Upper and Lower Bounds for Essential Norm of Weighted Composition Operators from Bergman Spaces with Békollé Weights

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1. Introduction and Preliminaries

Let $D$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(D)$ the space of all holomorphic functions on $D$. For a $\psi \in H(D)$ and $\varphi$ a holomorphic self-map of $D$, the weighted composition operator $W_{\psi, \varphi}$ is a linear operator on $H(D)$ defined by $W_{\psi, \varphi}f = \psi(f \circ \varphi)$, $f \in H(D)$. Several authors have studied these weighted composition operators on different spaces of analytic functions, see for example, [1–12] and the related references therein. Recently, Stevic and Sharma [12] characterized boundness and compactness of $W_{\psi, \varphi}$ acting from weighted Bergman spaces $A^p(\sigma)$ to Bloch-type spaces $B_\mu$.

Motivated by results in [12], in this paper, we give upper and lower bounds for essential norm of weighted composition operator $W_{\psi, \varphi}$ acting from weighted Bergman spaces $A^p(\sigma)$ to Bloch-type spaces $B_\mu$.

Let $\sigma$ be a weight function such that $\sigma/(1 - |z|^2)^{\delta}$ is in the class $B_\mu'\psi(\alpha)$ of Békollé weights, $\mu$ a normal weight function, $\psi$ a holomorphic map on $D$, and $\varphi$ a holomorphic self-map on $D$. In this paper, we give upper and lower bounds for essential norm of weighted composition operator $W_{\psi, \varphi}$ acting from weighted Bergman spaces $A^p(\sigma)$ to Bloch-type spaces $B_\mu$.

A continuous function $\sigma : [0, 1) \rightarrow [0, \infty)$ is called a weight or a weight function. We extend it on $D$ by defining $\sigma(z) = \sigma(|z|)$ for all $z \in D$. For $0 < p < \infty$ and $\sigma$ a weight, denoted by $A^p(\sigma)$ the weighted Bergman space consisting of holomorphic functions $f$ on $D$ such that

$$\|f\|_{A^p(\sigma)} = \left( \int_D |f(z)|^p \sigma(z) dA(z) < \infty \right),$$

where $dA$ is the normalized area measure in $D$. If $\sigma(z) = \sigma_\psi(\gamma)(z) = (1 - |z|^2)^\gamma$, then $A^p(\sigma)$ is the well-known weighted Bergman space $A^p_\gamma$. For $\rho_\psi > 1$ and $\alpha > -1$, the class $B_\rho_\psi'\psi(\alpha)$ of Békollé weights consists of weights $\sigma$ with the property that there exists a constant $C > 0$ such that

$$\left( \int_{S(\theta, h)} \sigma dA_u \right) \leq C [A_u(S(\theta, h))]^{\rho_\psi} \left( \int_{S(\theta, h)} \sigma^{1/\rho_\psi} dA_u \right)^{-\rho_\psi/\rho_\psi'}. \quad (2)$$

Here, $\alpha > -1, dA_u(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z)$ is the probability measure on $D, S(\theta, h) = \{ z = re^{i\theta} : 1 - h < r < 1, |\theta - \phi| < h/2 \}, \theta \in [0, 2\pi], h \in (0, 1)$ is the Carleson square in $D$, and $\rho_\psi'$ is the conjugate exponent of $\rho_\psi$, that is, $1/\rho_\psi + 1/\rho_\psi' = 1$.

Recall that a weight $\mu$ is normal if there exist positive numbers $\eta$ and $r$, $0 < \eta < r$, and $\delta \in [0, 1)$ such that

$$\frac{\mu(r)}{(1 - r)^{\eta}} \text{ is decreasing on } [\delta, 1) \text{ and } \lim_{r \to 1} \frac{\mu(r)}{(1 - r)^{\eta}} = 0,$$

$$\frac{\mu(r)}{(1 - r)^{\eta}} \text{ is increasing on } [\delta, 1) \text{ and } \lim_{r \to 1} \frac{\mu(r)}{(1 - r)^{\eta}} = \infty. \quad (3)$$
It is well known that classical weights $\sigma_\alpha(z) = (1 - |z|^2)^\alpha$, $\alpha > -1$ are normal weights.

For a normal weight $\mu$, the weighted Bloch-type space $\mathcal{B}_\mu$ on $D$ is the space of all functions $f$ in $H(D)$ such that
\[ \sup_{z \in D} \mu(z) |f'(z)| < \infty. \]
The space $\mathcal{B}_\mu$ is a Banach space with the norm
\[ \|f\|_{\mathcal{B}_\mu} = |f(0)| + \sup_{z \in D} \mu(z) |f'(z)|. \]

Throughout this paper, $r \in (0,1)$ is fixed, $p > 0$, $p_0 > 1$, and $\eta > -1$. We also assume that $p_0 \geq p$, $\sigma$ a weight function such that $\sigma/(1 - |z|^2)^\alpha$ belongs to $B_{p_0}(\alpha)$, $\gamma \geq (\eta + 2)p_0/p - 2$, and $K_1 = 1/(1 - \lambda z)^{\gamma+2}$ be the reproducing kernel of the Bergman space $A^p(\sigma)$. Constants are denoted by $C$; they are positive and not necessarily the same at each occurrence. The notation $A \lesssim B$ means that $A$ is less than or equal to a constant multiple of $B$, and $D \geq C$ is a constant multiple of $D$ that is greater than or equal to $E$. When $A \lesssim B$ as well as $A \geq B$, then we write $A \sim B$.

2. Essential Norm of $W_{\psi\phi} : \mathcal{A}^p(\sigma) \longrightarrow \mathcal{B}_\mu$

In this section, we give upper and lower bounds for the essential norm of weighted composition operator $W_{\psi\phi} : \mathcal{A}^p(\sigma) \longrightarrow \mathcal{B}_\mu$.

Recall that if $X$ and $Y$ are two Banach spaces, then the essential norm $\|T\|_{eX \rightarrow Y}$ of a bounded linear operator $T : X \longrightarrow Y$ is defined as
\[ \|T\|_{eX \rightarrow Y} = \inf \{ \|T - K\| : K \text{ is compact from } X \text{ to } Y \}, \]
where $\|T\|$ denotes the usual operator norm. Clearly, $T$ is compact if and only if $\|T\|_{eX \rightarrow Y} = 0$.

Theorem 1. Let $\psi, \phi \in H(D)$, and $\phi$ be a holomorphic self-map of $D$ such that $|\phi|_{\infty} = 1$. Assume that $W_{\psi\phi} : \mathcal{A}^p(\sigma) \longrightarrow \mathcal{B}_\mu$ is bounded. Then,
\[ \|W_{\psi\phi}\|_{e\mathcal{A}^p(\sigma) \rightarrow \mathcal{B}_\mu} \lesssim \limsup_{\psi(z) \to z} \mu(z) |\psi'(z)| \left( \int_{D(\psi(z))} |\sigma dA| \right)^{-1/p} \]
\[ \quad + \limsup_{|\psi(z)| \to 1} \frac{\mu(z) |\psi'(z)|}{1 - |\psi(z)|^2} \left( \int_{D(\psi(z))} |\sigma dA| \right)^{-1/p}, \]

where $D_{\psi(z)}(r) = \{ w \in D : |w - \psi(z)| < r(1 - |\psi(z)|) \}$.

To prove the main result of this paper, we need the following lemmas. The next two lemmas can be found in [12].

Lemma 2. The following estimates hold:

1. For each $z \in D$, we have that
\[ |f^{(k)}(z)| \leq C \left( \int_{D(z)} \sigma dA \right)^{-1/p} \|f\|_{A^p(\sigma)}^{1/p} \text{ for all } f \in A^p_\sigma \]
\[ (7) \]

2. For each $\lambda \in \mathbb{D}$, we have that
\[ \|K_\lambda\|_{A^p(\sigma)} \lesssim \left( \int_{D(\lambda)} |\sigma dA| \right)^{1/p} \left( \frac{(1 - |\lambda|^2)^{\gamma+2}}{1 - \lambda z} \right)^{1/p}, \]
\[ (8) \]

where $D_\lambda(r) = \{ z \in D : |z - \lambda| < r(1 - |\lambda|) \}$.

Lemma 3. For each $\lambda \in \mathbb{D}$, the function $f_\lambda$ defined as
\[ f_\lambda(z) = \frac{(1 - |\lambda|^2)^{1/2(p+2)p/p}}{(1 - \lambda z)^{1/2(p+2)p/p}} \left( \int_{D(\lambda)} |\sigma dA| \right)^{-1/p}, \]

is in $A^p(\sigma)$. Moreover, $\sup_{|\sigma| \geq 1} \|f_\lambda\|_{A^p(\sigma)} = 1$ and $f_\lambda$ converges to zero, uniformly on compact subsets of $D$ as $|\lambda| \longrightarrow 1$.

The next lemma can be found in [8].

Lemma 4. Let $p \in (1, \infty)$. If a bounded sequence $\{f_j\}_{j \in \mathbb{N}}$ in $A^p(\sigma)$ converges to zero uniformly on compact subsets of $D$, then $\{f_j\}_{j \in \mathbb{N}}$ also converges to 0 weakly in $A^p(\sigma)$.

Proof of Theorem 1. Lower bound. Let $\{\zeta_j\}_{j \in \mathbb{N}}$ be a sequence in $D$ such that $|\phi(\zeta_j)| \longrightarrow 1$ as $j \longrightarrow \infty$ and $\limsup_{j \to \infty} \mu(\zeta_j) |\psi'(\zeta_j)| \left( \int_{D(\zeta_j)} |\sigma dA| \right)^{-1/p} = \limsup_{|\psi(z)| \to 1} \mu(z) |\psi'(z)| \left( \int_{D(\psi(z))} |\sigma dA| \right)^{-1/p}.
\[ (10) \]

For each $j \in \mathbb{N}$, let $y_j$ be defined as $y_j(z) = 1 - (1 - |\phi(\zeta_j)|^2)(1 - (\phi(\zeta_j))^{-1}. Then, $y_j \in H^\infty$ and $\sup_{|\sigma| \geq 1} |y_j(z)| \leq 3$. Consider the family of functions defined as $g_j(z) = y_j(z) f_{\phi(\zeta_j)(z)}$, where $f_{\phi(\zeta_j)}$ is defined as in (9). Also, by Lemma 3, $\sup_{|\sigma| \geq 1} \|g_j\|_{A^p(\sigma)} = 1$ and $\{g_j\}_{j \in \mathbb{N}}$ converges to zero uniformly on compact subsets of $D$ as $j \longrightarrow \infty$. By Lemma 4, $g_j$ converges to zero weakly in $A^p(\sigma)$. Thus, for any compact operator $K : A^p(\sigma) \longrightarrow \mathcal{B}_\mu$, we have that $\|Kg_j\|_{\mathcal{B}_\mu} \longrightarrow 0$ as $j \longrightarrow \infty$. Moreover,
\[
\gamma_j(\varphi(\zeta_j)) = 0, \quad \gamma_j'(z) = -\varphi(\zeta_j) \frac{1 - |\varphi(\zeta_j)|^2}{(1 - \varphi(\zeta_j)z)^2},
\]
(11)

Also, \(g_j(\varphi(\zeta_j)) = 0 \) and

\[
f_{\varphi}(\zeta_j)(\varphi(\zeta_j)) = \left(\int_{D_{\phi}(r)} \sigma dA\right)^{-1/p}.
\]
(12)

Now

\[
f_{\varphi}(\zeta_j)(z) = \varphi(\zeta_j) \left(\left(\eta + 2\right) \frac{p_0}{p} + 1\right) \left(\int_{D_{\phi}(r)} \sigma dA\right)^{-1/p} \left(1 - |\varphi(\zeta_j)|^2\right)^{1 + (\eta + 2)p_0/p}
\]
\[
\times \left(1 - \varphi(\zeta_j)z\right)^{2 + (\eta + 2)p_0/p} f_{\varphi}(\zeta_j)(\varphi(\zeta_j))
\]
\[
= \varphi(\zeta_j) \left(\left(\eta + 2\right) \frac{p_0}{p} + 1\right) \left(\int_{D_{\phi}(r)} \sigma dA\right)^{-1/p} \frac{1 - |\varphi(\zeta_j)|^2}{1 - |\varphi(\zeta_j)|^2}.
\]
(13)

Therefore, from (12) and (13), we have that

\[
g_j'(\varphi(\zeta_j)) = \gamma_j'(\varphi(\zeta_j)) f_{\varphi}(\zeta_j) (\varphi(\zeta_j)) + \gamma_j(\varphi(\zeta_j)) f_{\varphi}(\zeta_j)'(\varphi(\zeta_j))
\]
\[
= -\varphi(\zeta_j) \left(\int_{D_{\phi}(r)} \sigma dA\right)^{-1/p} \frac{1 - |\varphi(\zeta_j)|^2}{1 - |\varphi(\zeta_j)|^2}.
\]
(14)

Using the facts that \(\|W_{\psi,\varphi}\|_{L^p(\sigma) \to \mathcal{B}_p} \geq \limsup_{j \to \infty} \|W_{\psi,\varphi}\|_{L^p(\sigma) \to \mathcal{B}_p} \geq \limsup_{j \to \infty} K g_j\|_{\mathcal{B}_p} = 0\), and \(\limsup_{j \to \infty} K g_j\|_{\mathcal{B}_p} = 0\), we have that

\[
\|W_{\psi,\varphi}\|_{L^p(\sigma) \to \mathcal{B}_p} \geq \limsup_{j \to \infty} \mu(\zeta_j) |\psi'(\zeta_j) g_j(\varphi(\zeta_j)) + \psi(\zeta_j) f_{\varphi}(\zeta_j)'(\varphi(\zeta_j))| \geq \limsup_{j \to \infty} \mu(\zeta_j) |\psi(\zeta_j) f_{\varphi}(\zeta_j)'(\varphi(\zeta_j))| \left(\int_{D_{\phi}(r)} \sigma dA\right)^{-1/p}.
\]
(15)

Again, let \(\zeta_j\) be a sequence in \(\mathcal{D}\) such that \(|\varphi(\zeta_j)| \to 1\) as \(j \to \infty\) and

\[
\lim_{j \to \infty} \mu(\zeta_j) |\psi(\zeta_j) f_{\varphi}'(\zeta_j)(\varphi(\zeta_j))| \left(\int_{D_{\phi}(r)} \sigma dA\right)^{-1/p} = \limsup_{|\psi(\zeta)| \to 1} \mu(z) |\psi(z) f_{\varphi}'(z)| \left(\int_{D_{\phi}(r)} \sigma dA\right)^{-1/p}.
\]
(16)

For each \(j \in \mathbb{N}\), let \(h_j\) be defined as

\[
h_j(z) = \left\{ \begin{array}{ll}
1 & \text{if } \left(1 - |\varphi(\zeta_j)|^2\right)^{1 + (\eta + 2)p_0/p} \\
\frac{1}{1 + (\eta + 2)p_0/p} \left(1 - |\varphi(\zeta_j)|^2\right)^{2 + (\eta + 2)p_0/p} & \text{if } |\varphi(\zeta_j)| > 1
\end{array} \right.
\]
(17)

Then, by Lemma 3, sup \(\|h_j\|_{L^p(\sigma)} \leq 1\) and \(\{h_j\}_{j \in \mathbb{N}}\) converges to zero uniformly on compact subsets of \(\mathcal{D}\) as \(j \to \infty\). By Lemma 4, \(h_j\) converges to zero weakly in \(A^p(\sigma)\). Thus, for any compact operator \(K : A^p(\sigma) \to \mathcal{B}_p\), we have that \(\|K h_j\|_{\mathcal{B}_p} \to 0\) as \(j \to \infty\). Moreover, \(h_j'(\varphi(\zeta_j)) = 0\) and

\[
h_j(\varphi(\zeta_j)) = \left(\frac{1}{1 + (\eta + 2)p_0/p} + \frac{1}{1 + (\eta + 2)p_0/p} \left(1 - |\varphi(\zeta_j)|^2\right)^{2 + (\eta + 2)p_0/p}\right) \left(\int_{D_{\phi}(r)} \sigma dA\right)^{-1/p}.
\]
(18)

Thus, using (18), we have that

\[
\|W_{\psi,\varphi}\|_{L^p(\sigma) \to \mathcal{B}_p} \geq \limsup_{j \to \infty} \mu(\zeta_j) |\psi'(\zeta_j) h_j(\varphi(\zeta_j)) + \psi(\zeta_j) f_{\varphi}'(\zeta_j)(\varphi(\zeta_j))| \geq \limsup_{j \to \infty} \mu(\zeta_j) |\psi'(\zeta_j)| \left(\int_{D_{\phi}(r)} \sigma dA\right)^{-1/p}.
\]
(19)

Combining (15) and (19), we have that

\[
\|W_{\psi,\varphi}\|_{L^p(\sigma) \to \mathcal{B}_p} \geq \limsup_{j \to \infty} \mu(\zeta_j) |\psi'(\zeta_j)| \left(\int_{D_{\phi}(r)} \sigma dA\right)^{-1/p} + \limsup_{|\psi(\zeta)| \to 1} \mu(z) |\psi(z) f_{\varphi}'(z)| \left(\int_{D_{\phi}(r)} \sigma dA\right)^{-1/p}.
\]
(20)
Upper bound. Let $f \in \mathcal{A}^p(\sigma)$ be such that $\|f\|_{\mathcal{A}^p(\sigma)} \leq 1$. Let
\[ Lf(z) = f(a_jz), \quad (21) \]
where $a_j = j/(j+1)$.

Then, by Theorem 6.1 in [3], we have that $L_j : \mathcal{A}^p(\sigma) \rightarrow \mathcal{B}_\mu$ is compact. Since $W_{\psi, \varphi} : \mathcal{A}^p(\sigma) \rightarrow \mathcal{B}_\mu$ is bounded, so $W_{\psi, \varphi} L_j : \mathcal{A}^p(\sigma) \rightarrow \mathcal{B}_\mu$ is compact. Thus, for fixed $r$ in $(0,1)$, we have that
\[ \|W_{\psi, \varphi}\|_{\mathcal{L}(\mathcal{A}^p(\sigma) \rightarrow \mathcal{B}_\mu)} \leq \sup_{\|\varphi\|_{\mathcal{H}(\sigma)} \leq 1} \|W_{\psi, \varphi}(I - L_j)f\|_{\mathcal{B}_\mu} \]
\[ \leq \sup_{\|\varphi\|_{\mathcal{H}(\sigma)} \leq 1} \left[ |W_{\psi, \varphi}(I - L_j)f(0)| + \sup_{|z| \leq r} \|\varphi(z)\| \cdot |W_{\psi, \varphi}(I - L_j)f'(z)| + \sup_{|z| > r} \|\varphi(z)\| \cdot |W_{\psi, \varphi}(I - L_j)f'(z)| \right], \quad (22) \]
where $I$ is the identity operator on $\mathcal{A}^p(\sigma)$. Now
\[ |W_{\psi, \varphi}(I - L_j)f'(z)| \leq |\psi(z)| |f(\varphi(z)) - f(a_j \varphi(z))| + |\varphi(z)\psi'(z)| \cdot |f'(\varphi(z)) - f'(a_j \varphi(z))| \]
\[ \leq |\psi(z)| |f(\varphi(z)) - f(a_j \varphi(z))| + |\varphi(z)| \cdot |\psi'(z)| |f'(\varphi(z)) - f'(a_j \varphi(z))| \]
\[ + \frac{|\psi(z)| \cdot |\psi'(z)|}{j+1} |f'(a_j \varphi(z))| + |\varphi(z)\psi'(z)| |f'(\varphi(z)) - f'(a_j \varphi(z))|. \]
\[ (23) \]
Let $|\varphi(z)| \leq r$ and $\zeta = \varphi(z)$. Let $\Gamma = [a_j \zeta, \zeta]$ be the line segment from $a_j \zeta$ to $\zeta$. Then, $\Gamma \subset D(0,r)$, where $D(0,r) = \{ z : |z| \leq r \}$. Thus, by Lemma 2, we have that
\[ |f'(\zeta) - f'(a_j \zeta)| = \left| \int_\Gamma f''(\lambda)d\lambda \right| \leq \left| \frac{\| \varphi \|_{\mathcal{L}(\mathcal{B}_0, \mathcal{B}_0)}}{j+1} \int_{\mathcal{A}(D(0,r))} |f''(\lambda)| \right| \]
\[ \leq \left| \frac{\| \varphi \|_{\mathcal{L}(\mathcal{B}_0, \mathcal{B}_0)}}{j+1} \int_{\mathcal{A}(D(0,r))} \left[ \left( \int_{D_j(r)} \| \sigma d\lambda \| \right)^{-1/p} \right] \right| \|f\|_{\mathcal{A}^p(\sigma)}. \]
\[ (24) \]
Again, let $\gamma_0 = (\eta + 2)p_0/p - 2$. Then, by Lemma 2, we have that
\[ 1 = |K_{\chi}^{[\eta]}(0)| \leq C \left( \int_{D_j(1/2)} \| \sigma d\lambda \| \right)^{-1/p} \left| \int_{D_j(r)} \| \sigma d\lambda \| \right| \left( \frac{1}{1 - |\lambda|^2} \right)_r, \quad (25) \]
Thus,
\[ \left| \frac{1 - |\zeta|^2}{(1 - |\lambda|^2)^{(\eta+2)p_0/p}} \right| \leq \left| \frac{1 - |\zeta|^2}{(1 - |\lambda|^2)^{\gamma_0 \eta+2}} \right| \]
\[ \leq \left| \frac{1 - |\zeta|^2}{(1 - |\lambda|^2)^{\gamma_0 \eta+2}} \right| \leq 1. \quad (26) \]
Thus, from (24) and (26), we have that
\[ |f'(\zeta) - f'(a_j \zeta)| \leq \left| \frac{\| \varphi \|_{\mathcal{L}(\mathcal{B}_0, \mathcal{B}_0)}}{j+1} \left( \frac{\|f\|_{\mathcal{A}^p(\sigma)}}{1 - |\lambda|^2} \right)^{(\eta+2)p_0/p} \right| \|f\|_{\mathcal{A}^p(\sigma)}. \]
\[ (27) \]
Similarly, we can show that
\[ |f'(\zeta) - f(a_j \zeta)| \leq \frac{1}{j+1} \left( \frac{\|f\|_{\mathcal{A}^p(\sigma)}}{1 - |\lambda|^2} \right)^{(\eta+2)p_0/p} \|f\|_{\mathcal{A}^p(\sigma)}. \]
\[ (28) \]
Also, by Lemma 2 and equation (26), we have that
\[ \sup_{|\varphi(z)| \leq r} |f'(a_j \varphi(z))| \leq \sup_{|\varphi(z)| \leq r} \left( \frac{\|f\|_{\mathcal{A}^p(\sigma)}}{1 - |\lambda|^2} \right)^{(\eta+2)p_0/p} \|f\|_{\mathcal{A}^p(\sigma)}. \]
\[ (29) \]
Since $W_{\psi, \varphi} : \mathcal{A}^p(\sigma) \rightarrow \mathcal{B}_\mu$ is bounded, so
\[ \|W_{\psi, \varphi}f\|_{\mathcal{B}_\mu} \leq \|f\|_{\mathcal{A}^p(\sigma)}. \]
\[ (30) \]
for each $f \in \mathcal{A}^p(\sigma)$. By taking, respectively, $f(z) = 1$ and $f(z) = z$ and using the fact that $|\varphi(z)| < 1$, we have that
\[ M_1 = \sup_{z \in \mathcal{D}} \mu(|\varphi(z)|) < \coand M_2 = \sup_{z \in \mathcal{D}} \mu(|\psi(z)|) < \infty. \]
\[ (31) \]
Combining (23) and (27)-(29), we have that
\[ \sup_{\|\varphi(z)\| \leq r} \mu(z) \cdot |\int_{\mathcal{A}(D(0,r))} \left[ \int_{D_j(r)} \| \sigma d\lambda \| \right] \left( \frac{1}{1 - |\lambda|^2} \right)_r | \rightarrow 0 \quad \text{as} \quad j \rightarrow \infty. \]
\[ (32) \]
Using (28), we have that

\[
|W_{\psi\phi}(I-L) f(0)| \geq |(\psi(0)f(\phi(0)) - \psi(0)f(a,\phi(0)))| \leq \frac{\|\psi(0)f(0)\|}{j+1} \frac{\|f\|_{dP(\sigma)}}{(1-r^2)^{3\gamma \|2j\|_{P_\mu}}}.
\]

Using (31), (32), and (33), we have that

\[
\sup_{\|f\|_{dP(\sigma)} \leq 1} \left| \sup_{|\psi(\phi)|<r} \left[ |W_{\psi\phi}(I-L) f(0)| + |W_{\psi\phi}(I-L) f'(z)| \right] \right| \rightarrow 0
\]

as \(j \rightarrow \infty\). The last term in the right-hand side of (22) is dominated by

\[
\sup_{|\psi(\phi)|<r} \mu(z) |\psi'(z)| \left\{ \left| f(\psi(z)) + f'(a,\phi(z)) \right| \right\} + \sup_{|\psi(\phi)|<r} \mu(z) |\psi'(z)| \left\{ \left| f'(\psi(z)) + a, f'(a,\phi(z)) \right| \right\},
\]

which is further dominated by a constant multiple of

\[
\sup_{|\psi(\phi)|<r} \mu(z) |\psi'(z)| \left\{ \left( \int_{D_{\psi(\phi)}} \sigma dA \right)^{-1/p} \right\} \left( \int_{D_{\psi(\phi)}} \sigma dA \right)^{-1/p} + \sup_{|\psi(\phi)|<r} \mu(z) |\psi'(z)| \left\{ \left( \int_{D_{\psi(\phi)}} \sigma dA \right)^{-1/p} \right\} \left( \int_{D_{\psi(\phi)}} \sigma dA \right)^{-1/p}
\]

Using (34) and (37) in (22), we have that

\[
\|W_{\psi\phi}\|_{L^p(\sigma) \rightarrow \mathcal{B}_\mu} \leq \sup_{|\psi(\phi)|<r} \mu(z) |\psi'(z)| \left( \int_{D_{\psi(\phi)}} \sigma dA \right)^{-1/p} + \sup_{|\psi(\phi)|<r} \mu(z) |\psi'(z)| \left( \int_{D_{\psi(\phi)}} \sigma dA \right)^{-1/p}.
\]

Finally, letting \(r \rightarrow 1\), then we get

\[
\|W_{\psi\phi}\|_{L^p(\sigma) \rightarrow \mathcal{B}_\mu} \leq \limsup_{|\psi(\phi)|<1} \mu(z) |\psi'(z)| \left( \int_{D_{\psi(\phi)}} \sigma dA \right)^{-1/p} + \limsup_{|\psi(\phi)|<1} \mu(z) |\psi'(z)| \left( \int_{D_{\psi(\phi)}} \sigma dA \right)^{-1/p}.
\]

Combining (20) and (39), we get the desired result.

**Corollary 5.** Let \(p \in (1,\infty)\), \(\psi \in H(D)\), and \(\phi\) be a holomorphic self-map of \(D\), such that \(\|\psi\|_{C^\infty} = 1\). Let \(W_{\psi\phi} : \mathcal{A}^p(\sigma) \rightarrow \mathcal{B}_\mu\) be a holomorphic self-map of \(D\), such that \(\|\psi\|_{C^\infty} = 1\). Let \(W_{\psi\phi} : \mathcal{A}^p(\sigma) \rightarrow \mathcal{B}_\mu\) is compact if and only if the following conditions are satisfied:

1. \(\limsup_{|\psi(\phi)|<1} \mu(z) |\psi'(z)| \left( \int_{D_{\psi(\phi)}} \sigma dA \right)^{-1/p} = 0\)
2. \(\limsup_{|\psi(\phi)|<1} \mu(z) |\psi'(z)| \left( \int_{D_{\psi(\phi)}} \sigma dA \right)^{-1/p} = 0\)

**Data Availability**

The data used to support the findings of this study are included within the article.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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