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Research Article

Weighted Composition Operators from Besov Zygmund-Type Spaces into Zygmund-Type Spaces

Xiangling Zhu [1] and Nanhui Hu [2,3]

[1] University of Electronic Science and Technology of China, Zhongshan Institute, 528402 Zhongshan, Guangdong, China
[2] Department of Mathematics, Shantou University, Shantou, 515063 Guangdong, China
[3] Department of Mathematics, Jiaying University, Meizhou, 514015 Guangdong, China

Correspondence should be addressed to Nanhui Hu; hunhjyu@163.com

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The boundedness, compactness, and essential norm of weighted composition operators from Besov Zygmund-type spaces into Zygmund-type spaces are investigated in this paper.

1. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the space of all analytic functions in $\mathbb{D}$. For an analytic self-map φ of $\mathbb{D}$ and $u \in H(\mathbb{D})$, the weighted composition operator uC_φ is defined as follows:

$$\left(uC_\varphi f\right)(z) = u(z)f(\varphi(z)), \quad f \in H(\mathbb{D}), z \in \mathbb{D}. \tag{1}$$

When $u = 1$, uC_φ is just the composition operator, denoted by C_φ. In the past several decades, composition operators and weighted composition operators have received much attention and appear in various settings in the literature (see, for example, [2–5, 8, 10, 13, 15, 16, 19]).

Let $\alpha \in (0,\infty)$. The Bloch type space $\mathscr{B}^\alpha$ consists of those functions $f \in H(\mathbb{D})$ for which

$$\|f\|_{\mathscr{B}^\alpha} = |f(0)| + \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\alpha |f'(z)| < \infty. \tag{2}$$

$\mathscr{B}^\alpha$ is a Banach space under the above norm. It is known that when $\alpha = 1$, $\mathscr{B}^1 = \mathscr{B}$ is the classical Bloch space.

For $0 < \beta < \infty$, an $f \in H(\mathbb{D})$ is said to be in the Zygmund-type space $\mathscr{Z}^\beta$, if

$$\|f\|_{\mathscr{Z}^\beta} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\beta |f''(z)| < \infty. \tag{3}$$

It is easy to check that $\mathscr{Z}^\beta$ is a Banach space under the norm $\|\cdot\|_{\mathscr{Z}^\beta}$. When $\beta = 1$, $\mathscr{Z}^1 = \mathscr{Z}$ is the Zygmund space. When $\beta > 1$, $\mathscr{Z}^\beta$ is just the Bloch type space $\mathscr{B}^{\beta-1}$. In particular, when $\beta = 2$, $\mathscr{Z}^\beta$ is just the Bloch space $\mathscr{B}$. Hence, the Zygmund space is the space of all $f \in H(\mathbb{D})$ such that $f' \in \mathscr{B}$ with norm

$$\|f\|_{\mathscr{Z}} = |f(0)| + \|f'\|_{\mathscr{B}}. \tag{4}$$

Let dA be the normalized area measure on $\mathbb{D}$. For $1 < p < \infty$, the Besov space, denoted by B_p, is the space of all $f \in H(\mathbb{D})$ such that

$$b_p(f) := \left(\int_{\mathbb{D}} \left|f'(z)\right|^p \left(1-|z|^2\right)^{p-2} dA(z)\right)^{1/p} < \infty. \tag{5}$$

This space is a Banach space with the following norm $\|f\|_{B_p} = |f(0)| + b_p(f)$. In particular, B_2 is the classical Dirichlet space. Besov spaces are Möbius invariant in the sense that $b_p(f \circ \psi) = b_p(f)$ for all $f \in B_p$ and $\psi \in \mathrm{Aut}(\mathbb{D})$, the set of all Möbius maps of $\mathbb{D}$ (see [1, 19]).

In [4], Colonna and Tjani introduced a new class type space $\mathcal{Z}^p_{p-2}$, called the Besov Zygmund-type space, which consists of all $f \in H(\mathbb{D})$ such that $f' \in B_p$. Since the Besov space is contained in the Bloch space, it follows that the Besov Zygmund-type space is a subset of the Zygmund space, and hence, it is contained in the disk algebra.

Colonna and Li studied the boundedness and compactness of the operator $uC_\varphi : H^\infty \longrightarrow \mathcal{Z}$ and $uC_\varphi : \mathcal{B}^\alpha (0 < \alpha < 1) \longrightarrow \mathcal{Z}$ in [2, 3], respectively. Here, H^∞ is the space of bounded analytic functions. (See [2, 3, 6–14, 16, 17] for more results of composition operators, weighted composition operators, and related operators on the Zygmund space and Zygmund-type spaces.) Colonna and Tjani characterized the boundedness and compactness of $uC_\varphi : \mathcal{Z}^p_{p-2} \longrightarrow \mathcal{B}^\alpha$ in [4].

In this work, we give some characterizations for the boundedness, compactness, and the essential norm of the operator $uC_\varphi : \mathcal{Z}^p_{p-2} \longrightarrow \mathcal{Z}^\beta$.

Throughout the paper, we denote by C a positive constant which may differ from one occurrence to the next. In addition, we say that $A \lesssim B$ if there exists a constant C such that $A \leq CB$. The symbol $A \approx B$ means that $A \lesssim B \lesssim A$.

2. Main Results and Proofs

In this section, we formulate and prove our main results in this paper. For this purpose, we need the following lemmas.

Lemma 1. *Suppose $1 < p < \infty$. Then, there exists a positive constant C such that*

$$|f'(z)| \leq C\|f\|_{\mathcal{Z}^p_{p-2}} \left(\log \frac{2}{1-|z|^2}\right)^{1-(1/p)}, \tag{6}$$

$$\left|f''(z)\right| \leq \frac{C\|f\|_{\mathcal{Z}^p_{p-2}}}{1-|z|^2}, \tag{7}$$

$$\|f\|_\infty \leq C\|f\|_{\mathcal{Z}^p_{p-2}} \tag{8}$$

for every $f \in \mathcal{Z}^p_{p-2}$.

Proof. For $f \in B_p$, it is well known that

$$|f(z)| \lesssim \|f\|_{B_p} \left(\log \frac{2}{1-|z|^2}\right)^{1-(1/p)}, \quad z \in \mathbb{D},$$
$$\|f\|_{\mathcal{B}} \lesssim \|f\|_{B_p}. \tag{9}$$

Then, the inequalities in (6) follow from the definition of the Besov Zygmund-type space. Since the Zygmund space is continuously embedded into H^∞, as shown in Lemma 2.1 of [18], we get that $\|f\|_\infty \leq C\|f\|_{\mathcal{Z}^p_{p-2}}$. The proof is complete.

Lemma 2 (see [4]). *Let $1 < p < \infty$. Every sequence in $\mathcal{Z}^p_{p-2}$ bounded in norm has a subsequence which converges uniformly in $\bar{\mathbb{D}}$ to a function in $\mathcal{Z}^p_{p-2}$.*

Lemma 3 (see [4]). *Let X be a Banach space that is continuously contained in the disk algebra, and let Y be any Banach space of analytic functions on $\mathbb{D}$. Suppose that*

(i) the point evaluation functionals on Y are continuous

(ii) for every sequence $\{f_n\}$ in the unit ball of X that exists $f \in X$ and a subsequence $\{f_{n_j}\}$ such that $f_{n_j} \longrightarrow f$ uniformly on $\bar{\mathbb{D}}$

(iii) the operator $T : X \longrightarrow Y$ is continuous if X has the supremum norm and Y is given the topology of uniform convergence on compact sets

Then, T is a compact operator if and only if given a bounded sequence $\{f_n\}$ in X such that $f_n \longrightarrow 0$ uniformly on $\bar{\mathbb{D}}$, then the sequence $\|Tf_n\|_Y \longrightarrow 0$ as $n \to \infty$.

The following result is a direct consequence of Lemmas 2 and 3.

Lemma 4. *Let $1 < p < \infty$ and $0 < \beta < \infty$. If $T : \mathcal{Z}^p_{p-2} \longrightarrow \mathcal{Z}^\beta$ is bounded, then T is compact if and only if $\|Tf_k\|_{\mathcal{Z}^\beta} \longrightarrow 0$ as $k \longrightarrow \infty$ for any sequence $\{f_k\}$ in $\mathcal{Z}^p_{p-2}$ bounded in norm which converge to 0 uniformly in $\bar{\mathbb{D}}$.*

The following estimates are fundamental in operator theory and function spaces on the unit disk (see ([19], Lemma 3.10)).

Lemma 5 (see [19]). *Suppose that $z \in \mathbb{D}$,c is real, $t > -1$, and*

$$I_{c,t}(z) = \int_{\mathbb{D}} \frac{\left(1-|w|^2\right)^t}{|1-\bar{w}z|^{2+t+c}} dA(w). \tag{10}$$

(i) If $c < 0$, then as a function of z,$I_{c,t}$ is bounded on $\mathbb{D}$

(ii) If $c = 0$, then $I_{c,t}(z) \approx \log(1/(1-|z|^2))$, as $|z| \longrightarrow 1$

(iii) If $c > 0$, then $I_{c,t}(z) \approx 1/(1-|z|^2)^c$, as $|z| \longrightarrow 1$

Now, we are in a position to give the following characterization of bound composition operators from $\mathcal{Z}^p_{p-2}$ to $\mathcal{Z}^\beta$.

Theorem 6. *Let $1 < p < \infty$, $0 < \beta < \infty$, $u \in H(\mathbb{D})$, and φ be an analytic self-map of $\mathbb{D}$. Then, $uC_\varphi : \mathcal{Z}^p_{p-2} \longrightarrow \mathcal{Z}^\beta$ is bounded if and only if $u \in \mathcal{Z}^\beta$,*

$$\sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\beta |2u'(z)\varphi'(z) + u(z)\varphi''(z)| \left(\log \frac{2}{1-|\varphi(z)|^2}\right)^{1-(1/p)} < \infty, \tag{11}$$

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^{\beta}|u(z)||\varphi'(z)|^2}{1-|\varphi(z)|^2} < \infty. \quad (12)$$

Proof. First, suppose that $u \in \mathscr{Z}^{\beta}$, (11) and (12) hold. For arbitrary $z \in \mathbb{D}$ and $f \in \mathscr{Z}^{p}_{p-2}$, by Lemma 1, we have

$$\left|(uC_{\varphi}f)(0)\right| \lesssim |u(0)|\|f\|_{\mathscr{Z}^{p}_{p-2}},$$

$$|(uC_{\varphi}f)'(0)| \lesssim \left(|u'(0)|+|u(0)\varphi'(0)| \cdot \left(\log\frac{2}{1-|\varphi(0)|^2}\right)^{1-(1/p)}\right)\|f\|_{\mathscr{Z}^{p}_{p-2}},$$

$$\begin{aligned}
&\sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}|(uC_{\varphi}f)''(z)| \\
&\leq \sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}|u''(z)||f(\varphi(z))| \\
&\quad + \sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}\left|u(z)\left(\varphi'(z)\right)^2\right||f''(\varphi(z))| \\
&\quad + \sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}|2u'(z)\varphi'(z)+u(z)\varphi''(z)||f'(\varphi(z))| \\
&\lesssim \sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}|u''(z)|\|f\|_{\mathscr{Z}^{p}_{p-2}} \\
&\quad + \sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^{\beta}|u(z)||\varphi'(z)|^2}{1-|\varphi(z)|^2}\|f\|_{\mathscr{Z}^{p}_{p-2}} \\
&\quad + \sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}|2u'(z)\varphi'(z) \\
&\quad + u(z)\varphi''(z)|\left(\log\frac{2}{1-|\varphi(z)|^2}\right)^{1-(1/p)}\|f\|_{\mathscr{Z}^{p}_{p-2}} < \infty.
\end{aligned} \quad (13)$$

Therefore, $uC_{\varphi}: \mathscr{Z}^{p}_{p-2} \longrightarrow \mathscr{Z}^{\beta}$ is bounded.

Conversely, suppose that $uC_{\varphi}: \mathscr{Z}^{p}_{p-2} \longrightarrow \mathscr{Z}^{\beta}$ is bounded. Applying the operator uC_{φ} to z^j with $j = 0, 1, 2$, and using the boundedness of uC_{φ}, we get that $u \in \mathscr{Z}^{\beta}$, $u\varphi \in \mathscr{Z}^{\beta}$, and $u\varphi^2 \in \mathscr{Z}^{\beta}$. Hence,

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}|2u'(z)\varphi'(z)+u(z)\varphi''(z)| < \infty, \quad (14)$$

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}|u(z)\left(\varphi'(z)\right)^2| < \infty. \quad (15)$$

For $a \in \mathbb{D}$ such that $|a| > \sqrt{1-2/e}$, set

$$\begin{aligned}
f_a(z) = {}& \frac{(\bar{a}z-1)}{\bar{a}}\left[\left(1+\log\frac{2}{1-\bar{a}z}\right)^2+1\right]\left(\log\frac{2}{1-|a|^2}\right)^{-1-(1/p)} \\
&- 2\int_0^z \log\frac{2}{1-\bar{a}w}dw\left(\log\frac{2}{1-|a|^2}\right)^{-(1/p)}.
\end{aligned} \quad (16)$$

Then,

$$\begin{aligned}
f_a'(z) = {}& \left(\log\frac{2}{1-\bar{a}z}\right)^2\left(\log\frac{2}{1-|a|^2}\right)^{-1-(1/p)} \\
&- 2\log\frac{2}{1-\bar{a}z}\left(\log\frac{2}{1-|a|^2}\right)^{-(1/p)}
\end{aligned}$$

$$\begin{aligned}
f_a''(z) = {}& \frac{2\bar{a}}{1-\bar{a}z}\left(\log\frac{2}{1-\bar{a}z}\right)\left(\log\frac{2}{1-|a|^2}\right)^{-1-(1/p)} \\
&- \frac{2\bar{a}}{1-\bar{a}z}\left(\log\frac{2}{1-|a|^2}\right)^{-(1/p)}.
\end{aligned} \quad (17)$$

Thus, for $\sqrt{1-2/e} < |a| < 1$, we have

$$|f_a''(z)| \leq \frac{2}{|1-\bar{a}z|}\left(\log\frac{2}{1-|a|^2}\right)^{-(1/p)}, \quad (18)$$

and by Lemma 5,

$$\begin{aligned}
&\int_{\mathbb{D}}|f_a''(z)|^p(1-|z|^2)^{p-2}dA(z) \\
&\lesssim \frac{1}{\log\left(2/(1-|a|^2)\right)}\int_{\mathbb{D}}\frac{(1-|z|^2)^{p-2}}{|1-\bar{a}z|^p}dA(z) \lesssim 1.
\end{aligned} \quad (19)$$

So,

$$\sup_{\sqrt{1-2/e}<|a|<1}\|f_a\|_{\mathscr{Z}^{p}_{p-2}} < \infty. \quad (20)$$

Therefore, by the boundedness of $uC_{\varphi}: \mathscr{Z}^{p}_{p-2} \longrightarrow \mathscr{Z}^{\beta}$, we get

$$\sup_{\sqrt{1-2/e}<|a|<1}\|uC_{\varphi}f_a\|_{\mathscr{Z}^{\beta}} < \infty. \quad (21)$$

Since $f_a''(a) = 0$, $|f_a'(a)| = (\log(2/(1-|a|^2)))^{1-(1/p)}$, for any $w \in \mathbb{D}$ such that $\sqrt{1-2/e} < |\varphi(w)| < 1$, we have

$$\begin{aligned}
&(1-|w|^2)^{\beta}|\left(uC_{\varphi}f_{\varphi(w)}\right)''(w)| \\
&= (1-|w|^2)^{\beta}\Big|u''(w)f_{\varphi(w)}(\varphi(w)) \\
&\quad + \left(2u'(w)\varphi'(w)+u(w)\varphi''(w)\right)f'_{\varphi(w)}(\varphi(w))\Big|,
\end{aligned} \quad (22)$$

which implies that

$$\begin{aligned}
&(1-|w|^2)^{\beta}|2u'(w)\varphi'(w) \\
&\quad + u(w)\varphi''(w)|\left(\log\left(2/(1-|\varphi(w)|^2)\right)\right)^{1-(1/p)} \\
&\leq uC_{\varphi}f_{\varphi(w)}\|_{\mathscr{Z}^{\beta}} + (1-|w|^2)^{\beta}|u''(w)f_{\varphi(w)}(\varphi(w))| \\
&\lesssim \|uC_{\varphi}f_{\varphi(w)}\|_{\mathscr{Z}^{\beta}} + \|u\|_{\mathscr{Z}^{\beta}}\|f_{\varphi(w)}\|_{\mathscr{Z}^{p}_{p-2}} < \infty.
\end{aligned} \quad (23)$$

By (14), we get

$$\sup_{|\varphi(w)|\le\sqrt{1-2/e}}\left(1-|w|^2\right)^{\beta}|2u'(w)\varphi'(w)+u(w)\varphi''(w)|\left(\log\left(2/\left(1-|\varphi(w)|^2\right)\right)\right)^{1-(1/p)} \lesssim \sup_{w\in\mathbb{D}}\left(1-|w|^2\right)^{\beta}|2u'(w)\varphi'(w)+u(w)\varphi''(w)|<\infty. \tag{24}$$

From (23) and (24), we see that (11) holds.
For $a\in\mathbb{D}$, define

$$g_a(z)=\frac{1}{2}\frac{\left(1-|a|^2\right)^2}{(1-\bar{a}z)}-\frac{\left(1-|a|^2\right)^3}{(1-\bar{a}z)^2}+\frac{1}{2}\frac{\left(1-|a|^2\right)^4}{(1-\bar{a}z)^3}. \tag{25}$$

So,

$$g_a'(z)=\frac{\bar{a}}{2}\frac{\left(1-|a|^2\right)^2}{(1-\bar{a}z)^2}-2\bar{a}\frac{\left(1-|a|^2\right)^3}{(1-\bar{a}z)^3}+\frac{3}{2}\bar{a}\frac{\left(1-|a|^2\right)^4}{(1-\bar{a}z)^4},$$
$$g_a''(z)=\bar{a}^2\frac{\left(1-|a|^2\right)^2}{(1-\bar{a}z)^3}-6\bar{a}^2\frac{\left(1-|a|^2\right)^3}{(1-\bar{a}z)^4}+6\bar{a}^2\frac{\left(1-|a|^2\right)^4}{(1-\bar{a}z)^5}. \tag{26}$$

By Lemma 5, we see that $g_a\in\mathscr{Z}^p_{p-2}$ and $\sup_{a\in\mathbb{D}}\|g_a\|_{\mathscr{Z}^p_{p-2}}<\infty$. By the boundedness of $uC_\varphi:\mathscr{Z}^p_{p-2}\longrightarrow\mathscr{Z}^\beta$, we get $\sup_{a\in\mathbb{D}}\|uC_\varphi g_a\|_{\mathscr{Z}^\beta}<\infty$. After a calculation,

$$g_a(a)=0,\quad g_a'(a)=0,\quad |g_a''(a)|=\frac{|a|^2}{1-|a|^2}. \tag{27}$$

Hence, for any $w\in\mathbb{D}$,

$$\frac{\left(1-|w|^2\right)^{\beta}|u(w)||\varphi(w)|^2|\varphi'(w)|^2}{1-|\varphi(w)|^2}\lesssim\|uC_\varphi g_{\varphi(w)}\|_{\mathscr{Z}^\beta}<\infty. \tag{28}$$

On the one hand, from (28), we obtain

$$\sup_{|\varphi(w)|>(1/2)}\frac{\left(1-|w|^2\right)^{\beta}|u(w)||\varphi'(w)|^2}{1-|\varphi(w)|^2}<\infty. \tag{29}$$

On the other hand, by (15), we get

$$\sup_{|\varphi(w)|\le(1/2)}\frac{\left(1-|w|^2\right)^{\beta}|u(w)||\varphi'(w)|^2}{1-|\varphi(w)|^2}\lesssim\sup_{w\in\mathbb{D}}\left(1-|w|^2\right)^{\beta}|u(w)\varphi'(w)|^2<\infty. \tag{30}$$

From (29) and (30), we see that (12) holds. The proof is complete.

Next, we estimate the essential norm of $uC_\varphi:\mathscr{Z}^p_{p-2}\longrightarrow\mathscr{Z}^\beta$. Recall that the essential norm of $uC_\varphi:\mathscr{Z}^p_{p-2}\longrightarrow\mathscr{Z}^\beta$ is defined as the distance from uC_φ to the set of compact operators $K:\mathscr{Z}^p_{p-2}\longrightarrow\mathscr{Z}^\beta$, that is,

$$\|uC_\varphi\|_{e,\mathscr{Z}^p_{p-2}\to\mathscr{Z}^\beta}=\inf\left\{\|uC_\varphi-K\|_{\mathscr{Z}^p_{p-2}\to\mathscr{Z}^\beta}:K\text{ is a compact operator}\right\}. \tag{31}$$

Theorem 7. *Let $1<p<\infty$, $0<\beta<\infty$, $u\in H(\mathbb{D})$, and φ be an analytic self-map of $\mathbb{D}$ such that $uC_\varphi:\mathscr{Z}^p_{p-2}\longrightarrow\mathscr{Z}^\beta$ is bounded. Then,*

$$\|uC_\varphi\|_{e,\mathscr{Z}^p_{p-2}\to\mathscr{Z}^\beta}\approx\max\{E,G\}. \tag{32}$$

Here,

$$E=\limsup_{|\varphi(z)|\to1}\left(1-|z|^2\right)^{\beta}|2u'(z)\varphi'(z)+u(z)\varphi''(z)|\left(\log\frac{2}{1-|\varphi(z)|^2}\right)^{1-(1/p)},$$
$$G=\limsup_{|\varphi(z)|\to1}\frac{\left(1-|z|^2\right)^{\beta}|u(z)||\varphi'(z)|^2}{1-|\varphi(z)|^2}. \tag{33}$$

Proof. First we prove that

$$\|uC_\varphi\|_{e,\mathscr{Z}^p_{p-2}\to\mathscr{Z}^\beta}\gtrsim\max\{E,G\}. \tag{34}$$

Let $\{z_j\}_{j\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_j)|\longrightarrow1$ as $j\longrightarrow\infty$. Define

$$k_j(z)=\frac{\left(\bar{\varphi(z_j)}z-1\right)}{\bar{\varphi(z_j)}}\left[\left(1+\log\frac{2}{1-\bar{\varphi(z_j)}z}\right)^2+1\right]\cdot\left(\log\frac{2}{1-|\varphi(z_j)|^2}\right)^{-1-1/p}-2\int_0^z\log\frac{2}{1-\bar{\varphi(z_j)}w}dw\cdot\left(\log\frac{2}{1-|\varphi(z_j)|^2}\right)^{-1/p},$$

$$l_j(z)=\frac{1}{2}\frac{\left(1-|\varphi(z_j)|^2\right)^2}{\left(1-\bar{\varphi(z_j)}z\right)}-\frac{\left(1-|\varphi(z_j)|^2\right)^3}{\left(1-\bar{\varphi(z_j)}z\right)^2}+\frac{1}{2}\frac{\left(1-|\varphi(z_j)|^2\right)^3}{\left(1-\bar{\varphi(z_j)}z\right)^3}. \tag{35}$$

From the proof of Theorem 6, we see that k_j and l_j belong to $\mathscr{Z}^p_{p-2}$. Moreover, k_j and l_j converge to 0 uniformly on $\bar{\mathbb{D}}$ as $j \longrightarrow \infty$. Hence, for any compact operator $K : \mathscr{Z}^p_{p-2} \longrightarrow \mathscr{Z}^\beta$, by Lemma 4, we obtain

$$\begin{aligned}
\|uC_\varphi - K\|_{\mathscr{Z}^p_{p-2}\to\mathscr{Z}^\beta} &\gtrsim \limsup_{j\to\infty}\|uC_\varphi(k_j)\|_{\mathscr{Z}^\beta} - \limsup_{j\to\infty}\|K(k_j)\|_{\mathscr{Z}^\beta} \\
&\gtrsim \limsup_{j\to\infty}\left(1-|z_j|^2\right)^\beta |2u'(z_j)\varphi'(z_j) \\
&\quad + u(z_j)\varphi''(z_j)|\left(\log\frac{2}{1-|\varphi(z_j)|^2}\right)^{1-(1/p)} \\
&\quad + \limsup_{j\to\infty}\left(1-|z_j|^2\right)^\beta |u''(z_j)||k_j(\varphi(z_j))| \\
&= \limsup_{j\to\infty}\left(1-|z_j|^2\right)^\beta |2u'(z_j)\varphi'(z_j) \\
&\quad + u(z_j)\varphi''(z_j)|\left(\log\frac{2}{1-|\varphi(z_j)|^2}\right)^{1-(1/p)},
\end{aligned}$$

$$\begin{aligned}
\|uC_\varphi - K\|_{\mathscr{Z}^p_{p-2}\to\mathscr{Z}^\beta} &\gtrsim \limsup_{j\to\infty}\|uC_\varphi(l_j)\|_{\mathscr{Z}^\beta} - \limsup_{j\to\infty}\|K(l_j)\|_{\mathscr{Z}^\beta} \\
&\gtrsim \limsup_{j\to\infty}\frac{\left(1-|z_j|^2\right)^\beta |u(z_j)||\varphi'(z_j)|^2|\varphi(z_j)|^2}{1-|\varphi(z_j)|^2}.
\end{aligned} \tag{36}$$

Here, we used the fact that $\limsup_{j\to\infty}(1-|z_j|^2)^\beta |u''(z_j)||k_j(\varphi(z_j))| = 0$ since $u \in \mathscr{Z}^\beta$ and k_j converges to 0 uniformly on $\bar{\mathbb{D}}$ as $j \longrightarrow \infty$.

Therefore, we get

$$\begin{aligned}
\|uC_\varphi\|_{e,\mathscr{Z}^p_{p-2}\to\mathscr{Z}^\beta} &= \inf_K \|uC_\varphi - K\|_{\mathscr{Z}^p_{p-2}\to\mathscr{Z}^\beta} \\
&\gtrsim \limsup_{|\varphi(z)|\to 1}\left(1-|z|^2\right)^\beta |2u'(z)\varphi'(z) \\
&\quad + u(z)\varphi''(z)|\left(\log\frac{2}{1-|\varphi(z)|^2}\right)^{1-(1/p)} = E,
\end{aligned}$$

$$\|uC_\varphi\|_{e,\mathscr{Z}^p_{p-2}\to\mathscr{Z}^\beta} \gtrsim \limsup_{|\varphi(z)|\to 1}\frac{\left(1-|z|^2\right)^\beta |u(z)||\varphi'(z)|^2}{1-|\varphi(z)|^2} = G, \tag{37}$$

as desired.

Next, we prove that

$$\|uC_\varphi\|_{e,\mathscr{Z}^p_{p-2}\to\mathscr{Z}^\beta} \lesssim \max\{E, G\}. \tag{38}$$

Let $r \in [0,1)$. Define $K_r : H(\mathbb{D}) \longrightarrow H(\mathbb{D})$ by

$$(K_r f)(z) = f_r(z) = f(rz), \quad f \in H(\mathbb{D}). \tag{39}$$

It is clear that K_r is compact on $\mathscr{Z}^p_{p-2}$ and $\|K_r\|_{\mathscr{Z}^p_{p-2}\to\mathscr{Z}^p_{p-2}} \le 1$; moreover, $f_r - f \longrightarrow 0$ uniformly on compact subsets of $\mathbb{D}$ as $r \longrightarrow 1$. Let $\{r_j\} \subset (0,1)$ such that $r_j \longrightarrow 1$ as $j \longrightarrow \infty$. Then, for each $j \in \mathbb{N}$, $uC_\varphi K_{r_j} : \mathscr{Z}^p_{p-2} \longrightarrow \mathscr{Z}^\beta$ is compact. Hence,

$$\left\|uC_\varphi\right\|_{e,\mathscr{Z}^p_{p-2}\to\mathscr{Z}^\beta} \le \limsup_{j\to\infty}\left\|uC_\varphi - uC_\varphi K_{r_j}\right\|_{\mathscr{Z}^p_{p-2}\to\mathscr{Z}^\beta}. \tag{40}$$

Thus, we only need to show that

$$\limsup_{j\to\infty}\left\|uC_\varphi - uC_\varphi K_{r_j}\right\|_{\mathscr{Z}^p_{p-2}\to\mathscr{Z}^\beta} \lesssim \max\{E, G\}. \tag{41}$$

For any $f \in \mathscr{Z}^p_{p-2}$ with $\|f\|_{\mathscr{Z}^p_{p-2}} \le 1$, from the facts that

$$\begin{aligned}
&\lim_{j\to\infty}|u(0)f(\varphi(0)) - u(0)f(r_j\varphi(0))| = 0, \\
&\lim_{j\to\infty}\left|u'(0)\left(f-f_{r_j}\right)'(\varphi(0)) + u(0)\left(f-f_{r_j}\right)'(\varphi(0))\varphi'(0)\right| = 0,
\end{aligned} \tag{42}$$

we have

$$\begin{aligned}
&\limsup_{j\to\infty}\left\|\left(uC_\varphi - uC_\varphi K_{r_j}\right)f\right\|_{\mathscr{Z}^\beta} \\
&= \limsup_{j\to\infty}\left(1-|z|^2\right)^\beta\left|\left(u\cdot\left(f-f_{r_j}\right)\circ\varphi\right)''(z)\right| \\
&\le \limsup_{j\to\infty}\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^\beta|u''(z)|\left|\left(f-f_{r_j}\right)(\varphi(z))\right| \\
&\quad + \limsup_{j\to\infty}\sup_{|\varphi(z)|\le r_t}\left(1-|z|^2\right)^\beta\left|2u'(z)\varphi'(z)\right. \\
&\quad \left.+ u(z)\varphi''(z)\right|\left|\left(f-f_{r_j}\right)'(\varphi(z))\right| \\
&\quad + \limsup_{j\to\infty}\sup_{|\varphi(z)|> r_t}\left(1-|z|^2\right)^\beta\left|2u'(z)\varphi'(z)\right. \\
&\quad \left.+ u(z)\varphi''(z)\right|\left|\left(f-f_{r_j}\right)'(\varphi(z))\right| + \limsup_{j\to\infty}\sup_{|\varphi(z)|\le r_t}\left(1\right. \\
&\quad \left.-|z|^2\right)^\beta|\varphi'(z)|^2|u(z)|\left|\left(f-f_{r_j}\right)''(\varphi(z))\right| \\
&\quad + \limsup_{j\to\infty}\sup_{|\varphi(z)|> r_t}\left(1-|z|^2\right)^\beta|\varphi'(z)|^2|u(z)|\left|\left(f\right.\right. \\
&\quad \left.\left.- f_{r_j}\right)''(\varphi(z))\right| := S_1 + S_2 + S_3 + S_4 + S_5,
\end{aligned} \tag{43}$$

where $t \in \mathbb{N}$ is large enough such that $r_j \ge 1/2$ for all $j \ge t$. Since $f_{r_j} - f \longrightarrow 0$ as $j \longrightarrow \infty$, by Lemma 2, we have

$$\begin{aligned}
S_1 &= \limsup_{j\to\infty}\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^\beta|u''(z)||\left(f-f_{r_j}\right)(\varphi(z))| \\
&\le \|u\|_{\mathscr{Z}^\beta}\limsup_{j\to\infty}\sup_{w\in\mathbb{D}}|f(w) - f(r_j w)| = 0.
\end{aligned} \tag{44}$$

Since $r_j f'_{r_j} - f' \longrightarrow 0$ and $r_j^2 f''_{r_j} - f'' \longrightarrow 0$ uniformly on compact subsets of $\mathbb{D}$ as $j \longrightarrow \infty$, we have

$$S_2 = \limsup_{j\to\infty} \sup_{|\varphi(z)|\le r_t} \left(1-|z|^2\right)^\beta \left|2u'(z)\varphi'(z) + u(z)\varphi''(z)\right| \left|\left(f-f_{r_j}\right)'(\varphi(z))\right| \lesssim (\|u\varphi\|_{\mathscr{Z}^\beta}+\|u\|_{\mathscr{Z}^\beta}) \limsup_{j\to\infty} \sup_{|w|\le r_t} \left|f'(w) - r_j f'(r_j w)\right| = 0, \tag{45}$$

$$S_4 = \limsup_{j\to\infty} \sup_{|\varphi(z)|\le r_t} \left(1-|z|^2\right)^\beta |\varphi'(z)|^2 |u(z)| \left|\left(f-f_{r_j}\right)''(\varphi(z))\right| \lesssim \left(\left\|u\varphi^2\right\|_{\mathscr{Z}^\beta} + \|u\varphi\|_{\mathscr{Z}^\beta} + \|u\|_{\mathscr{Z}^\beta}\right) \limsup_{j\to\infty} \sup_{|w|\le r_t} \left|f''(w) - r_j^2 f''(r_j w)\right| = 0. \tag{46}$$

Now, we estimate S_3. Using Lemma 1 and the fact that $\|f\|_{\mathscr{Z}^p_{p-2}} \le 1$, we have

$$S_3 = \limsup_{j\to\infty} \sup_{|\varphi(z)|> r_t} \left(1-|z|^2\right)^\beta \left|2u'(z)\varphi'(z) + u(z)\varphi''(z)\right| \left|\left(f-f_{r_j}\right)'(\varphi(z))\right| \lesssim \limsup_{j\to\infty} \left\|f-f_{r_j}\right\|_{\mathscr{Z}^p_{p-2}} \sup_{|\varphi(z)|>r_t} \left(1-|z|^2\right)^\beta \times \left|2u'(z)\varphi'(z) + u(z)\varphi''(z)\right| \left(\log \frac{2}{1-|\varphi(z)|^2}\right)^{1-(1/p)}. \tag{47}$$

Taking the limit as $t \longrightarrow \infty$, we get

$$S_3 \lesssim E. \tag{48}$$

Similarly, again by Lemma 1,

$$S_5 = \limsup_{j\to\infty} \sup_{|\varphi(z)|> r_t} \left(1-|z|^2\right)^\beta |\varphi'(z)|^2 |u(z)| \left|\left(f-f_{r_j}\right)''(\varphi(z))\right| \lesssim \limsup_{j\to\infty} \left\|f-f_{r_j}\right\|_{\mathscr{Z}^p_{p-2}} \sup_{|\varphi(z)|>r_t} \frac{\left(1-|z|^2\right)^\beta |\varphi'(z)|^2 |u(z)|}{1-|\varphi(z)|^2}. \tag{49}$$

Taking the limit as $t \longrightarrow \infty$, we get

$$S_5 \lesssim G. \tag{50}$$

Hence, by (43), (44), (45), (46), (48), and (50), we get

$$\limsup_{j\to\infty} \left\|uC_\varphi - uC_\varphi K_{r_j}\right\|_{\mathscr{Z}^p_{p-2}\to\mathscr{Z}^\beta} \lesssim \max\{E, G\}, \tag{51}$$

which with (40) implies the desired result. The proof is complete.

From Theorem 7 and the result that $\|uC_\varphi\|_{e,\mathscr{Z}^p_{p-2}\to\mathscr{Z}^\beta} = 0$ if and only if $uC_\varphi : \mathscr{Z}^p_{p-2} \longrightarrow \mathscr{Z}^\beta$ is compact, we get the following corollary.

Corollary 8. *Let $1<p<\infty$, $0<\beta<\infty$, $u \in H(\mathbb{D})$, and φ be an analytic self-map of $\mathbb{D}$ such that $uC_\varphi : \mathscr{Z}^p_{p-2} \longrightarrow \mathscr{Z}^\beta$ is bounded. Then, $uC_\varphi : \mathscr{Z}^p_{p-2} \longrightarrow \mathscr{Z}^\beta$ is compact if and only if*

$$\limsup_{|\varphi(z)|\to 1} \frac{\left(1-|z|^2\right)^\beta |u(z)| |\varphi'(z)|^2}{1-|\varphi(z)|^2} = 0,$$

$$\limsup_{|\varphi(z)|\to 1} \left(1-|z|^2\right)^\beta \left|2u'(z)\varphi'(z) + u(z)\varphi''(z)\right| \left(\log \frac{2}{1-|\varphi(z)|^2}\right)^{1-(1/p)} = 0. \tag{52}$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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