i(v,x,t)\right](u-v)_{x_i}h_\eta(u-v)\,dx\right|$$
$$\le\int_{\{\Omega:|u-v|<\eta\}}\left|a_i(x,t)^{1/p_i(x,t)}(u-v)_{x_i}\right|a_i(x,t)^{-1/p_i(x,t)}\cdot\left|(u-v)h_\eta(u-v)\right|dx. \tag{56}$$

If the set $\{\Omega:|u-v|=0\}$ has a positive measure, by condition (34), we have

$$\lim_{\eta\to 0}\left|\int_{\{\Omega:|u-v|<\eta\}}\left[b_i(u,x,t)-b_i(v,x,t)\right](u-v)_{x_i}h_\eta(u-v)\,dx\right|$$
$$\le c\left(\int_{\{\Omega:|u-v|=0\}}\left(a_i(x,t)^{1/p_i(x,t)}\left|u_{x_i}-v_{x_i}\right|\right)^{p_i(x,t)}dx\right)^{1/p^{+}_{it}}\cdot\left(\int_\Omega a_i(x,t)^{-1/(p_i(x,t)-1)}dx\right)^{1/p'_{1it}}=0, \tag{57}$$

where $p'_{1it}=p'^{+}_{it}$ or p'^{-}_{it} according to $\int_\Omega a_i(x,t)^{-1/(p_i(x,t)-1)}dx\le 1$ or $\int_\Omega a_i(x,t)^{-1/(p_i(x,t)-1)}dx>1$ from Lemma 3.

If the set $\{\Omega:|u-v|=0\}$ only has zero measure, since $a_i(x,t)^{-1/(p_i(x,t)-1)}\in L^1(\Omega)$,

$$\lim_{\eta\to 0}\left|\int_{\{\Omega:|u-v|<\eta\}}\left[b_i(u,x,t)-b_i(v,x,t)\right](u-v)_{x_i}h_\eta(u-v)\,dx\right|$$
$$\le c\left(\int_\Omega\left(a_i^{1/p_i}\left|u_{x_i}-v_{x_i}\right|\right)^{p_i}dx\right)^{1/p_i}\cdot\left(\int_{\{\Omega:|u-v|=0\}}a_i(x,t)^{-1/(p_i-1)}dx\right)^{(p_i-1)/p_i}$$
$$\le c\left(\int_\Omega a_i(x)\left(\left|u_{x_i}\right|^{p_i}+\left|v_{x_i}\right|^{p_i}\right)dx\right)^{1/p_i}\cdot\left(\int_{\{\Omega:|u-v|=0\}}a_i(x,t)^{-1/(p_i-1)}dx\right)^{(p_i-1)/p_i}=0. \tag{58}$$

Finally, by condition (34), we have

$$\lim_{\eta\to 0}\left|\int_\Omega\left[b_i(u,x,t)-b_i(v,x,t)\right]\frac{\partial\phi_{\eta t}}{\partial x_i}S_\eta(u-v)\,dx\right|$$
$$=\lim_{\eta\to 0}\left|\int_{\Omega\setminus\Omega_{\eta t}}\left[b_i(u,x,t)-b_i(v,x,t)\right]\frac{\partial\phi_{\eta t}}{\partial x_i}S_\eta(u-v)\,dx\right|$$
$$\le\lim_{\eta\to 0}\frac{1}{\eta}\int_{\Omega\setminus\Omega_{\eta t}}a_i^{1/p_i(x,t)}\left|\left(\prod_{j=1}^N a_j(x,t)\right)_{x_i}\right|\left|S_\eta(u-v)(u-v)\right|dx$$
$$\le\lim_{\eta\to 0}\frac{1}{\eta}\left(\int_{\Omega\setminus\Omega_{\eta t}}a_i(x,t)\left|\left(\prod_{j=1}^N a_j(x,t)\right)_{x_i}\right|^{p_i(x,t)}dx\right)^{1/p^{+}_{it}}\cdot\left(\int_{\Omega\setminus\Omega_{\eta t}}\left|S_\eta(u-v)(u-v)\right|^{p_i(x,t)/(p_i(x,t)-1)}dx\right)^{1/p'^{+}_{it}}$$
$$\le c\lim_{\eta\to 0}\left(\int_{\Omega\setminus\Omega_{\eta t}}\left|S_\eta(u-v)(u-v)\right|^{p_i(x,t)/(p_i(x,t)-1)}dx\right)^{1/p'^{+}_{it}}$$
$$\le c\left(\int_\Omega|u-v|\,dx\right)^{1/p'^{+}_{it}}, \tag{59}$$

since $u(x,t),v(x,t)\in L^\infty(Q_T)$.

Now, let $\eta\to 0$ in (42). We easily obtain that

$$\int_\Omega|u(x,s)-v(x,s)|\,dx\le\int_\Omega|u(x,\tau)-v(x,\tau)|\,dx+c\left(\int_0^t\int_\Omega|u-v|\,dx\,dt\right)^l. \tag{60}$$

where $l<1$.

Using Lemma 6, we have

$$\int_\Omega|u(x,s)-v(x,s)|\,dx\le c\int_\Omega|u(x,\tau)-v(x,\tau)|\,dx. \tag{61}$$

Then by the arbitrary of τ,

$$\int_{\Omega}|u(x,s)-v(x,s)|dx \le c\int_{\Omega}|u_0(x)-v_0(x)|dx. \quad (62)$$

4. The Uniqueness of Weak Solution

Lemma 7. *Let $u_t \in L^{p'}(0,T;W^{-1,p'}(\Omega))$. For any continuous function $h(s)$, $H(s)=\int_0^s h(\sigma)d\sigma$, a.e.$t_1,t_2\in(0,T)$, it holds*

$$\int_{t_1}^{t_2}\int_{\Omega} h(u)u_t dxdt = \int_{\Omega}(H(u)(x,t_2)-H(u)(x,t_1))dx. \quad (63)$$

This lemma can be generalized from of Lemma 2.2 in [28] simply; we do not give the details here.

Theorem 8. *Let $p_- > 1$, $a_i(x,t)\in C^1(\overline{Q_T})$ satisfy (8)(9) and $(\partial a_i(x,t))/\partial t \le 0$, $b_i(s,x,t)$ be a Lipschitz function on $\mathbb{R}\times\overline{\Omega}\times[0,T]$. If $u(x,t)$ and $v(x,t)$ are two weak solutions of equation (1) with the initial values $u_0(x)$ and $v_0(x)$, respectively, then, there exists a positive constant $\beta_j \ge 2$ such that*

$$\begin{aligned}&\int_{\Omega}\left(\prod_{j=1}^{N} a_j^{\beta_j}(x,t)\right)|u(x,t)-v(x,t)|^2dx\\ &\quad\le c\int_{\Omega}\left(\prod_{j=1}^{N} a_j^{\beta_j}(x,0)\right)|u_0(x)-v_0(x)|^2dx.\end{aligned} \quad (64)$$

Proof. Let $u(x,t)$ and $v(x,t)$ be two weak solutions of equation (1) with the initial values $u_0(x)$ and $v_0(x)$, respectively. By a process of limitation, we may choose $\varphi=\chi_{[\tau,s]}\prod_{j=1}^{N}a_j^{\beta_j}(u-v)$ as a test function. Denoting that $Q_{\tau s}=\Omega\times[\tau,s]$, then,

$$\begin{aligned}&\iint_{Q_{\tau s}}(u-v)\prod_{j=1}^{N}a_j^{\beta_j}\frac{\partial(u-v)}{\partial t}dxdt\\ &\quad=-\sum_{i=1}^{N}\iint_{Q_{\tau s}}a_i(x)\left(|u_{x_i}|^{p_i-2}u_{x_i}-|v_{x_i}|^{p_i-2}v_{x_i}\right)\\ &\qquad\cdot\left[(u-v)\prod_{j=1}^{N}a_j^{\beta_j}\right]_{x_i}dxdt-\sum_{i=1}^{N}\iint_{Q_{\tau s}}\\ &\qquad\cdot[b_i(u,x,t)-b_i(v,x,t)]\left[(u-v)\prod_{j=1}^{N}a_j^{\beta_j}\right]_{x_i}dxdt.\end{aligned} \quad (65)$$

At first, we have

$$\begin{aligned}&\iint_{Q_{\tau s}}a_i(x,t)\left(|u_{x_i}|^{p_i(x,t)-2}u_{x_i}-|v_{x_i}|^{p_i(x,t)-2}v_{x_i}\right)\\ &\quad\cdot(u-v)_{x_i}\prod_{j=1}^{N}a_j^{\beta_j}dxdt\ge 0,\end{aligned} \quad (66)$$

$$\begin{aligned}&\left|\iint_{Q_{\tau s}}(u-v)a_i(x,t)\left(|u_{x_i}|^{p_i(x,t)-2}u_{x_i}-|v_{x_i}|^{p_i(x,t)-2}v_{x_i}\right)\left(\prod_{j=1}^{N}a_j^{\beta_j}\right)_{x_i}dxdt\right|\\ &\quad\le\iint_{Q_{\tau s}}|u-v|a_i(x,t)\left(|u_{x_i}|^{p_i(x,t)-1}+|v_{x_i}|^{p_i(x,t)-1}\right)\left|\left(\prod_{j=1}^{N}a_j^{\beta_j}\right)_{x_i}\right|dxdt\\ &\quad\le c\left(\int_{\tau}^{s}\int_{\Omega}a_i(x,t)\left(|u_{x_i}|^{p_i(x,t)}+|v_{x_i}|^{p_i(x,t)}\right)dxdt\right)^{1/p'_{1i}}\\ &\qquad\cdot\left(\int_{\tau}^{s}\int_{\Omega}a_i(x,t)\left|\left(\prod_{j=1}^{N}a_j^{\beta_j}\right)_{x_i}\right|^{p_i(x,t)}|u-v|^{p_i(x,t)}dxdt\right)^{1/p_{1i}}\\ &\quad\le c\left(\int_{\tau}^{s}\int_{\Omega}a_i(x)\left(|u_{x_i}|^{p_i}+|v_{x_i}|^{p_i}\right)dxdt\right)^{(p_i-1)/p_i}\\ &\qquad\cdot\left(\int_{\tau}^{s}\int_{\Omega}a_i(x)\prod_{j=1}^{N}\left|a_j^{\beta_j-1}a_{jx_i}\right|\right)^{p_i}|u-v|^{p_i}dxdt\Bigg)^{1/p_i}\\ &\quad\le c\left(\int_{\tau}^{s}\int_{\Omega}a_i(x)\prod_{j=1}^{N}\left|a_j^{\beta_j-1}a_{jx_i}\right|^{p_i}|u-v|^{p_i}dxdt\right)^{1/p_i}.\end{aligned} \quad (67)$$

Here, we have used the fact that $|a_{x_i}|\le c$ and $p_{1i}=p_i^+$ or p_i^- according to

$$\int_{\tau}^{s}\int_{\Omega}a_i(x,t)\left|\left(\prod_{j=1}^{N}a_j^{\beta_j}\right)_{x_i}\right|^{p_i(x,t)}|u-v|^{p_i(x,t)}dxdt\le 1, \quad (68)$$

or

$$\int_{\tau}^{s}\int_{\Omega}a_i(x,t)\left|\left(\prod_{j=1}^{N}a_j^{\beta_j}\right)_{x_i}\right|^{p_i(x,t)}|u-v|^{p_i(x,t)}dxdt> 1. \quad (69)$$

p'_{1i} has a similar meaning. Now, if we denote that $Q_1=\{(x,t)\in Q_T: 1<p(x,t)<2\}$ and $Q_2=\{(x,t)\in Q_T: p(x,t)\ge 2\}$, by that $\beta_i\ge 2$, we have

$$\begin{aligned}&\left(\iint_{Q_{\tau s}\cap Q_2}a_i(x,t)\prod_{j=1}^{N}\left|a_j^{\beta_j-1}a_{jx_i}\right|^{p_i(x,t)}|u-v|^{p_i(x,t)}dxdt\right)^{1/p_{1i}}\\ &\quad\le c\left(\int_{\tau}^{s}\int_{\Omega}\prod_{j=1}^{N}a_j^{\beta_j}|u-v|^2dxdt\right)^{1/p_{1i}}.\end{aligned} \quad (70)$$

and by the Hölder inequality,

$$\begin{aligned}&\left(\iint_{Q_{\tau s}\cap Q_1}a_i(x,t)\prod_{j=1}^{N}\left|a_j^{\beta_j-1}a_{jx_i}\right|^{p_i(x,t)}|u-v|^{p_i(x,t)}dxdt\right)^{1/p_{1i}}\\ &\quad\le c\left(\int_{\tau}^{s}\int_{\Omega}\prod_{j=1}^{N}a_j^{\beta_j}|u-v|^2dxdt\right)^{1/k}.\end{aligned} \quad (71)$$

where $k<1$.

Combing (67)-(71), we obtain

$$\left|\iint_{Q_{\tau s}}(u-v)a_i(x,t)\left(\left|u_{x_i}\right|^{p_i(x,t)-2}u_{x_i}-\left|v_{x_i}\right|^{p_i(x,t)-2}v_{x_i}\right)\left(\prod_{j=1}^{N}a_j^{\beta_j}\right)_{x_i}dxdt\right|$$
$$\leq c\left(\int_\tau^s\int_\Omega\prod_{j=1}^{N}a_j^{\beta_j}|u-v|^2dxdt\right)^l, \tag{72}$$

where $l<1$.

Secondly,

$$\iint_{Q_{\tau s}}\left[b_i(u,x,t)-b_i(v,x,t)\right]\left[(u-v)\prod_{j=1}^{N}a_j^{\beta_j}\right]_{x_i}dxdt$$
$$=\iint_{Q_{\tau s}}\left[b_i(u,x,t)-b_i(v,x,t)\right](u-v)\left(\prod_{j=1}^{N}a_j^{\beta_j}\right)_{x_i}dxdt$$
$$+\iint_{Q_s}\left[b_i(u,x,t)-b_i(v,x,t)\right](u-v)_{x_i}\prod_{j=1}^{N}a_j^{\beta_j}dxdt. \tag{73}$$

For the first term on the right hand side of (73), by that $\beta_j\geq 2$, $|a_{jx_i}|\leq c$, using the Hölder inequality, we have

$$\iint_{Q_{\tau s}}\left[b_i(u,x,t)-b_i(v,x,t)\right](u-v)\left(\prod_{j=1}^{N}a_j^{\beta_j}\right)_{x_i}dxdt$$
$$=\int_\tau^s\int_\Omega\left[b_i(u,x,t)-b_i(v,x,t)\right]$$
$$\cdot(u-v)\sum_{k=1}^{N}\left(\beta_k a_k^{\beta_k-1}a_{kx_i}\prod_{j=1,j\neq k}^{N}a_j^{\beta_j}\right)dxdt \tag{74}$$
$$\leq c\int_\tau^s\int_\Omega|u-v|\sum_{k=1}^{N}\left(\beta_k a_k^{\beta_k-1}a_{kx_i}\prod_{j=1,j\neq k}^{N}a_j^{\beta_j}\right)dxdt$$
$$\leq c\left(\int_\tau^s\int_\Omega\prod_{j=1}^{N}a_j^{\beta_j}|u-v|^2dxdt\right)^{1/2}.$$

For the second term on the right hand side of (73), since $\beta_i\geq 2$, denoting $p_i'(x,t)=p_i(x,t)/(p_i(x,t)-1)$ as usual, we have

$$\left(\beta_i-\frac{1}{p_i(x,t)}\right)p_i'(x,t)\geq\beta_i. \tag{75}$$

By this inequality, we deduce

$$\left|\iint_{Q_{\tau s}}\left[b_i(u,x,t)-b_i(v,x,t)\right](u-v)_{x_i}\prod_{j=1}^{N}a_j^{\beta_j}dxdt\right|$$
$$\leq c\left(\int_\tau^s\int_\Omega a_i^{(\beta_i-1/p_i(x,t))p_i'(x,t)}\left(\prod_{j=1,j\neq i}^{N}a_j^{\beta_j}\left|b_i(u,x,t)-b_i(v,x,t)\right|\right)^{p_{i'}(x,t)}dxdt\right)^{1/p_{1i}'}$$
$$\cdot\left(\int_\tau^s\int_\Omega a_i(x,t)\left(\left|u_{x_i}\right|^{p_i(x,t)}+\left|v_{x_i}\right|^{p_i(x,t)}\right)dxdt\right)^{1/p_{1i}}$$
$$\leq c\left(\int_\tau^s\int_\Omega a_i^{(\beta_i-1/p_i(x,t))p_i'(x,t)}\left(\prod_{j=1,j\neq i}^{N}a_j^{\beta_j}\left|b_i(u,x,t)-b_i(v,x,t)\right|\right)^{p_{i'}(x,t)}dxdt\right)^{1/p_{1i}'}$$
$$\leq c\left(\int_\tau^s\int_\Omega\prod_{j=1}^{N}a_j^{\beta_j}|u-v|^{p_{i'}(x,t)}dxdt\right)^{1/p_{1i}'}. \tag{76}$$

Defining Q_1,Q_2 as above, if $p_i(x,t)\geq 2$, then $1<p_i'(x,t)\leq 2$. By the Hölder inequality,

$$\left(\iint_{Q_{\tau s}\bigcap Q_2}\prod_{j=1}^{N}a_j^{\beta_j}|u-v|^{p_{i'}(x,t)}dxdt\right)^{1/p_{1i}'}$$
$$\leq c\left(\int_\tau^s\int_\Omega\prod_{j=1}^{N}a_j^{\beta_j}|u-v|^2dxdt\right)^{1/k}, \tag{77}$$

where $k<1$.

If $1<p_i(x,t)<2$, then $p_i'(x,t)>2$,

$$\left(\iint_{Q_{\tau s}\bigcap Q_1}\int_\Omega\prod_{j=1}^{N}a_j^{\beta_j}|u-v|^{p_{i'}(x,t)}dxdt\right)^{1/p_{1i}'}$$
$$\leq c\left(\int_\tau^s\int_\Omega\prod_{j=1}^{N}a_j^{\beta_j}|u-v|^2dxdt\right)^{1/p_{1i}'}. \tag{78}$$

From (76)-(78), we obtain

$$\left|\iint_{Q_{\tau s}}\left[b_i(u,x,t)-b_i(v,x,t)\right](u-v)_{x_i}\prod_{j=1}^{N}a_j^{\beta_j}dxdt\right|$$
$$\leq c\left(\int_\tau^s\int_\Omega\prod_{j=1}^{N}a_j^{\beta_j}|u-\mathrm{v}|^2dxdt\right)^l, \tag{79}$$

where $l<1$.

Once more, by Lemma 7, we have

$$\begin{aligned}
&\iint_{Q_{\tau s}} (u-v)\prod_{j=1}^{N} a_j^{\beta_j}\frac{\partial(u-v)}{\partial t}dxdt\\
&= \iint_{Q_{\tau s}} (u-v)\sqrt{\prod_{j=1}^{N} a_j^{\beta_j}}\frac{\partial\left[\sqrt{\prod_{j=1}^{N} a_j^{\beta_j}}(u-v)\right]}{\partial t}dxdt\\
&\quad - \iint_{Q_{\tau s}} (u-v)^2\sqrt{\prod_{j=1}^{N} a_j^{\beta_j}}\frac{\partial\sqrt{\prod_{j=1}^{N} a_j^{\beta_j}}}{\partial t}dxdt\\
&= \int_\Omega \prod_{j=1}^{N} a_j^{\beta_j}[u(x,s)-v(x,s)]^2dx\\
&\quad - \int_\Omega \prod_{j=1}^{N} a_j^{\beta_j}[u(x,\tau)-v(x,\tau)]^2dx\\
&\quad - \iint_{Q_{\tau s}} (u-v)^2\sqrt{\prod_{j=1}^{N} a_j^{\beta_j}}\frac{\partial\sqrt{\prod_{j=1}^{N} a_j^{\beta_j}}}{\partial t}dxdt.
\end{aligned} \tag{80}$$

According to (65), (66), (72), (74), (79), and (80), since $(\partial a_i(x,t))/\partial t \le 0$ and $\beta_j \ge 2$, we have

$$\begin{aligned}
&\int_\Omega \prod_{j=1}^{N} a_j^{\beta_j}[u(x,s)-v(x,s)]^2dx - \int_\Omega \prod_{j=1}^{N} a_j^{\beta_j}[u(x,\tau)-v(x,\tau)]^2dx\\
&\le c\left(\int_\tau^s\int_\Omega \prod_{j=1}^{N} a_j^{\beta_j}|u(x,t)-v(x,t)|^2dxdt\right)^l\\
&\quad + \iint_{Q_{\tau s}} (u-v)^2\sqrt{\prod_{j=1}^{N} a_j^{\beta_j}}\frac{\partial\sqrt{\prod_{j=1}^{N} a_j^{\beta_j}}}{\partial t}dxdt\\
&\le c\left(\int_\tau^s\int_\Omega \prod_{j=1}^{N} a_j^{\beta_j}|u(x,t)-v(x,t)|^2dxdt\right)^l,
\end{aligned} \tag{81}$$

where $l<1$. By (81), using the generalized Gronwall inequality, we deduce

$$\begin{aligned}
&\int_\Omega \prod_{j=1}^{N} a_j(x,s)^{\beta_j}|u(x,s)-v(x,s)|^2dx\\
&\quad\le \int_\Omega \prod_{j=1}^{N} a_j(x,\tau)^{\beta_j}|u(x,\tau)-v(x,\tau)|^2dx.
\end{aligned} \tag{82}$$

Thus, by the arbitrary of τ, we have

$$\begin{aligned}
&\int_\Omega \prod_{j=1}^{N} a_j(x,s)^{\beta_j}|u(x,s)-v(x,s)|^2dx\\
&\quad\le \int_\Omega \prod_{j=1}^{N} a_j(x,0)^{\beta_j}|u_0(x)-v_0(x)|^2dx.
\end{aligned} \tag{83}$$

By (83), we have the conclusion. The proof is complete.

By this theorem, Theorem 5 is true clearly.

5. The Strong Solutions Dependent on the Initial Value

For the completeness of the paper, we will give a basic theorem about the existence of strong solutions.

Theorem 9. *If $p_->2$, $a_i(x,t)\in C(\bar{\Omega})$ satisfies (8) and (9), $b_i(s,x,t)$ is a C^1 function on $\mathbb{R}\times\bar{\Omega}\times[0,T]$,*

$$u_0\in L^\infty(\Omega),\ |u_{0x_i}|\in L^{p_i}(\Omega),\quad i=1,2,\cdots,N, \tag{84}$$

$$\int_\Omega a_i(x,t)^{-1/(p_i(x,t)-1)}dx<\infty,\quad i=1,2,\cdots,N, \tag{85}$$

$$\begin{aligned}
&|b_i(s,x,t)|\le ca_i(x,t)^{1/p_i(x,t)},\ |b_{is}(s,x,t)|\\
&\quad\le ca_i(x,t)^{1/p_i(x,t)},\quad i=1,2,\cdots,N,
\end{aligned} \tag{86}$$

then equation (1) with initial value (2) has a weak solution.

Here, $b_{is}(s,x,t)=(\partial b_i(s,x,t))/\partial s$. Before we give the proof of Theorem 9, we would like to point out that condition (86) is just a sufficient condition; we also can use other conditions to replace them. For example, when $a_i(x,t)\equiv a(x)$, $p_i(x,t)=p_i$, by the conditions

$$\begin{aligned}
&\int_\Omega a_i(x)^{-2/(p_i-2)}dx<c,\\
&|b_{is}(s,x,t)|\le ca(x)^{1/p_i}.
\end{aligned} \tag{87}$$

Theorem 9 had been obtained in [19].

Proof of Theorem 9. Consider the following regularized problem

$$\begin{aligned}
&u_{\varepsilon t}-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left((a_i(x,t)+\varepsilon)|u_{\varepsilon x_i}|^{p_i(x,t)-2}u_{\varepsilon x_i}\right)\\
&\quad-\sum_{i=1}^{N}\frac{\partial b_i(u_\varepsilon,x,t)}{\partial x_i}=0,\quad (x,t)\in Q_T,
\end{aligned} \tag{88}$$

$$u_\varepsilon(x,t)=0,\quad (x,t)\in\partial\Omega\times(0,T), \tag{89}$$

$$u_\varepsilon(x,0)=u_{\varepsilon0}(x),\quad x\in\Omega. \tag{90}$$

Here, $u_{\varepsilon0}\in C_0^\infty(\Omega)$, $|u_{\varepsilon0}|_{L^\infty(\Omega)}\le|u_0|_{L^\infty(\Omega)}$, and $|\nabla u_{\varepsilon0}|$ converges to $|\nabla u_0(x)|$ in $L^{p_+}(\Omega)$. It is well-known that the above problem has a unique weak solution $u_\varepsilon\in L^\infty(Q_T)$ and $a_i(x,t)|u_{x_i}|^{p_i(x,t)}\in L^1(Q_T)$ [8], and

$$\|u_\varepsilon\|_{L^\infty(Q_T)}\le c(M), \tag{91}$$

where $M=\|u_0(x)\|_{L^\infty(\Omega)}$.

Multiplying (88) by u_ε and integrating it over Q_T, then

$$\frac{1}{2}\int_\Omega u_\varepsilon^2 dx + \sum_{i=1}^{N}\iint_{Q_T}(a_i(x,t)+\varepsilon)\left|u_{\varepsilon x_i}\right|^{p_i(x,t)}dxdt + \sum_{i=1}^{N}\iint_{Q_T}\frac{\partial b_i(u_\varepsilon,x,t)}{\partial x_i}u_\varepsilon dxdt = \frac{1}{2}\int_\Omega u_{\varepsilon 0}^2 dx. \tag{92}$$

If $|b_i(u_\varepsilon,x,t)| \le c a_i(x,t)^{1/(p_i(x,t))}$, by that $\int_\Omega a_i (x,t)^{-1/(p_i(x,t)-1)}dx < \infty$, then

$$\left|\int_\Omega \frac{\partial b_i(u_\varepsilon,x,t)}{\partial x_i}u_\varepsilon dx\right| = \left|-\int_\Omega \frac{\partial u_\varepsilon}{\partial x_i}b_i(u_\varepsilon,x,t)dx\right| \le c\int_\Omega a_i(x,t)^{1/p_i(x,t)}\left|\frac{\partial u_\varepsilon}{\partial x_i}\right|dx \le c\int_\Omega a_i(x,t)\left|u_{x_i}\right|^{p_i(x,t)}dx + c. \tag{93}$$

Accordingly, by (92), we have

$$\int_\Omega u_\varepsilon^2 dx + \sum_{i=1}^{N}\iint_{Q_T}(a_i(x,t)+\varepsilon)\left|u_{\varepsilon x_i}\right|^{p_i(x,t)}dxdt \le c. \tag{94}$$

Multiplying (88) by $u_{\varepsilon t}$, integrating it over Q_T, then it yields

$$\iint_{Q_T}|u_{\varepsilon t}|^2 dxdt = \sum_{i=1}^{N}\iint_{Q_T}\frac{\partial}{\partial x_i}\left((a_i(x,t)+\varepsilon)\left|u_{\varepsilon x_i}\right|^{p_i(x,t)-2}u_{\varepsilon x_i}\right)u_{\varepsilon t}dxdt + \sum_{i=1}^{N}\iint_{Q_T}u_{\varepsilon t}\frac{\partial b_i(u_\varepsilon,x,t)}{\partial x_i}dxdt. \tag{95}$$

Note that $p_- > 2$,

$$\begin{aligned}&-\left(\left|u_{\varepsilon x_i}\right|^2+\varepsilon\right)^{(p_i(x,t)-2)/2}u_{\varepsilon x_i}\cdot\frac{\partial}{\partial x_i}u_{\varepsilon t}\\ &= -\frac{1}{2}\frac{\partial}{\partial t}\int_0^{\left|u_{\varepsilon x_i}\right|^2+\varepsilon}s^{(p_i(x,t)-2)/2}ds + \frac{1}{2}\int_0^{\left|u_{\varepsilon x_i}\right|^2+\varepsilon}\frac{\partial}{\partial t}s^{(p_i(x,t)-2)/2}ds\\ &= -\frac{1}{2}\frac{\partial}{\partial t}\int_0^{\left|u_{\varepsilon x_i}\right|^2+\varepsilon}s^{(p_i(x,t)-2)/2}ds + \frac{1}{4}\int_0^{\left|u_{\varepsilon x_i}\right|^2+\varepsilon}s^{(p_i(x,t)-2)/2}\ln s\frac{\partial p_i(x,t)}{\partial t}ds\\ &= -\frac{1}{2}\frac{\partial}{\partial t}\int_0^{\left|u_{\varepsilon x_i}\right|^2+\varepsilon}s^{(p_i(x,t)-2)/2}ds + \frac{1}{4}\int_0^{\left|u_{\varepsilon x_i}\right|^2+\varepsilon}s^{(p_i(x,t)-2)/2}\ln s\frac{\partial p_i(x,t)}{\partial t}ds,\end{aligned} \tag{96}$$

and by the Young inequality,

$$\begin{aligned}&\left|\iint_{Q_T}(a_i(x,t)+\varepsilon)\int_0^{\left|u_{\varepsilon x_i}\right|^2+\varepsilon}s^{(p_i(x,t)-2)/2}\ln s\frac{\partial p_i(x,t)}{\partial t}dsdxdt\right|\\ &\le c\iint_{Q_T}(a_i(x,t)+\varepsilon)\left(\left|u_{\varepsilon x_i}\right|^2+\varepsilon\right)dxdt\\ &\le c\iint_{Q_T}(a_i(x,t)+\varepsilon)\left(\left|u_{\varepsilon x_i}\right|^{p_i(x,t)}+1\right)dxdt \le c.\end{aligned} \tag{97}$$

We have

$$\begin{aligned}&\sum_{i=1}^{N}\iint_{Q_T}\frac{\partial}{\partial x_i}\left((a_i(x,t)+\varepsilon)\left(\left|u_{\varepsilon x_i}\right|^2+\varepsilon\right)^{(p_i(x,t)-2)/2}u_{\varepsilon x_i}\right)u_{\varepsilon t}dxdt\\ &= -\sum_{i=1}^{N}\iint_{Q_T}(a_i(x,t)+\varepsilon)\left(\left|u_{\varepsilon x_i}\right|^2+\varepsilon\right)^{(p_i(x,t)-2)/2}u_{\varepsilon x_i}\cdot\frac{\partial}{\partial x_i}u_{\varepsilon t}dxdt = -\sum_{i=1}^{N}\frac{1}{2}\iint_{Q_T}(a_i(x,t)+\varepsilon)\frac{d}{dt}\\ &\int_0^{\left|u_{\varepsilon x_i}\right|^2+\varepsilon}s^{(p_i(x,t)-2)/2}dsdxdt + \frac{1}{4}\sum_{i=1}^{N}\iint_{Q_T}(a_i(x,t)+\varepsilon)\\ &\int_0^{\left|u_{\varepsilon x_i}(x,t)\right|^2+\varepsilon}s^{(p_i(x,t)-2)/2}\ln s\frac{\partial p_i(x,t)}{\partial t}dsdxdt.\end{aligned} \tag{98}$$

If $|b_{iu}(u_\varepsilon,x,t)| \le c a_i(x,t)^{1/p_i(x,t)}$,

$$\begin{aligned}&\iint_{Q_T}u_{\varepsilon t}\frac{\partial b_i(u_\varepsilon,x,t)}{\partial x_i}dxdt\\ &\le \iint_{Q_T}|b_{iu}(u_\varepsilon,x,t)|\left|u_{\varepsilon x_i}\right||u_{\varepsilon t}|dxdt + \iint_{Q_T}\left|b_{ix_i}(u_\varepsilon,x,t)\right||u_{\varepsilon t}|dxdt\\ &\le \frac{1}{2}\iint_{Q_T}|u_{\varepsilon t}|^2dxdt + c\iint_{Q_T}a_i(x,t)\cdot\left|u_{\varepsilon x_i}\right|^{p_i(x,t)}dxdt + c.\end{aligned} \tag{99}$$

Combining (95), (98), and (99), we have

$$\iint_{Q_T}|u_{\varepsilon t}|^2dxdt + \sum_{i=1}^{N}\iint_{Q_T}(a_i(x,t)+\varepsilon)\frac{d}{dt}\int_0^{\left|u_{\varepsilon x_i}\right|^2}s^{(p_i(x,t)-2)/2}dsdxdt \le c. \tag{100}$$

By the above inequality, we have

$$\iint_{Q_T}|u_{\varepsilon t}|^2dxdt \le c + c\sum_{i=1}^{N}\int_\Omega(a_i(x,0)+\varepsilon)\left|u_{\varepsilon 0x_i}\right|^{p_i}dx \le c. \tag{101}$$

Thus, by (91), (94), and (101), there exist a function u and a $n-$dimensional vector function $\overleftarrow{\zeta} = (\zeta_1,\cdots,\zeta_n)$ satisfying that $u_\varepsilon \to u$ a.e. in Q_T, and

$$
\begin{gathered}
u \in L^{\infty}(Q_T), |\zeta_i| \in L^1\left(0, T; L^{p_i(x,t)/(p_i(x,t)-1)}(\Omega)\right), \\
u_\varepsilon \rightharpoonup *u, \text{ in } L^{\infty}(Q_T), \\
b_i(u_\varepsilon, x, t) \to b_i(u, x, t), \text{ a.e.in } Q_T, \\
a_i(x,t)\left|u_{\varepsilon x_i}\right|^{p_i(x,t)-2} u_{\varepsilon x_i} \rightharpoonup \zeta_i, \text{ in } L^1\left(0, T; L^{p_i(x,t)/(p_i(x,t)-1)}(\Omega)\right).
\end{gathered} \tag{102}
$$

Similar as with Theorem 2.1 of Chapter 2 in [1] (also, one can refer to [19] in which $p_i(x,t) = p_i$ is just a constant), we are able to show that

$$
\begin{aligned}
&\lim_{\varepsilon\to 0} \sum_{i=1}^{N} \iint_{Q_T} (a_i(x,t)+\varepsilon)\left|u_{\varepsilon x_i}\right|^{p_i(x,t)-2} u_{\varepsilon x_i}\varphi_{x_i} dxdt \\
&= \lim_{\varepsilon\to 0} \sum_{i=1}^{N} \iint_{Q_T} a_i(x,t)\left|u_{\varepsilon x_i}\right|^{p_i(x,t)-2} u_{\varepsilon x_i}\varphi_{x_i} dxdt \\
&= \iint_{Q_T} \overleftarrow{\zeta} \cdot \nabla\varphi dxdt = \sum_{i=1}^{N} \iint_{Q_T} a_i(x,t) \\
&\quad \cdot \left|u_{x_i}\right|^{p_i(x,t)-2} u_{x_i}\varphi_{x_i} dxdt,
\end{aligned} \tag{103}
$$

for any $\varphi \in C_0^1(Q_T)$. At last, by a process of limit, we can choose the test function $\varphi(x,t) = \chi_{[t_1,t_2]}\phi(x)$ in which $\phi(x) \in C_0^{\infty}(\Omega)$ and $\chi_{[t_1,t_2]}$ is the characteristic function of $[t_1, t_2] \subset (0, T)$. Then,

$$
\begin{aligned}
&\sum_{i=1}^{N} \int_{t_1}^{t_2}\int_{\Omega} \left[a_i(x,t)\left|u_{x_i}\right|^{p_i(x,t)-2} u_{x_i}\phi_{x_i} + b_i(u,x,t)\phi_{x_i}\right] dxdt \\
&\quad = \int_{\Omega} (u(x,t_2) - u(x,t_1))\phi(x)dx,
\end{aligned} \tag{104}
$$

Let $t = t_2$ and $t_1 \to 0$. Then, we have (23) and u is a strong solution of equation (1) with the initial value (2) in the sense of Definition 2.

At last, we give a simple comment. The condition $a_i u_{x_i} \in L^1(\Omega)$ can not assure the boundary value condition

$$
u(x,t) = 0, \quad (x,t) \in \Sigma_p \times (0,T) \subseteq \partial\Omega \times (0,T) \tag{105}
$$

is imposed in the sense of the trace. In fact, we have the following proposition.

Proposition 10. *If $a_i(x,t)$ satisfies (85), i.e.,*

$$
\int_{\Omega} a_i(x,t)^{-1/(p_i(x,t)-1)} dx < \infty, \quad i = 1, 2, \cdots, N. \tag{106}
$$

Then,

$$
\int_{\Omega} \left|u_{x_i}\right| dx \le c(T) < \infty, \quad i = 1, 2, \cdots, N. \tag{107}
$$

Thus, if $a_i(x,t)$ satisfies (85), the partial boundary value condition (105) can be imposed in the sense of trace. However, in this paper, we pay our attention on the studying the stability (or the uniqueness) of solutions independent of the boundary value condition (105), so Proposition 10 is not important.

6. Conclusion

It is clear of that Lemma 6 has a wider applications than the classical Gronwall inequality. Moreover, compared with reference [18], there is at least the essential difference in two aspects. The first one is that condition (9), i.e.,

$$
\prod_{i=1}^{N} a_i(x,t) = 0, (x,t) \in \partial\Omega \times [0,T], \tag{108}
$$

is much weaker than condition (5) appearing in [19], i.e.

$$
a_i(x) = 0, \quad x \in \partial\Omega, i = 1, 2, \cdots, N. \tag{109}
$$

Such a degeneracy is the special nature of the anisotropic equation. The second one is that we have not used boundary value condition (3) throughout this paper; in other words, condition (9) may replace boundary value condition (3) in some way. Moreover, using some techniques developed by the second author in his work [10], in which the well-posedness of weak solutions to equation,

$$
\begin{aligned}
\left(|v|^{\beta-1}v\right)_t &= \operatorname{div}\left(b(x,t)|\nabla v|^{p(x,t)-2}\nabla v\right) \\
&\quad + \sum_{i=1}^{N} g_i(x,t)\frac{\partial \gamma_i(v)}{\partial x_i}, \quad (x,t) \in Q_T
\end{aligned} \tag{110}
$$

has been discussed; the method used in this paper can be applied to study a more general equation

$$
\begin{aligned}
\left(|u|^{\beta-1}u\right)_t &= \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(a_i(x,t)\left|u_{x_i}\right|^{p_i(x,t)-2}u_{x_i}\right) \\
&\quad + f(x,t,u,\nabla u), \quad (x,t) \in Q_T,
\end{aligned} \tag{111}
$$

in the future.

Data Availability

There is not any data in the paper.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

The authors read and approve the final manuscript.

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