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Research Article

Upper and Lower Bounds for Essential Norm of Weighted Composition Operators from Bergman Spaces with Békollé Weights

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Let σ be a weight function such that $\sigma/(1-|z|^2)^\alpha$ is in the class $B_{p_0}(\alpha)$ of Békollé weights, μ a normal weight function, ψ a holomorphic map on $\mathbb{D}$, and φ a holomorphic self-map on $\mathbb{D}$. In this paper, we give upper and lower bounds for essential norm of weighted composition operator $W_{\psi,\varphi}$ acting from weighted Bergman spaces $\mathscr{A}^p(\sigma)$ to Bloch-type spaces $\mathscr{B}_\mu$.

1. Introduction and Preliminaries

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the space of all holomorphic functions on $\mathbb{D}$. For a $\psi \in H(\mathbb{D})$ and φ a holomorphic self-map of $\mathbb{D}$, the *weighted composition operator* $W_{\psi,\varphi}$ is a linear operator on $H(\mathbb{D})$ defined by $W_{\psi,\varphi}f = \psi(f \circ \varphi)$, $f \in H(\mathbb{D})$. Several authors have studied these weighted composition operators on different spaces of analytic functions, see for example, [1–12] and the related references therein. Recently, Stevic and Sharma [12] characterized boundness and compactness of $W_{\psi,\varphi}$ acting from weighted Bergman spaces $\mathscr{A}^p(\sigma)$ to Bloch-type spaces $\mathscr{B}_\mu$. Motivated by results in [12], in this paper, we give upper and lower bounds for essential norm of a weighted composition operator acting between these spaces.

A continuous function $\sigma : [0,1) \longrightarrow [0,\infty)$ is called a *weight* or a *weight function*. We extend it on $\mathbb{D}$ by defining $\sigma(z) = \sigma(|z|)$ for all $z \in \mathbb{D}$. For $0 < p < \infty$ and σ a weight, denoted by $\mathscr{A}^p(\sigma)$ the weighted Bergman space consisting of holomorphic functions f on $\mathbb{D}$ such that

$$\|f\|^p_{\mathscr{A}^p(\sigma)} = \int_{\mathbb{D}} |f(z)|^p \sigma(z) dA(z) < \infty, \qquad (1)$$

where dA is the normalized area measure in $\mathbb{D}$. If $\sigma(z) = \sigma_\gamma(z) = (1-|z|^2)^\gamma (\gamma > -1)$, then $\mathscr{A}^p(\sigma)$ is the well-known weighted Bergman space A^p_γ.

For $p_0 > 1$ and $\alpha > -1$, the class $B_{p_0}(\alpha)$ of Békollé weights consists of weights σ with the property that there exists a constant $C > 0$ such that

$$\left(\int_{S(\theta,h)} \sigma dA_\alpha\right) \le C[A_\alpha(S(\theta,h))]^{p_0} \left(\int_{S(\theta,h)} \sigma^{p'_0/p_0} dA_\alpha\right)^{-p_0/p'_0}. \qquad (2)$$

Here, $\alpha > -1, dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z)$ is the probability measure on $\mathbb{D}, S(\theta,h) = \{z = re^{i\phi} : 1-h < r < 1, |\theta - \phi| < h/2\}, \theta \in [0,2\pi], h \in (0,1)\}$ is the Carleson square in $\mathbb{D}$, and p'_0 is the conjugate exponent of p_0, that is, $1/p_0 + 1/p'_0 = 1$. Recall that a weightμ is normal if there exist positive numbers η and τ, $0 < \eta < \tau$, and $\delta \in [0,1)$ such that

$$\begin{aligned} &\frac{\mu(r)}{(1-r)^\eta} \text{ is decreasing on } [\delta,1) \text{ and } \lim_{r\to1} \frac{\mu(r)}{(1-r)^\eta} = 0, \\ &\frac{\mu(r)}{(1-r)^\tau} \text{ is increasing on } [\delta,1) \text{ and } \lim_{r\to1} \frac{\mu(r)}{(1-r)^\tau} = \infty. \end{aligned} \qquad (3)$$

It is well known that classical weights $\sigma_\alpha(z)=(1-|z|^2)^\alpha$, $\alpha>-1$ are normal weights.

For a normal weight μ, the *weighted Bloch-type space* $\mathscr{B}_\mu$ on $\mathbb{D}$ is the space of all functions f in $H(\mathbb{D})$ such that $\sup_{z\in\mathbb{D}}\mu(z)|f'(z)|<\infty$. The space $\mathscr{B}_\mu$ is a Banach space with the norm

$$\|f\|_{\mathscr{B}_\mu}=|f(0)|+\sup_{z\in\mathbb{D}}\mu(z)\left|f'(z)\right|. \tag{4}$$

Throughout this paper, $r\in(0,1)$ is fixed, $p>0$, $p_0>1$, and $\eta>-1$. We also assume that $p_0\geq p$, σ a weight function such that $\sigma/(1-|z|^2)^\alpha$ belongs to $B_{p_0}(\alpha)$, $\gamma\geq(\eta+2)p_0/p-2$, and $K_\lambda^\gamma=1/(1-\bar{\lambda}z)^{\gamma+2}$ be the reproducing kernel of the Bergman space $A^p(\sigma_\gamma)$. Constants are denoted by C; they are positive and not necessarily the same at each occurrence. The notation $A\lesssim B$ means that A is less than or equal to a constant multiple of B, and $D\gtrsim E$ means that a constant multiple of D is greater than or equal to E. When $A\lesssim B$ as well as $A\gtrsim B$, then we write $A\asymp B$.

2. Essential Norm of $W_{\psi,\varphi}:\mathscr{A}^p(\sigma)\longrightarrow\mathscr{B}_\mu$

In this section, we give upper and lower bounds for the essential norm of weighted composition operator $W_{\psi,\varphi}:\mathscr{A}^p(\sigma)\longrightarrow\mathscr{B}_\mu$.

Recall that if X and Y are two Banach spaces, then the essential norm $\|T\|_{e,X\to Y}$ of a bounded linear operator $T:X\longrightarrow Y$ is defined as

$$\|T\|_{e,X\to Y}=\inf\left\{\|T-K\|:K\text{ is compact from }X\text{ to }Y\right\}, \tag{5}$$

where $\|T\|$ denotes the usual operator norm. Clearly, T is compact if and only if $\|T\|_{e,X\to Y}=0$.

Theorem 1. *Let $\in(1,\infty)$, $\psi\in H(\mathbb{D})$, and φ be a holomorphic self-map of $\mathbb{D}$ such that $\|\varphi\|_\infty=1$. Assume that $W_{\psi,\varphi}:\mathscr{A}^p(\sigma)\longrightarrow\mathscr{B}_\mu$ is bounded. Then,*

$$\begin{aligned}\|W_{\psi,\varphi}\|_{e,\mathscr{A}^p(\sigma)\to\mathscr{B}_\mu}\asymp&\limsup_{|\varphi(z)|\to1}\mu(z)|\psi'(z)|\left(\int_{D_{\varphi(z)}(r)}\sigma dA\right)^{-1/p}\\&+\limsup_{|\varphi(z)|\to1}\frac{\mu(z)\,|\,\psi(z)\varphi'(z)\,|}{1-|\varphi(z)|^2}\left(\int_{D_{\varphi(z)}(r)}\sigma dA\right)^{-1/p},\end{aligned} \tag{6}$$

where $D_{\varphi(z)}(r)=\{w\in\mathbb{D}:|w-\varphi(z)|<r(1-|\varphi(z)|)\}$.

To prove the main result of this paper, we need the following lemmas. The next two lemmas can be found in [12].

Lemma 2. *The following estimates hold:*

(1) For each $z\in\mathbb{D}$, we have that

$$\left|f^{(k)}(z)\right|\leq C\frac{\left(\int_{D_z(r)}\sigma dA\right)^{-1/p}}{\left(1-|z|^2\right)^k}\|f\|_{\mathscr{A}^p(\sigma)}\text{for all }f\in A_\sigma^p \tag{7}$$

(2) For each $\lambda\in\mathbb{D}$, we have that

$$\|K_\lambda^\gamma\|_{A^p(\sigma)}\asymp\frac{\left(\int_{D_\lambda(r)}\sigma dA\right)^{1/p}}{\left(1-|\lambda|^2\right)^{\gamma+2}}, \tag{8}$$

where $D_\lambda(r)=\{z\in\mathbb{D}:|z-\lambda|<r(1-|\lambda|)\}$

Lemma 3. *For each $\lambda\in\mathbb{D}$, the function f_λ defined as*

$$f_\lambda(z)=\frac{\left(1-|\lambda|^2\right)^{1+(\eta+2)p_0/p}}{\left(1-\bar{\lambda}z\right)^{1+(\eta+2)p_0/p}}\left(\int_{D_\lambda(r)}\sigma dA\right)^{-1/p} \tag{9}$$

is in $A^p(\sigma)$. Moreover, $\sup_{\lambda\in\mathbb{D}}\|f_\lambda\|_{A^p(\sigma)}\asymp1$ and f_λ converges to zero, uniformly on compact subsets of $\mathbb{D}$ as $|\lambda|\longrightarrow1$.

The next lemma can be found in [8].

Lemma 4. *Let $p\in(1,\infty)$. If a bounded sequence $\{f_k\}_{k\in\mathbb{N}}$ in $\mathscr{A}^p(\sigma)$ converges to 0 uniformly on compact subsets of $\mathbb{D}$, then $\{f_k\}_{k\in\mathbb{N}}$ also converges to 0 weakly in $\mathscr{A}^p(\sigma)$.*

Proof of Theorem 1. Lower bound. Let $\{\zeta_j\}_{j\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(\zeta_j)|\longrightarrow1$ as $j\longrightarrow\infty$ and

$$\begin{aligned}&\limsup_{j\to\infty}\mu(\zeta_j)|\psi'(\zeta_j)|\left(\int_{D_{\varphi(\zeta_j)}(r)}\sigma dA\right)^{-1/p}\\&=\limsup_{|\varphi(z)|\to1}\mu(z)|\psi'(z)|\left(\int_{D_{\varphi(z)}(r)}\sigma dA\right)^{-1/p}.\end{aligned} \tag{10}$$

For each $j\in\mathbb{N}$, let γ_j be defined as $\gamma_j(z)=1-(1-|\varphi(\zeta_j)|^2)(1-\varphi(\bar{\zeta}_j)z)^{-1}$. Then, $\gamma_j\in H^\infty$ and $\sup_{j\in\mathbb{N}}|\gamma_j(z)|\leq3$. Consider the family of functions defined as $g_j(z)=\gamma_j(z)f_{\varphi(\zeta_j)}(z)$, where $f_{\varphi(\zeta_j)}$ is defined as in (9). Also, by Lemma 3, $\sup_j\|g_j\|_{A^p(\sigma)}\lesssim1$ and $\{g_j\}_{n\in\mathbb{N}}$ converges to zero uniformly on compact subsets of $\mathbb{D}$ as $j\longrightarrow\infty$. By Lemma 4, g_j converges to zero weakly in $A^p(\sigma)$. Thus, for any compact operator $K:A^p(\sigma)\longrightarrow\mathscr{B}_\mu$, we have that $\|Kg_j\|_{\mathscr{B}_\mu}\longrightarrow0$ as $j\longrightarrow\infty$. Moreover,

$$\gamma_j(\varphi(\zeta_j)) = 0, \gamma_j'(z) = -\bar{\varphi}(\zeta_j)\frac{1-|\varphi(\zeta_j)|^2}{\left(1-\bar{\varphi}(\zeta_j)z\right)^2},$$
$$\gamma'_j(\varphi(\zeta_j)) = \frac{-\bar{\varphi}(\zeta_j)}{1-|\varphi(\zeta_j)|^2}. \tag{11}$$

Also, $g_j(\varphi(\zeta_j)) = 0$ and

$$f_{\varphi(\zeta_j)}(\varphi(\zeta_j)) = \left(\int_{D_{\varphi(\zeta_j)}(r)} \sigma dA\right)^{-1/p}. \tag{12}$$

Now

$$f_{\varphi(\zeta_j)}{}'(z) = \bar{\varphi}(\zeta_j)\left((\eta+2)\frac{p_0}{p}+1\right)\left(\int_{D_{\varphi(\zeta_j)}(r)} \sigma dA\right)^{-1/p} \cdot \frac{\left(1-|\varphi(\zeta_j)|^2\right)^{1+(\eta+2)p_0/p}}{\left(1-\bar{\varphi}(\zeta_j)z\right)^{2+(\eta+2)p_0/p}} f_{\varphi(\zeta_j)}{}'(\varphi(\zeta_j))$$
$$= \bar{\varphi}(\zeta_j)\left((\eta+2)\frac{p_0}{p}+1\right)\frac{\left(\int_{D_{\varphi(\zeta_j)}(r)} \sigma dA\right)^{-1/p}}{1-|\varphi(\zeta_j)|^2}. \tag{13}$$

Therefore, from (12) and (13), we have that

$$g_j'(\varphi(\zeta_j)) = \gamma'_j(\varphi(\zeta_j)) f_{\varphi(\zeta_j)}(\varphi(\zeta_j)) + \gamma_j(\varphi(\zeta_j)) f_{\varphi(\zeta_j)}{}'(\varphi(\zeta_j))$$
$$= -\bar{\varphi}(\zeta_j)\frac{\left(\int_{D_{\varphi(\zeta_j)}(r)} \sigma dA\right)^{-1/p}}{1-|\varphi(\zeta_j)|^2}. \tag{14}$$

Using the facts that$\| W_{\psi,\varphi}\|_{e,\mathscr{A}^p(\sigma)\to\mathscr{B}_\mu} \gtrsim \limsup_{j\to\infty} \| W_{\psi,\varphi} g_j - K g_j\|_{\mathscr{B}_\mu}$and
$\limsup_{j\to\infty} \| K g_j\|_{\mathscr{B}_\mu} = 0$, we have that

$$\| W_{\psi,\varphi}\|_{e,\mathscr{A}^p(\sigma)\to\mathscr{B}_\mu} \gtrsim \limsup_{j\to\infty} \mu(\zeta_j)|\psi'(\zeta) g_j(\varphi(\zeta_j)) + \psi(\zeta_j)\varphi'(\zeta_j) g_j'(\varphi(\zeta_j))|$$
$$\gtrsim \limsup_{j\to\infty} \mu(\zeta_j)|\psi(\zeta_j)\varphi'(\zeta_j)| \frac{\left(\int_{D_{\varphi(\zeta_j)}(r)} \sigma dA\right)^{-1/p}}{1-|\varphi(\zeta_j)|^2}. \tag{15}$$

Again, let ζ_j be a sequence in $\mathbb{D}$ such that $|\varphi(\zeta_j)| \longrightarrow 1$ as $j \longrightarrow \infty$ and

$$\lim_{j\to\infty} \mu(\zeta_j)|\psi(\zeta_j)\varphi'(\zeta_j)\varphi(\zeta_j)| \left(\int_{D_{\varphi(\zeta_j)}(r)} \sigma dA\right)^{-1/p}$$
$$= \limsup_{|\varphi(z)|\to 1} \mu(z)|\psi(z)\varphi'(z)\varphi(z)| \left(\int_{D_{\varphi(z)}(r)} \sigma dA\right)^{-1/p}. \tag{16}$$

For each $j \in \mathbb{N}$, let h_j be defined as

$$h_j(z) = \left\{\frac{1}{1+(\eta+2)p_0/p}\frac{\left(1-|\varphi(\zeta_j)|^2\right)^{1+(\eta+2)p_0/p}}{\left(1-\bar{\varphi}(\zeta_j)z\right)^{1+(\eta+2)p_0/p}} - \frac{1}{2+(\eta+2)p_0/p}\frac{\left(1-|\varphi(\zeta_j)|^2\right)^{2+(\eta+2)p_0/p}}{\left(1-\bar{\varphi}(\zeta_j)z\right)^{2+(\eta+2)p_0/p}}\right\} \cdot \left(\int_{D_{\varphi(\zeta_j)}(r)} \sigma dA\right)^{-1/p}. \tag{17}$$

Then, by Lemma 3, $\sup_j \| h_j\|_{A^p(\sigma)} \lesssim 1$ and $\{h_j\}_{n\in\mathbb{N}}$ converges to zero uniformly on compact subsets of $\mathbb{D}$ as $j \longrightarrow \infty$. By Lemma 4, h_j converges to zero weakly in $A^p(\sigma)$. Thus, for any compact operator $K : A^p(\sigma) \longrightarrow \mathscr{B}_\mu$, we have that $\| K h_j\|_{\mathscr{B}_\mu} \longrightarrow 0$ as $j \longrightarrow \infty$. Moreover, $h_j'(\varphi(\zeta_j)) = 0$ and

$$h_j(\varphi(\zeta_j)) = \frac{1}{(1+(\eta+2)p_0/p)(2+(\eta+2)p_0/p)}\left(\int_{D_{\varphi(\zeta_j)}(r)} \sigma dA\right)^{-1/p}. \tag{18}$$

Thus, using (18), we have that

$$\| W_{\psi,\varphi}\|_{e,\mathscr{A}^p(\sigma)\to\mathscr{B}_\mu} \gtrsim \limsup_{j\to\infty} \mu(\zeta_j)|\psi'(\zeta_j) h_j(\varphi(\zeta_j)) + \psi(\zeta_j)\varphi'(\zeta_j) h_j'(\varphi(\zeta_j))|$$
$$\gtrsim \limsup_{j\to\infty} \mu(\zeta_j)|\psi'(\zeta_j)| \left(\int_{D_{\varphi(\zeta_j)}(r)} \sigma dA\right)^{-1/p}. \tag{19}$$

Combining (15) and (19), we have that

$$\| W_{\psi,\varphi}\|_{e,\mathscr{A}^p(\sigma)\to\mathscr{B}_\mu} \gtrsim \limsup_{|\varphi(z)|\to 1} \mu(z)|\psi'(z)| \left(\int_{D_{\varphi(z)}(r)} \sigma dA\right)^{-1/p} + \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)\,|\,\psi(z)\varphi'(z)\,|}{1-|\varphi(z)|^2} \cdot \left(\int_{D_{\varphi(z)}(r)} \sigma dA\right)^{-1/p}. \tag{20}$$

Upper bound. Let $f \in \mathscr{A}^p(\sigma)$ be such that $\|f\|_{\mathscr{A}^p(\sigma)} \le 1$. Let

$$L_j f(z) = f(a_j z), \tag{21}$$

where $a_j = j/(j+1)$.

Then, by Theorem 6.1 in [3], we have that $L_j : \mathscr{A}^p(\sigma) \longrightarrow \mathscr{A}^p(\sigma)$ is compact. Since $W_{\psi,\varphi} : \mathscr{A}^p(\sigma) \longrightarrow \mathscr{B}_\mu$ is bounded, so $W_{\psi,\varphi} L_j : \mathscr{A}^p(\sigma) \longrightarrow \mathscr{B}_\mu$ is compact. Thus, for fixed r in $(0,1)$, we have that

$$\begin{aligned}\|W_{\psi,\varphi}\|_{e,\mathscr{A}^p(\sigma)\to\mathscr{B}_\mu} &\le \sup_{\|f\|_{\mathscr{A}^p(\sigma)}\le 1} \|W_{\psi,\varphi}(I-L_j)f\|_{\mathscr{B}_\mu} \\ &\le \sup_{\|f\|_{\mathscr{A}^p(\sigma)}\le 1} \Big[|W_{\psi,\varphi}(I-L_j)f(0)| + \sup_{z\in\mathbb{D}} \mu(z)\, | \\ &\quad \cdot \big(W_{\psi,\varphi}(I-L_j)f\big)'(z)\,| \Big] \\ &\le \sup_{\|f\|_{\mathscr{A}^p(\sigma)}\le 1} \Big[|W_{\psi,\varphi}(I-L_j)f(0)| + \sup_{|\varphi(z)|\le r} \mu(z)\, | \\ &\quad \cdot \big(W_{\psi,\varphi}(I-L_j)f\big)'(z)| + \sup_{|\varphi(z)|>r} \mu(z)\, | \\ &\quad \cdot \big(W_{\psi,\varphi}(I-L_j)f\big)'(z)\,| \Big],\end{aligned} \tag{22}$$

where I is the identity operator on $\mathscr{A}^p(\sigma)$. Now

$$\begin{aligned}\big|\big(W_{\psi,\varphi}(I-L_j)f\big)'(z) &\le |\psi'(z)|\,|f(\varphi(z)) - f(a_j\varphi(z))| + |\psi(z)\varphi'(z)|\,| \\ &\quad \cdot \{f'(\varphi(z)) - a_j f'(a_j\varphi(z))\}| \\ &\le |\psi'(z)|\,|f(\varphi(z)) - f(a_j\varphi(z))| \\ &\quad + \frac{|\psi(z)\varphi'(z)|}{j+1} |f'(a_j\varphi(z))| \\ &\quad + |\psi(z)\varphi'(z)|\,|f'(\varphi(z)) - f'(a_j\varphi(z))|.\end{aligned} \tag{23}$$

Let $|\varphi(z)| \le r$ and $\zeta = \varphi(z)$. Let $\Gamma = [a_j\zeta, \zeta]$ be the line segment from $a_j\zeta$ to ζ. Then, $\Gamma \subset D(0,r)$, where $D(0,r) = \{z : |z| \le r\}$. Thus, by Lemma 2, we have that

$$\begin{aligned}|f'(\zeta) - f'(a_j\zeta)| &= \Big|\int_\Gamma f''(\lambda)\,d\lambda\Big| \le \frac{|\zeta|}{j+1} \sup_{\lambda\in D(0,r)} |f''(\lambda)| \\ &\lesssim \frac{|\zeta|}{j+1} \sup_{\lambda\in D(0,r)} \frac{\left(\int_{D_\lambda(r)} \sigma\, dA\right)^{-1/p}}{\left(1-|\lambda|^2\right)^2} \|f\|_{A^p(\sigma)}.\end{aligned} \tag{24}$$

Again, let $\gamma_0 = (\eta+2)p_0/p - 2$. Then, by Lemma 2, we have that

$$1 = |K_\zeta^{\gamma_0}(0)| \le C \left(\int_{D_0(1/2)} \sigma\, dA \right)^{-1/p} \|K_\zeta^{\gamma_0}\|_{A^p(\sigma)} \asymp \frac{\left(\int_{D_\zeta(r)} \sigma\, dA\right)^{1/p}}{\left(1-|\zeta|^2\right)^{\gamma_0+2}}. \tag{25}$$

Thus,

$$\frac{\left(1-|\zeta|^2\right)^{(\eta+2)p_0/p}}{\left(\int_{D_\zeta(r)} \sigma\, dA\right)^{1/p}} \le \frac{\left(1-|\zeta|^2\right)^{\gamma_0+2}}{\left(\int_{D_\zeta(r)} \sigma\, dA\right)^{1/p}} \lesssim 1. \tag{26}$$

Thus, from (24) and (26), we have that

$$\begin{aligned}&|f'(\zeta) - f'(a_j\zeta)| \\ &\lesssim \frac{|\zeta|}{j+1} \sup_{\lambda\in D(0,r)} \frac{1}{\left(1-|\lambda|^2\right)^{4+(\eta+2)p_0/p}} \frac{\left(1-|\lambda|^2\right)^{2+(\eta+2)p_0/p}}{\left(\int_{D_\lambda(r)} \sigma\, dA\right)^{1/p}} \|f\|_{A^p(\sigma)} \\ &\lesssim \frac{|\zeta|}{j+1} \frac{1}{\left(1-|r|^2\right)^{4+(\eta+2)p_0/p}} \|f\|_{A^p(\sigma)}.\end{aligned} \tag{27}$$

Similarly, we can show that

$$|f(\zeta) - f(a_j\zeta)| \lesssim \frac{|\zeta|}{j+1} \frac{1}{\left(1-|r|^2\right)^{3+(\eta+2)p_0/p}} \|f\|_{\mathscr{A}^p(\sigma)}. \tag{28}$$

Also, by Lemma 2 and equation (26), we have that

$$\begin{aligned}\sup_{|\varphi(z)|\le r} |f'(a_j\zeta)| &\lesssim \sup_{|\varphi(z)|\le r} \frac{\left(\int_{D_{a_j\varphi(z)}(r)} \sigma\, dA\right)^{-1/p}}{1 - a_j^2|\varphi(z)|^2} \|f\|_{\mathscr{A}^p(\sigma)} \\ &\lesssim \frac{1}{\left(1-a_j^2|r|^2\right)^{1+(\eta+2)p_0/p}} \|f\|_{\mathscr{A}^p(\sigma)}.\end{aligned} \tag{29}$$

Since $W_{\psi,\varphi} : \mathscr{A}^p(\sigma) \longrightarrow \mathscr{B}_\mu$ is bounded, so

$$\|W_{\psi,\varphi} f\|_{\mathscr{B}_\mu} \lesssim \|f\|_{\mathscr{A}^p(\sigma)} \tag{30}$$

for each $f \in \mathscr{A}^p(\sigma)$. By taking, respectively, $f(z) = 1$ and $f(z) = z$ and using the fact that $|\varphi(z)| < 1$, we have that

$$M_1 = \sup_{z\in\mathbb{D}} \mu(z)|\psi'(z)| < \infty \text{and}\, M_2 = \sup_{z\in\mathbb{D}} \mu(z)|\psi(z)\varphi'(z)| < \infty. \tag{31}$$

Combining (23) and (27)-(29), we have that

$$\sup_{\|f\|_{\mathscr{A}^p(\sigma)}\le 1} \sup_{|\varphi(z)|\le r} \mu(z) |\big(W_{\psi,\varphi}(I-L_j)f\big)'(z)| \longrightarrow 0 \text{ as } j \longrightarrow \infty. \tag{32}$$

Using (28), we have that

$$|W_{\psi,\varphi}(I-L_j)f(0)| = |\psi(0)f(\varphi(0)) - \psi(0)f(a_j\varphi(0))| \lesssim \frac{|\psi(0)\varphi(0)|}{j+1}\frac{\|f\|_{\mathscr{A}^p(\sigma)}}{(1-r^2)^{3+(\eta+2)p_0/p}}. \tag{33}$$

Using (31), (32), and (33), we have that

$$\sup_{\|f\|_{\mathscr{A}^p(\sigma)}\le 1}\sup_{|\varphi(z)|\le r}\left[|W_{\psi,\varphi}(I-L_j)f(0)| + |(W_{\psi,\varphi}(I-L_j)f)'(z)|\right] \longrightarrow 0 \tag{34}$$

as $j \longrightarrow \infty$. The last term in the right-hand side of (22) is dominated by

$$\sup_{|\varphi(z)|>r}\mu(z)|\psi'(z)|\{|f(\varphi(z))| + |f(a_j\varphi(z))|\} + \sup_{|\varphi(z)|>r}\mu(z)|\psi(z)\varphi'(z)|\{|f'(\varphi(z))| + a_j|f'(a_j\varphi(z))|\}, \tag{35}$$

which is further dominated by a constant multiple of

$$\begin{aligned}&\sup_{|\varphi(z)|>r}\mu(z)|\psi'(z)| \cdot \left\{\left(\int_{D_{\varphi(z)}(r)}\sigma dA\right)^{-1/p} + \left(\int_{D_{a_j\varphi(z)}(r)}\sigma dA\right)^{-1/p}\right\}\|f\|_{\mathscr{A}^p(\sigma)} \\ &+ \sup_{|\varphi(z)|>r}\mu(z)|\psi(z)\varphi'(z)| \left\{\frac{\left(\int_{D_{\varphi(z)}(r)}\sigma dA\right)^{-1/p}}{1-|\varphi(z)|^2} + \frac{\left(\int_{D_{a_j\varphi(z)}(r)}\sigma dA\right)^{-1/p}}{1-a_j^2|\varphi(z)|^2}\right\}\|f\|_{\mathscr{A}^p(\sigma)}.\end{aligned} \tag{36}$$

Letting $j \longrightarrow \infty$ in (36), we get

$$\begin{aligned}\limsup_{j\to\infty}\sup_{\|f\|_{\mathscr{A}^p(\sigma)}\le 1}\sup_{|\varphi(z)|>r}\mu(z)|(W_{\psi,\varphi}(I-L_k)f)'(z)| &\le \sup_{|\varphi(z)|>r}\mu(z)|\psi'(z)|\left(\int_{D_{\varphi(z)}(r)}\sigma dA\right)^{-1/p} \\ &+ \sup_{|\varphi(z)|>r}\mu(z)|\psi(z)\varphi'(z)|\frac{\left(\int_{D_{\varphi(z)}(r)}\sigma dA\right)^{-1/p}}{1-|\varphi(z)|^2}.\end{aligned} \tag{37}$$

Using (34) and (37) in (22), we have that

$$\begin{aligned}\|W_{\psi,\varphi}\|_{e,\mathscr{A}^p(\sigma)\to\mathscr{B}_\mu} &\lesssim \sup_{|\varphi(z)|>r}\mu(z)|\psi'(z)|\left(\int_{D_{\varphi(z)}(r)}\sigma dA\right)^{-1/p} \\ &+ \sup_{|\varphi(z)|>r}\mu(z)|\frac{\psi(z)\varphi'(z)|}{1-|\varphi(z)|^2}\left(\int_{D_{\varphi(z)}(r)}\sigma dA\right)^{-1/p}.\end{aligned} \tag{38}$$

Finally, letting $r \longrightarrow 1$, then we get

$$\begin{aligned}\|W_{\psi,\varphi}\|_{e,\mathscr{A}^p(\sigma)\to\mathscr{B}_\mu} &\lesssim \limsup_{|\varphi(z)|\to 1}\mu(z)|\psi'(z)|\left(\int_{D_{\varphi(z)}(r)}\sigma dA\right)^{-1/p} \\ &+ \limsup_{|\varphi(z)|\to 1}\mu(z)\frac{|\psi(z)\varphi'(z)|}{1-|\varphi(z)|^2}\left(\int_{D_{\varphi(z)}(r)}\sigma dA\right)^{-1/p}.\end{aligned} \tag{39}$$

Combining (20) and (39), we get the desired result.

Corollary 5. *Let $p \in (1,\infty)$, $\psi \in H(\mathbb{D})$, and φ be a holomorphic self-map of $\mathbb{D}$, such that $\|\varphi\|_\infty = 1$. Let $W_{\psi,\varphi} : \mathscr{A}^p(\sigma) \longrightarrow \mathscr{B}_\mu$ is bounded. Then, $W_{\psi,\varphi} : \mathscr{A}^p(\sigma) \longrightarrow \mathscr{B}_\mu$ is compact if and only if the following conditions are satisfied:*

(1) $\limsup_{|\varphi(z)|\to 1}\mu(z)\,|\,\psi'(z)\,|\,(\int_{D_{\varphi(z)}(r)}\sigma dA)^{-1/p} = 0$

(2) $\limsup_{|\varphi(z)|\to 1}\mu(z)\,|\,(\psi(z)\varphi'(z)|/1-|\varphi(z)|^2)$ $(\int_{D_{\varphi(z)}(r)}\sigma dA)^{-1/p} = 0$

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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