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Research Article

Existence of Solutions for a Schrödinger–Poisson System with Critical Nonlocal Term and General Nonlinearity

Jiafeng Zhang, Wei Guo, Changmu Chu, and Hongmin Suo

School of Data Science and Information Engineering, Guizhou Minzu University, Guiyang 550025, China

Correspondence should be addressed to Jiafeng Zhang; jiafengzhang@163.com

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We study the existence and multiplicity of nontrivial solutions for a Schrödinger–Poisson system involving critical nonlocal term and general nonlinearity. Based on the variational method and analysis technique, we obtain the existence of two nontrivial solutions for this system.

1. Introduction and Main Result

The Schrödinger–Poisson system is usually used to describe solitary waves for the nonlinear stationary Schrödinger equations interacting with an electromagnetic field. Since the introduction of the Schrödinger–Poisson system by Benci and Fortunato [1], it has been extensively studied. For more detailed information, we refer the interested readers to [2, 3] and the references therein.

In recent years, some researchers extensively studied the Schrödinger–Poisson system with critical growth in an unbounded domain and obtained interesting results under various suitable assumptions (see, e.g., [4–10]).

But there are currently only a few results for the following Schrödinger–Poisson systems with critical nonlocal terms in a bounded domain [11–13]:

$$\begin{cases} -\Delta u = q\eta\phi|u|^{2*-3}u + g(x,u), \text{ in } \Omega, \\ -\Delta\phi = q|u|^{2*-1}, \text{ in } \Omega, \\ \phi = u = 0, \text{ on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^n$; $N \geq 3$, q, η are real numbers; and g is a continuous function satisfying some suitable assumptions. In [11], assuming that $\eta = 1$ and $N = 3$, $g(x,u) = \lambda u$, the author proved that system (1) has a positive ground state solution for any $q > 0$ and $\lambda \in ((3/10)\lambda_1, \lambda_1)$, where λ is a real number and λ_1 is the first eigenvalue of $-\Delta$. Later, when $q = 1$, $\eta = 1$, $g(x,u) = \lambda u$, where λ is a real number, the authors in [12] studied system (1); they proved existence and nonexistence results of positive solutions when $N = 3$ and existence of solutions in both the resonance and the nonresonance case for higher dimensions. For the case $q = 1$, $g(x,u) = \lambda f(x)$, $\lambda > 0$ is a real number, $f \geq 0$, and $f \in L^{2*/(2*-1)}(\mathbb{R}^N)$, in [13]; when $\eta = 1$, authors proved that system (1) has at least two positive solutions if $0 < \lambda < \lambda*$ for some $\lambda* > 0$ small enough, and when $\eta = -1$, system (1) has at least one positive solution for any $\lambda > 0$.

On the basis of the above literature, this paper continues to study system (1), and intends to deal with the following Schrödinger–Poisson system with critical nonlocal term and general nonlinearity that without (AR) condition:

$$\begin{cases} -\Delta u = \phi|u|^{2*-3}u + g(x,u), \text{ in } \Omega, \\ -\Delta\phi = |u|^{2*-1}, \text{ in } \Omega, \\ \phi = u = 0, \text{ on } \partial\Omega, \end{cases} \tag{2}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is an open bounded domain with smooth boundary $\partial\Omega$,

$2* = 2N/(N-2)$ is the critical Sobolev exponent, and $g \in C(\Omega \times \mathbb{R}, \mathbb{R})$, $G(x,t) = \int_0^t g(x,s)ds$..

Throughout this paper, we make the following assumptions:

(g_1) $g \in C(\Omega \times \mathbb{R}, \mathbb{R})$, and there exist constants $c_1, c_2 > 0$ with c_1 which is small enough and $p \in (2, 2^*)$ such that $|g(x,t)| \le c_1 |t| + c_2 |t|^{p-1}, \forall (x,t) \in \Omega \times \mathbb{R}$.

(g_2) There exists a constant $K > 0$ big enough such that $G(x,t) \ge Kt^2$ for any $x \in \Omega$ and $|t| > 0$ large enough.

(g_3) There are constants $\rho > 2$ and $\nu > 0$ such that $\rho G(x,t) \le g(x,t)t + \nu t^2, \forall (x,t) \in \Omega \times \mathbb{R}$.

The main difficulties in the present paper are to estimate the critical value and prove the boundedness of (PS) sequence due to the lack of compactness. In order to overcome the above difficulties, by analytic techniques, we shall give the estimate of critical value of associated functional so that system (2) has at least two nontrivial solutions.

Throughout this paper, we use the following notations:

(i) The space $H_0^1(\Omega)$ has the inner product $(u,v) = \int_\Omega \nabla u \cdot \nabla v dx$and the norm $\|u\|^2 = \int_\Omega |\nabla u|^2 dx$, and the norm in $L^p(\Omega)$ is denoted by $|u|_p = (\int_\Omega |u|^p dx)^{1/p}$

(ii) B_r (respectively, ∂B_r) denotes the closed ball (respectively, the sphere) of center zero and radius r, i.e., $B_r = \{u \in H_0^1(\Omega) \colon \|u\| \le r\}, \partial B_r = \{u \in H_0^1(\Omega) \colon \|u\| = r\}$

(iii) C, C_0, C_1, C_2 ,... denote various positive constants, which may vary from line to line

(iv) Define the best constant $S = \inf\{\|u\|^2 : u \in H_0^1(\Omega) \int_\Omega |u|^{2^*} dx = 1\}$, which is attained by the functions $y_\varepsilon(x) = C_\varepsilon/(\varepsilon + |x|^2)^{(N-2)/2}$ for all $\varepsilon > 0$, where $C_\varepsilon = [N(N-2)\varepsilon]^{(N-2)/4}$.

Theorem 1. *Assume that $N \ge 3$, g satisfies (g_1), (g_2), and (g_3). Then, system (2) possesses at least two distinct nontrivial function pair solutions.*

Remark 2. Relative to [11, 12], the nonlinearity g is of a pure power form in [11, 12], and in the present paper, it is a general nonlinearity. Hence, we make a substantial improvement on the works of the former.

2. Proof of Main Result

Before proving our Theorem 1, we need the following lemmas.

Lemma 3 ([12, 13]). *For every fixed $u \in H_0^1(\Omega)$, there exists a unique $\phi_u \in H_0^1(\Omega)$ that solves the second equation of (2), and*

(i) $\phi_u \ge 0$ a.e.in Ω

(ii) For all $t > 0$, $\phi_{tu} = t^{2^*-1}\phi_u$

(iii) $\|\phi_u\| \le S^{-2*/2}\|u\|^{2*-1}$

(iv) $\int_\Omega |u|^{2*} dx = \int_\Omega |\nabla \phi_u||\nabla u| dx \le (1/2)\|\phi_u\|^2 + (1/2)\|u\|^2$

Hence, according to the standard arguments as those in [1], system (2) can be converted into the following boundary value problem:

$$\begin{cases} -\Delta u = \phi_u |u|^{2*-3} u + g(x,u), \text{ in } \Omega, \\ u = 0, \text{ on } \partial\Omega. \end{cases} \tag{3}$$

In order to study the existence of nontrivial solutions to problem (3), we shall firstly consider the existence of nontrivial solutions of the following problem:

$$\begin{cases} -\Delta u = \phi_u + (u^+)^{2*-3} + g^+(x,u), \text{ in } \Omega, \\ u = 0, \text{ on } \partial\Omega, \end{cases} \tag{4}$$

where

$$u^+ = \max\{u, 0\},$$
$$g^+(x,t) = \begin{cases} g(x,t), & t \ge 0, \\ 0, & t < 0. \end{cases} \tag{5}$$

The energy functional corresponding to (4) is

$$J(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2(2*-1)}\int_\Omega \phi_u + (u^+)^{2*-1} dx - \int_\Omega G^+(x,u) dx, \quad u \in H_0^1(\Omega), \tag{6}$$

where

$$G^+(x,t) = \int_0^t g^+(x,s) ds. \tag{7}$$

J is well defined with $J \in C^1(H_0^1(\Omega), \mathbb{R})$ and

$$\left\langle J'(u), v \right\rangle = (u,v) - \int_\Omega \phi_u + (u^+)^{2*-2} v dx - \int_\Omega g^+(x,u) v dx, \quad u, v \in H_0^1(\Omega). \tag{8}$$

The critical points of the functional J are just weak solutions of problem (4). Let $U_\varepsilon(x) = y_\varepsilon(x)/C_\varepsilon$ define a cutoff function $\varphi \in C_0^\infty(\Omega)$ such that

$$\varphi(x) = \begin{cases} 1, & |x| \le R, \\ 0, & |x| \ge 2R, \end{cases} \tag{9}$$

where $B_{2R}(0) \subset \Omega, 0 \le \varphi(x) \le 1$ for $R < |x| < 2R$.

Put $u_\varepsilon(x) = \varphi(x) U_\varepsilon(x), v_\varepsilon(x) = u_\varepsilon(x)/(\int_\Omega |u_\varepsilon|^{2*} dx)^{1/2*}$; hence, $\int_\Omega |v_\varepsilon|^{2*} dx = 1$.

Lemma 4 ([14, 15]). *$v_\varepsilon(x)$ satisfies the following estimates:*

$$\|v_\varepsilon\|^2 = S + O\left(\varepsilon^{(N-2)/2}\right), \tag{10}$$

$$\int_\Omega |v_\varepsilon|^q dx = \begin{cases} O\left(\varepsilon^{((N-2)q)/4}\right), & 1 \le q < \dfrac{N}{N-2}, \\ O\left(\varepsilon^{N/4}|\ln \varepsilon|\right), & q = \dfrac{N}{N-2}, \\ O\left(\varepsilon^{(2N-(N-2)q)/4}\right), & \dfrac{N}{N-2} < q < 2*. \end{cases} \tag{11}$$

Lemma 5. *Assume (g_1) and (g_3) hold; let $\{u_n\} \subset H_0^1(\Omega)$ be a sequence such that $J(u_n) \longrightarrow c, J'(u_n) \longrightarrow 0$, where $c \in (0, 2/(N+2)S^{N/2})$. Then, there exists $u \in H_0^1(\Omega)$ such that $u_n \longrightarrow u$, up to a subsequence. $J'(u) = 0$ and u is a nontrivial solution of problem (4).*

Proof. First, we prove that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. To prove the boundedness of $\{u_n\}$, arguing by contradiction, suppose that $\|u_n\| \longrightarrow \infty$. Set $v_n = u_n/\|u_n\|$. Then, $\|v_n\| = 1$ and $\|v_n\|_q \le C$ for $1 \le q \le 2^*$. By (g_3), we have

$$\begin{aligned} c + 1 + o(1)\|u_n\| &\ge J(u_n) - \frac{1}{\theta}\left\langle J'(u_n), u_n\right\rangle = \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_n\|^2 \\ &+ \left(\frac{1}{\theta} - \frac{1}{2(2^*-1)}\right)\int_\Omega \phi_{u_n^+}(u_n^+)^{2*-1}dx + \frac{1}{\theta}\int_\Omega g^+(x, u_n)u_n dx \\ &- \int_\Omega G^+(x, u_n)dx \ge \frac{\theta-2}{2\theta}\|u_n\|^2 \\ &- \frac{\nu}{\theta}\|u_n\|_2^2 = \|u_n\|^2\left(\frac{\theta-2}{2\theta} - \frac{\nu}{\theta}\|u_n\|_2^2\right), \end{aligned} \tag{12}$$

where $\theta = \min\{2^*, \rho\}$, which implies

$$1 \le \frac{2\nu}{\theta - 2}\liminf_{n\longrightarrow\infty}\|v_n\|_2^2. \tag{13}$$

Passing to a subsequence, we may assume that $v_n \longrightarrow v$ in $H_0^1(\Omega)$, then $v_n \longrightarrow v$ in $L^q(\Omega)$, $1 \le q < 2^*$, and $v_n \longrightarrow v$ a.e. in Ω. Hence, it follows from (13) that $v \ne 0$, and

$$\begin{aligned} 0 &= \lim_{n\longrightarrow\infty}\frac{c+o(1)}{\|u_n\|^2} = \lim_{n\longrightarrow\infty}\frac{J(u_n)}{\|u_n\|^2} \\ &= \lim_{n\longrightarrow\infty}\left[\frac{1}{2} - \frac{1}{2(2^*-1)\|u_n\|^2}\int_\Omega \phi_{u_n^+}(u_n^+)^{2*-1}dx - \int_\Omega \frac{G^+(x,u_n)}{u_n^2}v_n^2 dx\right] \\ &\le \lim_{n\longrightarrow\infty}\left[\frac{1}{2} - K\int_{\{x\in\Omega:|v(x)|\ge\sigma\}} v_n^2 dx\right] = \frac{1}{2} - K\int_\Omega v^2 dx < 0, \end{aligned} \tag{14}$$

where $\sigma > 0$ is chosen such that $|\{x \in \Omega : |v(x)| \ge \sigma\}| > 0$ and K is sufficiently big constant, which is a contradiction. Thus, $\{u_n\}$ is bounded in $H_0^1(\Omega)$ and there exists $u \in H_0^1(\Omega)$ such that $u_n \longrightarrow u$, up to a subsequence. Furthermore, $J'(u) = 0$ by the weak continuity of J'. If $u = 0$ in Ω, since the term $g(x, u)$ is subcritical, then $\langle J'(u_n), u_n\rangle = o(1)$ implies

$$\|u_n\|^2 - \int_\Omega \phi_{u_n^+}(u_n^+)^{2*-1}dx = o(1). \tag{15}$$

By Lemma 3-(iii), one has

$$\left\|\phi_{u_n^+}\right\| \le S^{-2*/2}\|u_n^+\|^{2*-1}. \tag{16}$$

It follows from (15) and (16) that

$$o(1) \ge \|u_n\|^2\left[1 - S^{-2^*}\|u_n\|^{2(2^*-2)}\right]. \tag{17}$$

If $\|u_n\| \longrightarrow 0$, it contradicts $c > 0$. Therefore,

$$\|u_n\|^2 \ge S^{N/2} + o(1). \tag{18}$$

By (15) and (18), we get

$$\begin{aligned} J(u_n) &= \frac{1}{2}\|u_n\|^2 - \frac{1}{2(2*-1)}\int_\Omega \phi_{u_n^+}(u_n^+)^{2*-1}dx + o(1) \\ &= \frac{2}{N+2}\|u_n\|^2 + o(1) \ge \frac{2}{N+2}S^{N/2} + o(1), \end{aligned} \tag{19}$$

which contradicts $c < 2/(N+2)S^{N/2}$. Thus, $u \ne 0$ and it is a nontrivial solution of problem (4).

Lemma 6. *Assume that g satisfies (g_1) and (g_2). Then, for $\varepsilon > 0$ small enough, $\sup_{t\ge 0} J(tv_\varepsilon) < 2/(N+2)S^{N/2}$.*

Proof. For $t \ge 0$, we consider the functions

$$\begin{aligned} h(t) &:= J(tv_\varepsilon) = \frac{t^2}{2}\|v_\varepsilon\|^2 - \frac{t^{2(2*-1)}}{2(2*-1)}\int_\Omega \phi_{v\varepsilon}|v_\varepsilon|^{2*-1}dx \\ &\quad - \int_\Omega G^+(x, tv_\varepsilon)dx \le \frac{Nt^2}{N+2}\|v_\varepsilon\|^2 - \frac{t^{2*}}{2*-1} \\ &\quad - \int_\Omega G^+(x, tv_\varepsilon)dx, \\ \bar{h}(t) &:= \frac{Nt^2}{N+2}\|v_\varepsilon\|^2 - \frac{t^{2*}}{2*-1}, \end{aligned} \tag{20}$$

where the above inequality comes from Lemma 3-(iv).

Notice that $\lim t \longrightarrow +\infty\ h(t) = -\infty, h(0) = 0$, and $h(t) > 0$ as t is sufficiently small. Therefore, $\sup_{t\ge 0} h(t) > 0$ is attained for some $t_\varepsilon > 0$. Since

$$0 = h'(t_\varepsilon) = \frac{2Nt_\varepsilon}{N+2}\|v_\varepsilon\|^2 - \frac{2 * t_\varepsilon^{2*-1}}{2*-1} - \int_\Omega g^+(x, tv_\varepsilon)v_\varepsilon dx, \tag{21}$$

we have

$$\|v_\varepsilon\|^2 = t_\varepsilon^{2*-2} + \frac{N+2}{2Nt_\varepsilon}\int_\Omega g^+(x, tv_\varepsilon)v_\varepsilon dx \geq t_\varepsilon^{2*-2}, \quad (22)$$

where the nonnegativity of $\int_\Omega g^+(x, tv_\varepsilon)v_\varepsilon dx$ comes from (g_2), (g_3), and the definition of g^+. Hence,

$$t_\varepsilon \leq \|v_\varepsilon\|^{2/(2*-2)} \triangleq t_\varepsilon^0, \quad (23)$$

It follows from (g_1) that

$$|g^+(x,t)| \leq c_1|t| + c_2|t|^{p-1}. \quad (24)$$

Hence, we have

$$\|v_\varepsilon\|^2 \leq t_\varepsilon^{2*-2} + \frac{c_1}{2}\int_\Omega |v_\varepsilon|^2 dx + \frac{c_2}{p}|t_\varepsilon|^{p-2}\int_\Omega |v_\varepsilon|^2 dx. \quad (25)$$

By (10), (11), and (25), when ε is sufficiently small, we have $t_\varepsilon^{2*-2} \geq \beta S$ with $\beta \in (0,1)$.

On the other hand, the function $h^-(t)$ attains its maximum at $t_\varepsilon^0 = \|v_\varepsilon\|^{2/(2*-2)}$ and it is increasing in the interval $[0, t_\varepsilon^0]$. By (10), (25), and $G^+(x,t) \geq Kt^2$ for $t \geq 0$, we deduce that

$$\begin{aligned} h(t_\varepsilon) &\leq \bar{h}(t_\varepsilon^0) - \int_\Omega G^+(x, t_\varepsilon v_\varepsilon)dx \leq \frac{2}{N+2}\|v_\varepsilon\|^N \\ &- \int_\Omega G^+(x, t_\varepsilon v_\varepsilon)dx \leq \frac{2}{N+2}\|v_\varepsilon\|^N - K\int_\Omega t_\varepsilon^2|v_\varepsilon|^2 dx \\ &\leq \frac{2}{N+2}S^{N/2} + O\left(\varepsilon^{(N-2)/2}\right) - KS^{2/(2*-2)}\int_\Omega |v_\varepsilon|^2 dx. \end{aligned} \quad (26)$$

From (11) and the fact that K is sufficiently large, by choosing sufficiently small ε, we can obtain

$$\sup_{t\geq 0} J(tv_\varepsilon) = h(t_\varepsilon) < \frac{2}{N+2}S^{N/2}. \quad (27)$$

Proof of Theorem 1. It follows from (g_1) that

$$|g^+(x,t)| \leq c_1|t| + c_2|t|^{p-1}, \quad (28)$$

$$|G^+(x,t)| \leq \frac{1}{2}c_1|t|^2 + \frac{1}{p}c_2|t|^p, \quad (29)$$

for all $t \in \mathbb{R}$ and $x \in \Omega$. By the Sobolev inequality, (28) and (29), for c_1 small enough, one has

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|^2 - \frac{1}{2(2*-1)}\int_\Omega \phi_{u_n^+}(u_n^+)^{2*-1}dx - \int_\Omega G^+(x,u)dx \\ &\geq \frac{1}{2}\|u\|^2 \frac{1}{2(2*-1)}S^{-2*}\|u\|^{2(2*-1)} - \frac{C}{2}\|u\|_2^2 - \frac{C}{p}\|u\|_p^p \\ &\geq \frac{1-C}{2}\|u\|^2 - \frac{1}{2(2*-1)}S^{-2*}\|u\|^{2(2*-1)} - \frac{C}{p}\|u\|^P. \end{aligned} \quad (30)$$

So, when $r > 0$ is sufficiently small, there is $\alpha > 0$ such that $J(u) \geq \alpha > 0$ for $u \in \partial B_r(0)$. Moreover, by the nonnegativity of $G^+(x,u)$, for $u_0 \in H_0^1(\Omega)\setminus\{0\}$, it holds that

$$\begin{aligned} J(tu_0) &= \frac{t^2}{2}\|u_0\|^2 - \frac{t^{2(2*-1)}}{2(2*-1)}\int_\Omega \phi_{u_0}|u_0|^{2*-1}dx - \int_\Omega G^+(x, tu_0) \\ &\leq \frac{t^2}{2}\|u_0\|^2 - \frac{t^{2(2*-1)}}{2(2*-1)}\int_\Omega \phi_{u_0}|u_0|^{2*-1}dx, \\ \lim_{t\to\infty} J(tu_0) &\longrightarrow -\infty, \qquad \text{as } t \longrightarrow \infty. \end{aligned} \quad (31)$$

Hence, we can choose $t_0 > 0$ such that $\|t_0u_0\| > r$ and $J(t_0u_0) \leq 0$. Using the Mountain Pass Lemma, there is a sequence $\{u_n\} \subset H_0^1(\Omega)$ satisfying

$$\begin{aligned} J(u_n) &\longrightarrow c \geq a, \\ J'(u_n) &\longrightarrow 0, \end{aligned} \quad (32)$$

where

$$\begin{aligned} c &= \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J(\gamma(t)), \\ \Gamma &= \left\{\gamma(t) \in C\left([0,1], H_0^1(\Omega)\right) \mid \gamma(0) = 0, J(\gamma(1)) < 0\right\}. \end{aligned} \quad (33)$$

According to Lemma 5 and Lemma 6, we can get a $(PS)_c$ sequence $\{u_n\} \subset H_0^1(\Omega)$, and $u \in H_0^1(\Omega)$ such that $J'(u) = 0$. Thus, u is a solution of problem (4). And then, $\langle J'(u), u^-\rangle = 0$, where $u^- = \min\{u, 0\}$. Hence, $u^- = 0$, that is, $u \geq 0$. We conclude from the strong maximum principle that u is a positive solution of problem (3).

Next, we give the proof of two nontrivial solutions to system (2).

From the above discussion, problem (3) has a positive solution u_1. Put $k(x,t) = -g(x,-t)$ for $t \in \mathbb{R}$. Note that if u_1 is a solution of (3), then, $-u_1$ is a solution of (3) replacing g with k. Hence, the equation

$$\begin{cases} -\Delta u = \phi_u|u|^{2*-3}u + k(x,u), \text{ in } \Omega, \\ u = 0, \text{ on } \partial\Omega \end{cases} \quad (34)$$

has at least a positive solution v. Let $u_2 = v$; then, u_2 is a solution of (3).

Obviously $u_1 \geq 0, u_2 \leq 0$. So, problem (3) has at least two distinct nontrivial solutions u_1 and u_1; therefore, similar to [4, 5], system (2) has at least two distinct nontrivial function pair solutions (u_1, ϕ_{u1}) and (u_2, ϕ_{u2}).

Data Availability

The findings in this research do not make use of data.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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