Research Article

Biharmonic Hypersurfaces in Pseudo-Riemannian Space Forms with at Most Two Distinct Principal Curvatures

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In this paper, we show that biharmonic hypersurfaces with at most two distinct principal curvatures in pseudo-Riemannian space form $N_s^{n+1}(c)$ with constant sectional curvature c and index s have constant mean curvature. Furthermore, we find that such biharmonic hypersurfaces M_r^{2k-1} in even-dimensional pseudo-Euclidean space $\mathbb{E}_s^{2k}$, M_{s-1}^{2k-1} in even-dimensional de Sitter space $\mathbb{S}_s^{2k}(c)$ ($c > 0$), and M_s^{2k-1} in even-dimensional anti-de Sitter space $\mathbb{H}_s^{2k}(c)$ ($c < 0$) are minimal.

1. Introduction

Let $N_s^{n+1}(c)$ be a $(n+1)$ -dimensional pseudo-Riemannian space form with index $1 \leq s \leq n+1$ and constant sectional curvature c . According to $c = 0$, $c > 0$, and $c < 0$, $N_s^{n+1}(c)$ is isometric to pseudo-Euclidean space $\mathbb{E}_s^{n+1}$, de Sitter space $\mathbb{S}_s^{n+1}(c)$, and anti-de Sitter space $\mathbb{H}_s^{n+1}(c)$.

Suppose that $\phi : M_r^n \rightarrow N_s^{n+1}(c)$, $r = s-1$, or s be an isometric immersion of a pseudo-Riemannian hypersurface M_r^n into $N_s^{n+1}(c)$. The hypersurface M_r^n is said to be biharmonic if its bitension field $\tau_2(\phi)$ vanishes identically, i.e.,

$$\tau_2(\phi) := \text{trace} \left(\nabla^\phi \nabla^\phi - \nabla_\nabla^\phi \right) \tau(\phi) - \text{trace} \tilde{R}(d\phi, \tau(\phi)) d\phi = 0, \quad (1)$$

where $\tau(\phi) := \text{div}(d\phi)$, $\tilde{R}$, ∇^ϕ , and ∇ are the curvature tensor of $N_s^{n+1}(c)$, the induced connection by ϕ on the bundle $\phi^{-1}TN_s^{n+1}(c)$, and the connection of M_r^n , respectively. If the mean curvature of the hypersurface M_r^n is zero, then we call M_r^n as minimal. It is generally known that minimal hypersurfaces are biharmonic ones. Conversely, the natural question is whether any biharmonic hypersurface is minimal.

For biharmonic hypersurfaces in pseudo-Euclidean spaces, there is a conjecture in [2] that every biharmonic hypersurface of pseudo-Euclidean space $\mathbb{E}_s^{n+1}$ is minimal. Up to now, this conjecture has been examined for many biharmonic hypersurfaces, such as M_r^2 of $\mathbb{E}_s^3$ (cf. [2, 3]), M_1^3 of $\mathbb{E}_1^4$ (cf. [1]), M_2^3 of $\mathbb{E}_2^4$ (cf. [10]), and M_r^n in $\mathbb{E}_s^{n+1}$ with at most three distinct principal curvatures and diagonalizable shape operator (cf. [5, 7]).

When the ambient space is de Sitter space $\mathbb{S}_s^{n+1}(c)$, there are also some papers that studied the above problem. Sasahara in [11] considered biharmonic hypersurfaces M_r^2 of $\mathbb{S}_1^3(c)$ and proved that it must be minimal when $r = 0$, but may not when $r = 1$. Investigators studied biharmonic hypersurfaces with at most two distinct principal curvatures in $\mathbb{S}_s^{n+1}(c)$ whose shape operator is diagonalizable in [6, 8] and showed that such hypersurface M_{s-1}^n is minimal, but the hypersurface M_s^n may not. Naturally, there is a question as to whether any biharmonic hypersurface M_{s-1}^n in de Sitter space $\mathbb{S}_s^{n+1}(c)$ is minimal.

The situation is quite different when the ambient space is anti-de Sitter space $\mathbb{H}_s^{n+1}(c)$. For biharmonic hypersurface M_r^2 of $\mathbb{H}_1^3(c)$, it must be minimal when $r = 1$ and may not when $r = 0$ (cf. [11]). For biharmonic hypersurface M_r^n with at most two distinct principal curvatures in $\mathbb{H}_s^{n+1}(c)$ whose shape operator is diagonalizable, it is minimal when $r = s$

and may not when $r = s - 1$ (cf. [6, 8]). A natural question is whether any biharmonic hypersurface M_s^n in anti-de Sitter space $\mathbb{H}_s^{n+1}(c)$ is minimal.

In this paper, we study biharmonic hypersurfaces with at most two distinct principal curvatures in pseudo-Riemannian space forms $N_s^{n+1}(c)$, without the restriction that the shape operator is diagonalizable. We proved such biharmonic hypersurfaces have constant mean curvature. Furthermore, we find that such biharmonic hypersurfaces M_r^{2k-1} in even-dimensional pseudo-Euclidean space $\mathbb{E}_s^{2k}$, M_{s-1}^{2k-1} in even-dimensional de Sitter space $\mathbb{S}_s^{2k}(c)$, and M_s^{2k-1} in even-dimensional anti-de Sitter space $\mathbb{H}_s^{2k}(c)$ are minimal.

2. Preliminaries

2.1. Notions and Formulas of Hypersurfaces in $N_s^{n+1}(c)$. Let $N_s^{n+1}(c)$ be a pseudo-Riemannian space form with index s and constant sectional curvature c . A nonzero vector X in $N_s^{n+1}(c)$ is called *time-like*, *space-like*, or *light-like*, according to whether $\langle X, X \rangle$ is negative, positive, or zero, respectively. Let M_r^n be a nondegenerate hypersurface in $N_s^{n+1}(c)$. $\vec{\xi}$ denotes a unit normal vector field to M_r^n , then $\varepsilon = \langle \vec{\xi}, \vec{\xi} \rangle = \pm 1$. Denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connections of M_r^n and $N_s^{n+1}(c)$, respectively. For any vector fields X, Y tangent to M_r^n , the Gauss formula and Weingarten formula are given by

$$\begin{aligned}\tilde{\nabla}_X Y &= \nabla_X Y + h(X, Y) \vec{\xi}, \\ \tilde{\nabla}_X \vec{\xi} &= -A(X),\end{aligned}\quad (2)$$

where h is the scalar-valued second fundamental form and A is the shape operator of M_r^n associated to $\vec{\xi}$. The mean curvature vector field H can be expressed as $\vec{H} = H \vec{\xi}$, with mean curvature $H = (1/n) \varepsilon \text{tr} A$. For any vector fields X, Y, Z tangent to M_r^n , the Codazzi and Gauss equations are (cf. [9])

$$(\nabla_X A)Y = (\nabla_Y A)X, \quad (3)$$

$$R(X, Y)Z = c(\langle Y, Z \rangle X - \langle X, Z \rangle Y) + \varepsilon \langle A(Y), Z \rangle A(X) - \varepsilon \langle A(X), Z \rangle A(Y), \quad (4)$$

here $R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z$.

A hypersurface M_r^n of $N_s^{n+1}(c)$ is biharmonic if and only if its mean curvature H satisfies the following two equations (cf. [4]):

$$A(\nabla H) = -\frac{n}{2} \varepsilon H(\nabla H), \quad (5)$$

$$\Delta H + \varepsilon H \text{tr} A^2 = n c H, \quad (6)$$

where

$$\Delta H = - \sum_{i=1}^n (e_i e_i H - \nabla_{e_i} e_i H), \quad (7)$$

here $\{e_1, e_2, \dots, e_n\}$ is a local orthonormal frame of $T_x M_r^n$.

2.2. The Shape Operator of M_r^n in $N_s^{n+1}(c)$. According to [9] (exercise 18, pp. 260-261), the tangent space $T_x M_r^n$ at $x \in M_r^n$ can be expressed as a direct sum of subspaces V_k , $1 \leq k \leq m$, that are mutually orthogonal and invariant under the shape operator A , and each $A|_{V_k}$ (the restriction of A on V_k) has form (a) or (b) as follows.

(a) $A|_{V_k}$ has the form

$$A|_{V_k} = \begin{pmatrix} \lambda_k & & & & \\ & 1 & \lambda_k & & \\ & & \ddots & \ddots & \\ & & & 1 & \lambda_k \\ & & & & 1 & \lambda_k \end{pmatrix}, \quad (8)$$

with respect to a basis $\mathfrak{B}_k = \{u_{k_1}, u_{k_2}, \dots, u_{k_{\alpha_k}}\}$ of V_k . The inner products of the basis elements in $\mathfrak{B}_k$ are all zero except

$$\langle u_{k_b}, u_{k_d} \rangle = \varepsilon_k = \pm 1, \quad b + d = \alpha_k + 1. \quad (9)$$

(b) $A|_{V_k}$ has the form

$$A|_{V_k} = \begin{pmatrix} \gamma_k & \tau_k & & & \\ -\tau_k & \gamma_k & & & \\ 1 & 0 & \gamma_k & \tau_k & \\ 0 & 1 & -\tau_k & \gamma_k & \\ & & 1 & 0 & \gamma_k & \tau_k \\ & & 0 & 1 & -\tau_k & \gamma_k \\ & & & & \ddots & \\ & & & & 1 & 0 & \gamma_k & \tau_k \\ & & & & 0 & 1 & -\tau_k & \gamma_k \end{pmatrix}, \quad \tau_k \neq 0, \quad (10)$$

with respect to a basis $\bar{\mathfrak{B}}_k = \{\bar{u}_{1_1}, \bar{v}_{1_1}, \dots, \bar{u}_{k_{\beta_k}}, \bar{v}_{k_{\beta_k}}\}$ of V_k . The inner products of the basis elements in $\bar{\mathfrak{B}}_k$ are all zero except

$$\langle \bar{u}_{k_b}, \bar{u}_{k_d} \rangle = 1 = -\langle \bar{v}_{k_b}, \bar{v}_{k_d} \rangle, \quad b + d = \beta_k + 1. \quad (11)$$

We denote by t the number of terms $A|_{V_k}$ having form (a). We adjust the order of V_k , $1 \leq k \leq m$, such that $A|_{V_k}$ have

form (a) for $1 \leq k \leq t$ and $A|_{V_k}$ have form (b) for $t+1 \leq k \leq m$. Denote $A_i = A|_{V_i}$, $1 \leq i \leq t$, and $\bar{A}_j = A|_{V_j}$, $t+1 \leq j \leq m$. Collecting all the vectors in $\mathfrak{B}_1, \dots, \mathfrak{B}_t, \mathfrak{B}_{t+1}, \dots, \mathfrak{B}_m$ in order, we get a basis $\mathfrak{B}$ of $T_x M_r^n$. With respect to this basis $\mathfrak{B}$, the shape operator A of the hypersurface M_r^n in $N_s^{n+1}(c)$ can be expressed as an almost diagonal matrix:

$$A = \text{diag} \{A_1, \dots, A_t, \bar{A}_{t+1}, \dots, \bar{A}_m\}, \quad (12)$$

and the inner products of the elements in $\mathfrak{B}$ are all zero except

$$\begin{aligned} \langle u_{i_a}, u_{i_b} \rangle &= \varepsilon_i = \pm 1, a + b = \alpha_i + 1, 1 \leq i \leq t, \\ \langle \bar{u}_{j_c}, \bar{u}_{j_d} \rangle &= 1 = -\langle \bar{v}_{j_c}, \bar{v}_{j_d} \rangle, c + d = \beta_j + 1, t+1 \leq j \leq m, \end{aligned} \quad (13)$$

where

$$\alpha_1 + \alpha_2 + \dots + \alpha_t + 2(\beta_{t+1} + \beta_{t+2} + \dots + \beta_m) = n. \quad (14)$$

Observe the forms (a) and (b); we see that A_i , $1 \leq i \leq t$, has only a simple eigenvalue λ_i and $\bar{A}_j$, $t+1 \leq j \leq m$, has eigenvalues $\gamma_j + \tau_j \sqrt{-1}$, $\gamma_j - \tau_j \sqrt{-1}$. It follows from the form of the shape operator A that M_r^n has principal curvatures

$$\lambda_1, \dots, \lambda_t, \quad \gamma_{t+1} \pm \tau_{t+1} \sqrt{-1}, \dots, \gamma_m \pm \tau_m \sqrt{-1}. \quad (15)$$

So, under the assumption that M_r^n has at most two distinct principal curvatures, the shape operator A has the following two possible forms:

- (I) $t = m$, i.e., $A = \text{diag} \{A_1, A_2, \dots, A_m\}$, and there are at most two distinct values among $\{\lambda_1, \lambda_2, \dots, \lambda_m\}$
- (II) $t = 0$, i.e., $A = \text{diag} \{\bar{A}_1, \bar{A}_2, \dots, \bar{A}_m\}$ and $\gamma_1 = \gamma_2 = \dots = \gamma_m = \gamma$, $\tau_1 = \tau_2 = \dots = \tau_m = \tau$, $\tau \neq 0$

For the form (I), we have

$$A(u_{i_a}) = \lambda_i u_{i_a} + u_{i_{a+1}}, A(\bar{u}_{i_{a_i}}) = \lambda_i u_{i_{a_i}}, \quad 1 \leq i \leq m, 1 \leq a \leq \alpha_i - 1. \quad (16)$$

And for the form (II), we have

$$\begin{aligned} A(\bar{u}_{j_b}) &= \nu \bar{u}_{j_b} - \tau \bar{v}_{j_b} + \bar{u}_{j_{b+1}}, A(\bar{u}_{j_{\beta_j}}) = \nu \bar{u}_{j_{\beta_j}} - \tau \bar{v}_{j_{\beta_j}}, \\ A(\bar{v}_{j_b}) &= \tau \bar{u}_{j_b} + \nu \bar{v}_{j_b} + \bar{v}_{j_{b+1}}, A(\bar{v}_{j_{\beta_j}}) = \tau \bar{u}_{j_{\beta_j}} + \nu \bar{v}_{j_{\beta_j}}, \end{aligned} \quad (17)$$

with $t+1 \leq j \leq m$, $1 \leq b \leq \beta_j$.

3. Theorems

Theorem 1. Let $N_s^{n+1}(c)$ be a $(n+1)$ -dimensional pseudo-Riemannian space form with index s and constant sectional

curvature c and M_r^n be a nondegenerate biharmonic hypersurface of $N_s^{n+1}(c)$ with at most two distinct principal curvatures, then M_r^n has constant mean curvature.

Proof. From Section 2, the shape operator A has the form (I) or (II). If A has the form (II), then its eigenvalues are not real and $-(n/2)\varepsilon H$ is not an eigenvalue. It follows from (5) that $\nabla H = 0$, which tells us H is a constant.

For the form (I), if we assume that H is not a constant, then (5) implies that $-(n/2)\varepsilon H$ is an eigenvalue of the shape operator A . When $\lambda_1 = \dots = \lambda_m$, then $\text{tr} A = -(n^2/2)\varepsilon H$. On the other hand, $\text{tr} A = n\varepsilon H$. These two expressions imply $H = 0$, a contradiction.

So, in the following, we need only to discuss the situation where there are two distinct values among $\{\lambda_1, \dots, \lambda_m\}$. Expression (5) also informs us that ∇H is an eigenvector of A with corresponding eigenvalue $-(n/2)\varepsilon H$. In view of (16), ∇H is one of the directions $u_{i_{a_i}}$, $1 \leq i \leq m$. Without loss of generality, we suppose ∇H is in the direction of $u_{1_{a_1}}$; it may be a light-like vector or not. We will follow different processes to lead contradictions for these two cases.

First of all, we give a lot of equations deduced from compatibility and symmetry of the connection, as well as the Codazzi equation.

Observe the inner products of the elements in basis $\mathfrak{B}$ given in Section 2, we can express

$$\nabla H = \sum_{i=1}^m \sum_{a=1}^{\alpha_i} \varepsilon_i u_{i_{a_i-a+1}}(H) u_{i_a}. \quad (18)$$

Since ∇H is in the direction of $u_{1_{a_1}}$, the above equation implies that

$$u_{1_1}(H) \neq 0, u_{i_a}(H) = 0, \quad i_a \neq 1_1. \quad (19)$$

Let $\nabla_{u_{i_a}} u_{j_b} = \sum_{k=1}^m \sum_{d=1}^{\alpha_k} \Gamma_{i_a j_b}^{k_d} u_{k_d}$. Applying compatibility condition to calculate

$$\nabla_{u_D} \langle u_{i_a}, u_{i_a} \rangle, \nabla_{u_D} \langle u_{i_a}, u_{i_b} \rangle, \nabla_{u_D} \langle u_{i_a}, u_{j_d} \rangle, \quad (20)$$

we conclude

$$\Gamma_{Di_a}^{i_{a_i-a+1}} = 0, \quad (21)$$

$$\Gamma_{Di_a}^{i_{a_i-b+1}} = -\Gamma_{Di_b}^{i_{a_i-a+1}}, \Gamma_{Di_a}^{j_{a_j-d+1}} = -\varepsilon_i \varepsilon_j \Gamma_{Dj_d}^{i_{a_i-a+1}}, \quad (22)$$

for $D \in \{k_e, 1 \leq k \leq m, 1 \leq e \leq \alpha_k\}$, $1 \leq i, j \leq m$, $1 \leq a, b \leq \alpha_i$, and $1 \leq d \leq \alpha_j$.

From the expression $(\nabla_{u_B} u_C - \nabla_{u_C} u_B)(H) = [u_B, u_C](H)$, $B, C \in \{k_d, 1 \leq k \leq m, 1 \leq d \leq \alpha_k\}$, and (19), we easily get

$$\Gamma_{BC}^{1_1} = \Gamma_{CB}^{1_1}, \quad B, C \neq 1_1. \quad (23)$$

We state that in the proof, if not otherwise specified, then for i_a in u_{i_a} and the connection's coefficients, the ranges of i

and a are as follows: $1 \leq i \leq m$ and $1 \leq a \leq \alpha_i$. For the equations about the connection's coefficients, when $a = \alpha_i$ (or $a = 1$), then the terms about i_{a+1} (or i_{a-1}) disappear. And when $b = \beta_j$ (or $b = 1$), then the terms about j_{b+1} (or j_{b-1}) and $\tilde{j}_{b+1}$ (or $\tilde{j}_{b-1}$) disappear.

It follows from the Codazzi equation (3) that for any vector fields X, Y, Z tangent to M_r^n ,

$$\langle (\nabla_X A)Y, Z \rangle = \langle (\nabla_Y A)X, Z \rangle. \quad (24)$$

Start with this equation, we can get a series of equations about the coefficients of connection.

- (i) For $(X, Y, Z) = (u_{1_a}, u_{1_b}, u_{1_{\alpha_1}})$ in (24), then combining (19), we obtain

$$\Gamma_{1_{a-1}1_{b+1}}^{1_1} = \Gamma_{1_b1_{a+1}}^{1_1}. \quad (25)$$

Applying (21), (22), and (23), we get from the above equation that

$$\Gamma_{1_{\alpha_1-1}1_{\alpha_1}}^{1_{a-1}} = \dots = \Gamma_{1_21_{\alpha_1}}^{1_2} = \Gamma_{1_11_{\alpha_1}}^{1_1} = 0. \quad (26)$$

Note that if $\alpha_1 = 1$, then (26) tells us nothing.

- (ii) For $(X, Y, Z) = (u_{1_a}, u_{1_b}, u_{1_{\alpha_1}})$, $2 \leq i \leq m$, in (24), then combining (19), we have

$$\left(\lambda_i + \frac{n}{2}\varepsilon H\right)\Gamma_{1_a1_b}^{1_1} + \Gamma_{1_a1_{b+1}}^{1_1} = \Gamma_{1_b1_{a+1}}^{1_1}, \quad 2 \leq i \leq m, \quad (27)$$

which together with (23) implies that if $\lambda_i = -(n/2)\varepsilon H$, then

$$\Gamma_{1_{a+1}i_{\alpha_i}}^{1_1} = 0, \quad (28)$$

and if $\lambda_i \neq -(n/2)\varepsilon H$, then

$$\Gamma_{1_a1_b}^{1_1} = 0. \quad (29)$$

- (iii) For $(X, Y, Z) = (u_{1_a}, u_{1_{\alpha_1}}, u_{1_{\alpha_1-b+1}})$, $2 \leq i \leq m$, in (24), then

$$-\Gamma_{1_a1_{\alpha_1}}^{1_{b-1}} = \left(\lambda_i + \frac{n\varepsilon H}{2}\right)\Gamma_{1_{\alpha_1}i_a}^{1_b} + \Gamma_{1_{\alpha_1}i_{a+1}}^{1_b} - \Gamma_{1_{\alpha_1}i_a}^{1_{b-1}}, \quad (30)$$

which together with (22), (23), (28), and (29) implies that if $\lambda_i \neq -(n/2)\varepsilon H$, then

$$\Gamma_{1_{\alpha_1}i_a}^{1_b} = 0, \quad 2 \leq i \leq m. \quad (31)$$

- (iv) For $(X, Y, Z) = (u_{1_a}, u_{1_b}, u_{1_{\alpha_1}})$, $2 \leq i \leq m$, in (24), then combining (23), we know

$$\Gamma_{1_a1_{b+1}}^{1_1} = \Gamma_{1_b1_{a+1}}^{1_1}. \quad (32)$$

From (22), (23), and the above relation, we conclude that

$$\Gamma_{1_{a_i}i_{b+1}}^{1_1} = 0, \quad 2 \leq i \leq m, \quad (33)$$

$$\Gamma_{1_a1_{\alpha_1}}^{i_b} = \Gamma_{1_{a+1}1_{\alpha_1}}^{i_{b+1}}, \quad 2 \leq i \leq m, \quad 1 \leq a, b \leq \alpha_i - 1. \quad (34)$$

It follows from (33) and (34) that if $a > b$, then

$$\Gamma_{1_a1_{\alpha_1}}^{i_b} = 0. \quad (35)$$

- (v) For $(X, Y, Z) = (u_{1_a}, u_{1_{\alpha_1}}, u_{1_{\alpha_i-a+1}})$, $2 \leq i \leq m$, in (24), then

$$\left(-\frac{n}{2}\varepsilon H - \lambda_i\right)\Gamma_{1_a1_{\alpha_1}}^{i_a} - \Gamma_{1_a1_{\alpha_1}}^{i_{a-1}} = u_{1_{\alpha_1}}(\lambda_i) + \Gamma_{1_{\alpha_1}i_{a+1}}^{i_a} - \Gamma_{1_{\alpha_1}i_a}^{i_{a-1}}, \quad (36)$$

which together with (22) and (34) gives that

$$\Gamma_{1_{\alpha_1}i_2}^{i_1} = \Gamma_{1_{\alpha_1}i_3}^{i_2} = \dots = \Gamma_{1_{\alpha_1}i_{\alpha_i}}^{i_{\alpha_i-1}} = 0, \quad 2 \leq i \leq m, \quad (37)$$

$$u_{1_{\alpha_1}}(\lambda_i) = \left(-\frac{n}{2}\varepsilon H - \lambda_i\right)\Gamma_{1_{\alpha_1}i_{\alpha_1}}^{i_{\alpha_i}}, \quad 2 \leq i \leq m. \quad (38)$$

- (vi) For $(X, Y, Z) = (u_{1_a}, u_{1_{\alpha_1}}, u_{1_{\alpha_i}})$, $2 \leq i \leq m$, $2 \leq a \leq \alpha_i$, in (24), then

$$\left(-\frac{n}{2}\varepsilon H - \lambda_i\right)\Gamma_{1_a1_{\alpha_1}}^{i_1} = \Gamma_{1_{\alpha_1}i_{a+1}}^{i_1}, \quad (39)$$

which combining (35) and (37), implies that

$$\Gamma_{1_{\alpha_1}i_b}^{i_1} = 0, \quad 2 \leq i \leq m, \quad 2 \leq b \leq \alpha_i. \quad (40)$$

- (vii) For $(X, Y, Z) = (u_{1_a}, u_{1_b}, u_{1_{\alpha_1}})$, $2 \leq i, j \leq m$, in (24), then combining (23), we have

$$(\lambda_j - \lambda_i)\Gamma_{1_a1_b}^{1_1} + \Gamma_{1_a1_{b+1}}^{1_1} = \Gamma_{1_b1_{a+1}}^{1_1}, \quad (41)$$

which together with (23) tells us that when $\lambda_i \neq \lambda_j$, then

$$\Gamma_{1_{ajb}}^{1_1} = 0, \quad 2 \leq i, j \leq m, \quad (42)$$

and when $\lambda_i = \lambda_j$, then

$$\Gamma_{i_a j_{b+1}}^{1_1} = 0, \quad (43)$$

$$\Gamma_{i_a 1_{\alpha_1}}^{j_b} = \Gamma_{i_{a+1} 1_{\alpha_1}}^{j_{b+1}}, \quad (44)$$

with $2 \leq i, j \leq m$, $1 \leq a \leq \alpha_i$, $1 \leq b \leq \alpha_j - 1$. (43) and (44) give that if $a > b$, then

$$\Gamma_{i_a 1_{\alpha_1}}^{j_b} = 0. \quad (45)$$

(viii) For $(X, Y, Z) = (u_i, u_{1_{\alpha_1}}, u_{j_{\alpha_j-b+1}})$ with $2 \leq i, j \leq m$, and $i \neq j$ in (24), then

$$\left(-\frac{n\epsilon H}{2} - \lambda_j\right) \Gamma_{i_a 1_{\alpha_1}}^{j_b} - \Gamma_{i_a 1_{\alpha_1}}^{j_{b-1}} = (\lambda_i - \lambda_j) \Gamma_{1_{\alpha_1} i_a}^{j_b} + \Gamma_{1_{\alpha_1} i_{a+1}}^{j_b} - \Gamma_{1_{\alpha_1} i_a}^{j_{b-1}}. \quad (46)$$

(ix) For $(X, Y, Z) = (u_i, u_{j_b}, u_{i_{\alpha_i-d+1}})$, $a \neq d$, in (24), with $\lambda_j = -(n/2)\epsilon H$ and $\lambda_i \neq -(n/2)\epsilon H$, then

$$\left(-\frac{n}{2}\epsilon H - \lambda_i\right) \Gamma_{i_a j_b}^{i_d} + \Gamma_{i_a j_{b+1}}^{i_d} - \Gamma_{i_a j_b}^{i_{d-1}} = \Gamma_{j_b i_{a+1}}^{i_d} - \Gamma_{j_b i_a}^{i_{d-1}}. \quad (47)$$

Let

$$\begin{cases} A_{i,1,j_b} = \Gamma_{i_2 j_b}^{i_1} + \Gamma_{i_3 j_b}^{i_2} + \dots + \Gamma_{i_{\alpha_i} j_b}^{i_{\alpha_i-1}}, \\ A_{i,2,j_b} = \Gamma_{i_3 j_b}^{i_1} + \Gamma_{i_4 j_b}^{i_2} + \dots + \Gamma_{i_{\alpha_i} j_b}^{i_{\alpha_i-2}}, \\ \vdots \\ A_{i,\alpha_i-1,j_b} = \Gamma_{i_{\alpha_i} j_b}^{i_1}, \end{cases} \quad (48)$$

with $1 \leq i, j \leq m$, $1 \leq b \leq \alpha_j$ and $\lambda_j = -(n/2)\epsilon H$, $\lambda_i \neq -(n/2)\epsilon H$. Since (47), we find

$$\begin{cases} \left(-\frac{n}{2}\epsilon H - \lambda_i\right) A_{i,e,j_b} + A_{i,e,j_{b+1}} - A_{i,e+1,j_b} = 0, & 1 \leq b \leq \alpha_j, \quad 1 \leq e \leq \alpha_i - 2, \\ \left(-\frac{n}{2}\epsilon H - \lambda_i\right) A_{i,\alpha_i-1,j_b} + A_{i,\alpha_i-1,j_{b+1}} = 0, & 1 \leq b \leq \alpha_j, \end{cases} \quad (49)$$

which implies that

$$A_{i,e,j_b} = 0, \quad 1 \leq b \leq \alpha_j, \quad 1 \leq e \leq \alpha_i - 1, \quad (50)$$

with $\lambda_j = -(n/2)\epsilon H$, $\lambda_i \neq -(n/2)\epsilon H$, and $1 \leq i, j \leq m$.

(x) For $(X, Y, Z) = (u_i, u_{j_b}, u_{i_{\alpha_i-a+1}})$ in (24), with $\lambda_j = -(n/2)\epsilon H$ and $\lambda_i \neq -(n/2)\epsilon H$, then

$$\left(-\frac{n}{2}\epsilon H - \lambda_i\right) \Gamma_{i_a j_b}^{i_a} + \Gamma_{i_a j_{b+1}}^{i_a} - \Gamma_{i_a j_b}^{i_{a-1}} = u_{j_b}(\lambda_i) + \Gamma_{j_b i_{a+1}}^{i_a} - \Gamma_{j_b i_a}^{i_{a-1}}, \quad (51)$$

which combining (50) gives that

$$\left(-\frac{n}{2}\epsilon H - \lambda_i\right) \sum_{a=1}^{\alpha_i} \Gamma_{i_a j_b}^{i_a} + \sum_{a=1}^{\alpha_i} \Gamma_{i_a j_{b+1}}^{i_a} = \alpha_i u_{j_b}(\lambda_i). \quad (52)$$

(xi) For $(X, Y, Z) = (u_i, u_{j_b}, u_{k_{\alpha_k-d+1}})$ in (24), with $\lambda_j = -(n/2)\epsilon H$, $\lambda_i = \lambda_k \neq -(n/2)\epsilon H$, and $i \neq k$, then

$$\left(-\frac{n}{2}\epsilon H - \lambda_k\right) \Gamma_{i_a j_b}^{k_d} + \Gamma_{i_a j_{b+1}}^{k_d} - \Gamma_{i_a j_b}^{k_{d-1}} = \Gamma_{j_b i_{a+1}}^{k_d} - \Gamma_{j_b i_a}^{k_{d-1}}. \quad (53)$$

Let

$$\begin{cases} B_{i,k,0,j_b} = \Gamma_{i_1 j_b}^{k_1} + \Gamma_{i_2 j_b}^{k_2} + \dots + \Gamma_{i_{\gamma_0} j_b}^{k_{\gamma_0}}, & \gamma_0 = \min\{\alpha_i, \alpha_k\}, \\ B_{i,k,1,j_b} = \Gamma_{i_2 j_b}^{k_1} + \Gamma_{i_3 j_b}^{k_2} + \dots + \Gamma_{i_{\gamma_1} j_b}^{k_{\gamma_1-1}}, & \gamma_1 = \min\{\alpha_i, \alpha_k + 1\}, \\ B_{i,k,0,j_b} = \Gamma_{i_3 j_b}^{k_1} + \Gamma_{i_4 j_b}^{k_2} + \dots + \Gamma_{i_{\gamma_2} j_b}^{k_{\gamma_2-2}}, & \gamma_2 = \min\{\alpha_i, \alpha_k + 2\}, \\ \vdots \\ B_{i,k,\alpha_i-1,j_b} = \Gamma_{i_{\alpha_i} j_b}^{k_1}, \end{cases} \quad (54)$$

with $\lambda_j = -(n/2)\epsilon H$, $\lambda_i = \lambda_k \neq -(n/2)\epsilon H$, and $i \neq k$. It follows from (53) that for $1 \leq b \leq \alpha_j$ and $0 \leq e \leq \alpha_i - 2$,

$$\begin{cases} \left(-\frac{n}{2}\epsilon H - \lambda_k\right) B_{i,k,e,j_b} + B_{i,k,e,j_{b+1}} - B_{i,k,e+1,j_b} = 0, \\ \left(-\frac{n}{2}\epsilon H - \lambda_k\right) B_{i,k,\alpha_i-1,j_b} + B_{i,k,\alpha_i-1,j_{b+1}} = 0, \end{cases} \quad (55)$$

which implies that for $\lambda_j = -(n/2)\epsilon H$, $\lambda_i = \lambda_k \neq -(n/2)\epsilon H$, and $i \neq k$,

$$B_{i,k,e,j_b} = 0, \quad 1 \leq b \leq \alpha_j, \quad 0 \leq e \leq \alpha_i - 1. \quad (56)$$

Now, we treat the two cases that ∇H is not light-like or light-like separately and get contradictions.

Case 1. ∇H is not light-like.

As ∇H is in the direction of $u_{1_{\alpha_1}}$, ∇H is not light-like means that $\alpha_1 = 1$. Observe equation (38); we find if $\lambda_i = -(n/2)\epsilon H$, then $u_{1_1}(H) = 0$, which contradicts with (19). So, we conclude that $\lambda_i \neq -(n/2)\epsilon H$, for $2 \leq i \leq m$. Since there are two distinct values among $\{\lambda_1, \dots, \lambda_m\}$ and $\text{tr}A = n\epsilon H$, we have $\lambda_2 = \dots = \lambda_m = 3n\epsilon H/2(n-1)$. It follows from (34) and (38) that

$$\Gamma_{i_a 1_1}^{i_a} = -\frac{3u_{1_1}(H)}{(n+2)H}, \quad 2 \leq i \leq m, \quad 1 \leq a \leq \alpha_i. \quad (57)$$

Denote $W = \Gamma_{i_a 1_1}^{i_a}$, we have the following lemmas.

Lemma 2. *We have*

$$u_{1_1}(W) + W^2 = \frac{3\varepsilon_1 n^2 H^2}{4(n-1)} - c\varepsilon_1. \quad (58)$$

Proof. Since $\lambda_i = 3n\varepsilon H/2(n-1)$ with $2 \leq i \leq m$, it follows from (46) that

$$\left(-\frac{n}{2}\varepsilon H - \lambda_j\right) \Gamma_{i_a 1_1}^{j_1} = \Gamma_{1_1 i_{a+1}}^{j_1}, \quad 2 \leq i, j \leq m, i \neq j, \quad (59)$$

combining (43), we have

$$\Gamma_{1_1 i_{a+1}}^{j_1} = 0, \quad (60)$$

with $2 \leq i, j \leq m, i \neq j$, and $2 \leq a \leq \alpha_i - 1$.

Using Gauss equation for $\langle R(u_{1_1}, u_{i_{a_i}})u_{1_1}, u_{i_1} \rangle$, with $2 \leq i \leq m$, combining (29), (31), (35), (40), (42), (43) and (60), we have

$$\begin{aligned} u_{1_1}(\Gamma_{i_{a_i} 1_1}^{i_{a_i}}) &= -(\Gamma_{i_{a_i} 1_1}^{i_{a_i}})^2 - \sum_{2 \leq j \leq m, j \neq i} \\ &\quad \cdot \left(\Gamma_{i_{a_i} 1_1}^{j_{a_j}} \Gamma_{1_1 j_{a_j}}^{i_{a_i}} - \Gamma_{j_{a_j} 1_1}^{i_{a_i}} \Gamma_{1_1 i_{a_i}}^{j_{a_j}} - \Gamma_{j_{a_j-1} 1_1}^{i_{a_i}} \Gamma_{1_1 i_{a_i}}^{j_{a_j-1}} + \Gamma_{i_{a_i} 1_1}^{j_{a_j}} \Gamma_{j_{a_j} 1_1}^{i_{a_i}} \right) \\ &\quad + \frac{3\varepsilon_1 n^2 H^2}{4(n-1)} - c\varepsilon_1. \end{aligned} \quad (61)$$

From (22) and (23), we know $\Gamma_{i_{a_i} 1_1}^{j_{a_j}} = \varepsilon_i \varepsilon_j \Gamma_{j_1 1_1}^{i_1}$, $\Gamma_{j_{a_j-1} 1_1}^{i_{a_i}} = \varepsilon_i \varepsilon_j \Gamma_{i_1 1_1}^{j_2}$, with $2 \leq i, j \leq m$. So, we can rewrite the above equation as

$$\begin{aligned} u_{1_1}(\Gamma_{i_{a_i} 1_1}^{i_{a_i}}) &= -(\Gamma_{i_{a_i} 1_1}^{i_{a_i}})^2 - \sum_{2 \leq j \leq m, j \neq i} \\ &\quad \cdot \left(\Gamma_{i_1 1_1}^{j_1} \Gamma_{1_1 j_1}^{i_1} + \Gamma_{1_1 1_1}^{j_2} \Gamma_{1_1 j_2}^{i_1} - \Gamma_{j_1 1_1}^{i_1} \Gamma_{1_1 i_1}^{j_1} + \Gamma_{i_1 1_1}^{j_1} \Gamma_{j_1 1_1}^{i_1} \right) \\ &\quad + \frac{3\varepsilon_1 n^2 H^2}{4(n-1)} - c\varepsilon_1, \quad 2 \leq i \leq m. \end{aligned} \quad (62)$$

For $2 \leq i, j \leq m$ and $i \neq j$, if $\alpha_j < \alpha_i$, then we have $\Gamma_{j_1 1_1}^{i_1} = \Gamma_{j_{a_j} 1_1}^{i_{a_j}} = 0$ from (22), (43), and (44). And if $\alpha_j = \alpha_i$, from (56), we know

$$\Gamma_{j_1 1_1}^{i_1} + \Gamma_{j_2 1_1}^{i_2} + \cdots + \Gamma_{j_{\alpha_j} 1_1}^{i_{\alpha_j}} = 0, \quad (63)$$

which together with (44), tells us that

$$\Gamma_{j_1 1_1}^{i_1} = 0. \quad (64)$$

So, for $2 \leq i, j \leq m$ and $i \neq j$, if $\alpha_j \leq \alpha_i$, then

$$\Gamma_{j_1 1_1}^{i_1} = 0. \quad (65)$$

For $2 \leq i, j \leq m$ and $i \neq j$, if $\alpha_j > \alpha_i$, combining (65), we get from (46) that

$$\begin{cases} -\frac{n(n+2)\varepsilon H}{2(n-1)} \Gamma_{i_1 1_1}^{j_2} = \Gamma_{1_1 i_2}^{j_2} - \Gamma_{1_1 i_1}^{j_1}, \\ -\frac{n(n+2)\varepsilon H}{2(n-1)} \Gamma_{i_2 1_1}^{j_3} = \Gamma_{1_1 i_3}^{j_3} - \Gamma_{1_1 i_2}^{j_2}, \\ \vdots \\ -\frac{n(n+2)\varepsilon H}{2(n-1)} \Gamma_{i_{\alpha_i-1} 1_1}^{j_{\alpha_i}} = \Gamma_{1_1 i_{\alpha_i}}^{j_{\alpha_i}} - \Gamma_{1_1 i_{\alpha_i-1}}^{j_{\alpha_i-1}}, \\ -\frac{n(n+2)\varepsilon H}{2(n-1)} \Gamma_{i_{\alpha_i} 1_1}^{j_{\alpha_i+1}} = -\Gamma_{1_1 i_{\alpha_i}}^{j_{\alpha_i}}. \end{cases} \quad (66)$$

If $\alpha_j > \alpha_i + 1$, then (43) and (44) give us that

$$\Gamma_{i_1 1_1}^{j_2} = \cdots = \Gamma_{i_{\alpha_i+1} 1_1}^{j_{\alpha_i+1}} = 0. \quad (67)$$

As (67), it follows from (66) that for $2 \leq i, j \leq m$ and $i \neq j$, if $\alpha_j > \alpha_i + 1$, then

$$\Gamma_{1_1 i_1}^{j_1} = 0. \quad (68)$$

For $2 \leq i, j \leq m$ and $i \neq j$, if $\alpha_j = \alpha_i + 1$, then combining (22) and (23), we get from (44) and (66) that

$$\begin{aligned} \Gamma_{i_1 1_{a_1}}^{j_2} &= \Gamma_{i_{a_i} 1_{a_1}}^{j_{a_j}} = \varepsilon_i \varepsilon_j \Gamma_{j_1 1_{a_1}}^{i_1}, \\ \Gamma_{1_{a_1} i_1}^{j_1} &= \alpha_i \Gamma_{1_{a_1} i_{a_i}}^{j_{a_i}} = \varepsilon_i \varepsilon_j \frac{n(n+2)\varepsilon H}{2(n-1)} \alpha_i \Gamma_{j_1 1_{a_1}}^{i_1}. \end{aligned} \quad (69)$$

Therefore, from (59), (65), (67), (68), and (69), (62) can be simplified to

$$\begin{aligned} &\frac{n(n+2)\varepsilon H}{2(n-1)} \varepsilon_i \left\{ \sum_{j=2, \alpha_j=\alpha_i-1}^m \varepsilon_j \alpha_j \left(\Gamma_{i_1 1_1}^{j_1} \right)^2 - \sum_{j=2, \alpha_j=\alpha_i+1}^m \varepsilon_j (\alpha_i + 1) \left(\Gamma_{j_1 1_1}^{i_1} \right)^2 \right\} \\ &+ u_{1_1}(\Gamma_{i_1 1_1}^{i_1}) + \left(\Gamma_{i_1 1_1}^{i_1} \right)^2 = \frac{3\varepsilon_1 n^2 H^2}{4(n-1)} - c\varepsilon_1, \quad 2 \leq i \leq m. \end{aligned} \quad (70)$$

Choosing $\{\alpha_{1_1}, \alpha_{1_2}, \dots, \alpha_{1_{\varepsilon_1}}, \alpha_{2_1}, \alpha_{2_2}, \dots, \alpha_{2_{\varepsilon_2}}, \dots, \alpha_{h_1}, \alpha_{h_2}, \dots, \alpha_{h_{\varepsilon_h}}\}$ in $\{\alpha_2, \dots, \alpha_m\}$, such that

$$\tilde{\alpha}_i := \alpha_{i_1} = \alpha_{i_2} = \cdots = \alpha_{i_{\varepsilon_i}}, \quad \tilde{\alpha}_i + 1 = \tilde{\alpha}_{i+1}, \quad 1 \leq i \leq h, \quad (71)$$

and $\tilde{\alpha}_1 - 1, \tilde{\alpha}_h + 1 \notin \{\alpha_2, \dots, \alpha_m\}$.

Taking sum in (70) for $i = k_1, k_2, \dots, k_{\xi_k}$, with $1 \leq k \leq h$, we have

$$\begin{cases} -(\tilde{\alpha}_1 + 1)Q_1 = \xi_1 U, \\ \tilde{\alpha}_1 Q_1 - (\tilde{\alpha}_2 + 1)Q_2 = \xi_2 U, \\ \tilde{\alpha}_2 Q_2 - (\tilde{\alpha}_3 + 1)Q_3 = \xi_3 U, \\ \vdots \\ \tilde{\alpha}_{h-2}Q_{h-2} - (\tilde{\alpha}_{h-1} + 1)Q_{h-1} = \xi_{h-1} U, \\ \tilde{\alpha}_{h-1}Q_{h-1} = \xi_h U, \end{cases} \quad (72)$$

where

$$\begin{aligned} U &= -u_{1_1}(W) - W^2 + \frac{3\varepsilon_1 n^2 H^2}{4(n-1)} - c\varepsilon_1, \\ Q_k &= \frac{n(n+2)\varepsilon H}{2(n-1)} \sum_{\substack{1 \leq p \leq \xi_k \\ 1 \leq q \leq \xi_{k+1}}} \varepsilon_\rho \varepsilon_\sigma \left(\Gamma_{\sigma_1 1_{a_1}}^{\rho_1} \right)^2, \quad \rho = k_p, \sigma = (k+1)_q. \end{aligned} \quad (73)$$

From (72), we can get $Q_k = 0$, $1 \leq k \leq h-1$, and $U = 0$, i.e., (58) holds.

Lemma 3. *We have*

$$\sum_{i=2}^m \sum_{a=1}^{\alpha_i} \Gamma_{i_a 1_1}^{i_a} u_{1_1}(H) + u_{1_1} u_{1_1}(H) - \varepsilon_1 \varepsilon \frac{(n+8)n^2 H^3}{4(n-1)} + n c \varepsilon_1 H = 0. \quad (74)$$

Proof. For $1 \leq k \leq m$, if α_k is an even number, we put

$$e_{k_a} = \frac{u_{k_a} - u_{k_{\alpha_k-a+1}}}{\sqrt{2}}, \quad k_{\alpha_k-a+1} = \frac{u_{k_a} + u_{k_{\alpha_k-a+1}}}{\sqrt{2}}, \quad (75)$$

with $1 \leq a \leq \alpha_k/2$. And if α_k is an odd number, we take

$$e_{k_a} = \frac{u_{k_a} - u_{k_{\alpha_k-a+1}}}{\sqrt{2}}, \quad k_{\alpha_k-a+1} = \frac{u_{k_a} + u_{k_{\alpha_k-a+1}}}{\sqrt{2}}, \quad 1 \leq a < \frac{\alpha_k + 1}{2}, \quad (76)$$

and $e_{k_{(\alpha_k+1)/2}} = u_{k_{(\alpha_k+1)/2}}$. We easily find $\mathfrak{E}_k = \{e_{k_1}, e_{k_2}, \dots, e_{k_{\alpha_k}}\}$ is an orthonormal basis of V_k with $1 \leq k \leq m$. Thus, $\mathfrak{E} = \{e_{k_a}, 1 \leq k \leq m, 1 \leq a \leq \alpha_k\}$ is an orthonormal basis of $T_x M_r^n$.

From (19), (75), and (76), we have when α_k is an even number,

$$\begin{aligned} \nabla_{e_{k_a}} e_{k_a}(H) &= \frac{1}{2} \left(\Gamma_{k_a k_a}^{1_1} + \Gamma_{k_{\alpha_k-a+1} k_{\alpha_k-a+1}}^{1_1} - \Gamma_{k_a k_{\alpha_k-a+1}}^{1_1} - \Gamma_{k_{\alpha_k-a+1} k_a}^{1_1} \right) u_{1_1}(H), \\ \nabla_{e_{k_{\alpha_k-a+1}}} e_{k_{\alpha_k-a+1}}(H) &= \frac{1}{2} \left(\Gamma_{k_a k_a}^{1_1} + \Gamma_{k_{\alpha_k-a+1} k_{\alpha_k-a+1}}^{1_1} + \Gamma_{k_a k_{\alpha_k-a+1}}^{1_1} + \Gamma_{k_{\alpha_k-a+1} k_a}^{1_1} \right) u_{1_1}(H), \end{aligned} \quad (77)$$

with $1 \leq a \leq \alpha_k/2$, and when α_k is an odd number,

$$\begin{aligned} \nabla_{e_{k_a}} e_{k_a}(H) &= \frac{1}{2} \left(\Gamma_{k_a k_a}^{1_1} + \Gamma_{k_{\alpha_k-a+1} k_{\alpha_k-a+1}}^{1_1} - \Gamma_{k_a k_{\alpha_k-a+1}}^{1_1} - \Gamma_{k_{\alpha_k-a+1} k_a}^{1_1} \right) u_{1_1}(H), \\ \nabla_{e_{k_{\alpha_k-a+1}}} e_{k_{\alpha_k-a+1}}(H) &= \frac{1}{2} \left(\Gamma_{k_a k_a}^{1_1} + \Gamma_{k_{\alpha_k-a+1} k_{\alpha_k-a+1}}^{1_1} + \Gamma_{k_a k_{\alpha_k-a+1}}^{1_1} + \Gamma_{k_{\alpha_k-a+1} k_a}^{1_1} \right) u_{1_1}(H), \\ \nabla_{e_{k_{(\alpha_k+1)/2}}} e_{k_{(\alpha_k+1)/2}}(H) &= \Gamma_{k_{(\alpha_k+1)/2} k_{(\alpha_k+1)/2}}^{1_1} u_{1_1}(H), \end{aligned} \quad (78)$$

with $1 \leq a < (\alpha_k + 1)/2$

Note that $e_{1_1} = u_{1_1}$; it follows from (19) that

$$e_{1_1}(H) \neq 0, e_{i_a}(H) = 0, \quad 2 \leq i \leq m, 1 \leq a \leq \alpha_i. \quad (79)$$

By calculating, we get

$$tr A^2 = \frac{(n+8)n^2 H^2}{4(n-1)}. \quad (80)$$

Since the above, we obtain from (6) that

$$\sum_{i=2}^m \sum_{a=1}^{\alpha_i} \Gamma_{i_a 1_1}^{i_a} u_{1_1}(H) + u_{1_1} u_{1_1}(H) - \varepsilon_1 \varepsilon \frac{(n+8)n^2 H^3}{4(n-1)} + n c \varepsilon_1 H = 0. \quad (81)$$

Now, we continue the proof of Theorem 1 for Case 1. Combining (57) and (58), we get

$$\begin{aligned} u_{1_1} u_{1_1}(H) - \frac{(n+2)(n+5)H}{9} W^2 - \frac{(n+2)n^2 \varepsilon_1 H^3}{4(n-1)} \\ + \frac{1}{3}(n+2)c\varepsilon_1 H. \end{aligned} \quad (82)$$

Substitute the above equation into (74), considering $\alpha_2 + \dots + \alpha_m = n(\alpha_1 = 1)$ and (57), we have

$$\left\{ \frac{2(n-4)(n+2)}{9} W^2 + \frac{(n+2+\varepsilon(n+8))n^2\varepsilon_1 H^2}{4(n-1)} - \frac{2}{3}(2n+1)c\varepsilon_1 \right\} H = 0. \quad (83)$$

Act on (83) with u_{1_1} , and using (57) and (58), we obtain

$$\begin{aligned} & \left\{ \frac{2(n-4)(n+2)}{9} W^2 + \frac{(n+2+\varepsilon(n+8))n^2\varepsilon_1 H^2}{4(n-1)} - \frac{2}{3}(2n+1)c\varepsilon_1 \right\} u_{1_1} \\ & \cdot (H) - 2(n+2) \\ & \cdot \left\{ \frac{2(n-4)}{9} W^2 + \frac{(n-10-\varepsilon(n+8))n^2\varepsilon_1 H^2}{12(n-1)} + \frac{2(n-4)}{9} c\varepsilon_1 \right\} HW = 0. \end{aligned} \quad (84)$$

As (83) and $W \neq 0$, the above equation implies that

$$\left\{ \frac{2(n-4)}{9} W^2 + \frac{(n-10-\varepsilon(n+8))n^2\varepsilon_1 H^2}{12(n-1)} + \frac{2(n-4)}{9} c\varepsilon_1 \right\} H = 0, \quad (85)$$

which together with (83) gives that

$$\begin{aligned} & \frac{[-(n+2)(n-13) + (n+8)(n+5)\varepsilon]n^2\varepsilon_1 H^3}{12(n-1)} \\ & - \frac{2}{9}(n-1)(n+5)c\varepsilon_1 H = 0, \end{aligned} \quad (86)$$

which implies that H is a constant, a contradiction.

Case 2. ∇H is light-like.

Notice that ∇H is light-like means $\alpha_1 \geq 2$. Since $trA = n\varepsilon H$, we can suppose $\lambda_1 = \dots = \lambda_p = -(n/2)\varepsilon H$ and $\lambda_{p+1} = \dots = \lambda_m = (2 + \alpha_1 + \dots + \alpha_p)n\varepsilon H / (2 + \alpha_{p+1} + \dots + \alpha_m)$. It follows from (19) and (38) that

$$\Gamma_{i_1 1 \alpha_1}^{i_1} = 0, \quad p+1 \leq i \leq m. \quad (87)$$

Calculate $\langle R(u_{1_2}, u_i)u_{1_{\alpha_1}}, u_{i_{\alpha_1-1}} \rangle$ and $\langle R(u_{1_1}, u_i)u_{1_{\alpha_1}}, u_{i_{\alpha_1-1}} \rangle$ with $p+1 \leq i \leq m$ by Gauss equation, and letting i and a for sum with $p+1 \leq i \leq m$ and $1 \leq a \leq \alpha_i$, combining (21), (22), (23) (26), (28), (29), (35), (42), (45), and (87), we get

$$\begin{aligned} & \sum_{i=p+1}^m \sum_{a=1}^{\alpha_i} \left\{ \sum_{d=3}^{\alpha_1-1} \Gamma_{1_2 1 \alpha_1}^{1_d} \Gamma_{i_a 1_d}^{i_a} + \sum_{k=2}^p \sum_{d=2}^{\alpha_k} \Gamma_{1_2 1 \alpha_1}^{k_d} \Gamma_{i_a k_d}^{i_a} \right\} \\ & - \sum_{i=p+1}^m \sum_{1 \leq b < a \leq \alpha_i} \Gamma_{i_a 1_2}^{i_b} \Gamma_{i_b 1_{\alpha_1}}^{i_a} - \sum_{\substack{p+1 \leq k \leq m \\ k \neq i}} \sum_{a=1}^{\alpha_i} \sum_{\substack{1 \leq b \leq \alpha_k \\ b < a}} \Gamma_{i_a 1_2}^{k_b} \Gamma_{k_b 1_{\alpha_1}}^{i_a} = 0, \end{aligned} \quad (88)$$

$$\begin{aligned} & \sum_{i=p+1}^m \sum_{a=1}^{\alpha_i} \left\{ \sum_{d=2}^{\alpha_1-1} \Gamma_{1_1 1 \alpha_1}^{1_d} \Gamma_{i_a 1_d}^{i_a} + \sum_{k=2}^p \sum_{d=1}^{\alpha_k} \Gamma_{1_2 1 \alpha_1}^{k_d} \Gamma_{i_a k_d}^{i_a} \right\} \\ & - \sum_{i=p+1}^m \sum_{1 \leq b < a \leq \alpha_i} \Gamma_{i_a 1_1}^{i_b} \Gamma_{i_b 1_{\alpha_1}}^{i_a} - \sum_{\substack{p+1 \leq k \leq m \\ k \neq i}} \sum_{a=1}^{\alpha_i} \sum_{\substack{1 \leq b \leq \alpha_k \\ b < a}} \Gamma_{i_a 1_1}^{k_b} \Gamma_{k_b 1_{\alpha_1}}^{i_a} \\ & = (\alpha_{p+1} + \dots + \alpha_m) \left(c\varepsilon_1 - \frac{1}{2} n\varepsilon_1 H \lambda_m \right). \end{aligned} \quad (89)$$

Since (50) and (56), we find

$$\sum_{i=p+1}^m \sum_{1 \leq b < a \leq \alpha_i} \Gamma_{i_a 1_d}^{i_b} \Gamma_{i_b 1_{\alpha_1}}^{i_a} = \sum_{\substack{p+1 \leq k \leq m \\ k \neq i}} \sum_{a=1}^{\alpha_i} \sum_{\substack{1 \leq b \leq \alpha_k \\ b < a}} \Gamma_{i_a 1_d}^{k_b} \Gamma_{k_b 1_{\alpha_1}}^{i_a} = 0. \quad (90)$$

So, (88) and (89) can be simplified to

$$\sum_{i=p+1}^m \sum_{a=1}^{\alpha_i} \left\{ \sum_{d=3}^{\alpha_1-1} \Gamma_{1_2 1 \alpha_1}^{1_d} \Gamma_{i_a 1_d}^{i_a} + \sum_{k=2}^p \sum_{d=2}^{\alpha_k} \Gamma_{1_2 1 \alpha_1}^{k_d} \Gamma_{i_a k_d}^{i_a} \right\} = 0, \quad (91)$$

$$\begin{aligned} & \sum_{i=p+1}^m \sum_{a=1}^{\alpha_i} \left\{ \sum_{d=2}^{\alpha_1-1} \Gamma_{1_1 1 \alpha_1}^{1_d} \Gamma_{i_a 1_d}^{i_a} + \sum_{k=2}^p \sum_{d=1}^{\alpha_k} \Gamma_{1_1 1 \alpha_1}^{k_d} \Gamma_{i_a k_d}^{i_a} \right\} \\ & = (\alpha_{p+1} + \dots + \alpha_m) \left(c\varepsilon_1 - \frac{1}{2} n\varepsilon_1 H \lambda_m \right). \end{aligned} \quad (92)$$

It follows from (52) that

$$\sum_{i=p+1}^m \sum_{a=1}^{\alpha_i} \left\{ \left(-\frac{n}{2} \varepsilon H - \lambda_m \right) \Gamma_{i_a k_d}^{i_a} + \Gamma_{i_a k_{d+1}}^{i_a} \right\} = 0, \quad (93)$$

with $1 \leq k \leq p$ and $k_d \neq 1_1$.

Multiply both sides of (92) with $-(n/2)\varepsilon H - \lambda_m$, we get

$$\begin{aligned} & \sum_{d=2}^{\alpha_1-1} \Gamma_{1_1 1 \alpha_1}^{1_d} \sum_{i=p+1}^m \sum_{a=1}^{\alpha_i} \left(-\frac{n}{2} \varepsilon H - \lambda_m \right) \Gamma_{i_a 1_d}^{i_a} \\ & + \sum_{k=2}^p \sum_{d=1}^{\alpha_k} \Gamma_{1_a 1 \alpha_1}^{k_d} \sum_{i=p+1}^m \sum_{a=1}^{\alpha_i} \left(-\frac{n}{2} \varepsilon H - \lambda_m \right) \Gamma_{i_a k_d}^{i_a} \\ & = \varepsilon_1 (\alpha_{p+1} + \dots + \alpha_m) \left(-\frac{n}{2} \varepsilon H - \lambda_m \right) \left(c - \frac{1}{2} n H \lambda_m \right). \end{aligned} \quad (94)$$

Combining (87) and (93), it gives us

$$\begin{aligned} & \sum_{d=2}^{\alpha_1-2} \Gamma_{1_1 1 \alpha_1}^{1_d} \sum_{i=p+1}^m \sum_{a=1}^{\alpha_i} \Gamma_{i_a 1_{d+1}}^{i_a} - \sum_{k=2}^p \sum_{d=1}^{\alpha_k-1} \Gamma_{1_a 1 \alpha_1}^{k_d} \sum_{i=p+1}^m \sum_{a=1}^{\alpha_i} \Gamma_{i_a k_{d+1}}^{k_d} \\ & = \varepsilon_1 (\alpha_{p+1} + \dots + \alpha_m) \left(-\frac{n}{2} \varepsilon H - \lambda_m \right) \left(c - \frac{1}{2} n H \lambda_m \right). \end{aligned} \quad (95)$$

Since (91), we get from (95) that

$$\left(-\frac{n}{2}\varepsilon H - \lambda_m\right)\left(c - \frac{1}{2}nH\lambda_m\right) = 0, \quad (96)$$

which combining $\lambda_m \neq -(n/2)\varepsilon H$ and $\lambda_m = (2 + \alpha_1 + \dots + \alpha_p)n\varepsilon H/2(\alpha_{p+1} + \dots + \alpha_m)$, tells us that $c - ((2 + \alpha_1 + \dots + \alpha_p)n^2\varepsilon H^2/4(\alpha_{p+1} + \dots + \alpha_m)) = 0$.

So, H is a constant, a contradiction.

In view of the two cases, we complete the proof of Theorem 1.

Remark 4. With the assumption that the shape operator is diagonalizable, the result of Theorem 1 was proved in [6, 8, 12] when the hypersurface M_r^n has at most three distinct principal curvatures.

Applying Theorem 1, we can get the following theorem.

Theorem 5. Let M_r^n be a nondegenerate biharmonic hypersurface of pseudo-Riemannian space form $N_s^{n+1}(c)$, with $c\varepsilon \leq 0$. Suppose that M_r^n has at most two distinct principal curvatures, which are all real numbers, then M_r^n is minimal.

Proof. We know from Theorem 1 that H is a constant; it follows from (6) that

$$HtrA^2 = n\varepsilon cH. \quad (97)$$

When $c\varepsilon \leq 0$, it is easy to see from (97) that $H = 0$. When $c\varepsilon \leq 0$, then (97) implies $H = 0$, or $trA^2 = 0$. Since trA^2 is equal to the sum of the squares of all principal curvatures and trA^2 is equal to the sum of all principal curvatures, so $trA^2 = 0$ tells us $trA = 0$. Combining $trA = n\varepsilon H$, we have $H = 0$, i.e., M_r^n is minimal.

For odd-dimensional hypersurfaces, under the assumption that hypersurfaces have at most two distinct principal curvatures, the principal curvatures are all real. So, from Theorem 5, the following theorems are true.

Theorem 6. Let M_r^{2k-1} be a nondegenerate biharmonic hypersurface of pseudo-Euclidean space $\mathbb{E}_s^{2k}$. Suppose that M_r^{2k-1} has at most two distinct principal curvatures, then M_r^{2k-1} is minimal.

Theorem 7. Let M_{s-1}^{2k-1} be a nondegenerate biharmonic hypersurface of de Sitter space $\mathbb{S}_s^{2k}(c)$. Suppose that M_{s-1}^{2k-1} has at most two distinct principal curvatures, then M_{s-1}^{2k-1} is minimal.

Theorem 8. Let M_s^{2k-1} be a nondegenerate biharmonic hypersurface of anti-de Sitter space $\mathbb{H}_s^{2k}(c)$. Suppose that M_s^{2k-1} has at most two distinct principal curvatures, then M_s^{2k-1} is minimal.

Remark 9. Under the assumption that the shape operator is diagonalizable, the results of Theorems 6, 7, and 8 were proven not only for odd-dimensional hypersurfaces but also for even-dimensional hypersurfaces (cf. [6, 8]).

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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