Research Article

The Regularity Criteria and the A Priori Estimate on the 3D Incompressible Navier-Stokes Equations in Orthogonal Curvilinear Coordinate Systems

Fan Geng¹, Shu Wang^{1,2} and Yongxin Wang²

¹School of Mathematics and Information Sciences, Guangzhou University, Guangzhou 510006, China

²School of Mathematics, Faculty of Science, Beijing University of Technology, Beijing 100124, China

Correspondence should be addressed to Fan Geng; gengfan@e.gzhu.edu.cn

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The paper considers the regularity problem on three-dimensional incompressible Navier-Stokes equations in general orthogonal curvilinear coordinate systems. We establish one regularity criteria of the weak solutions involving only in a vorticity component ω^3 and one a priori estimate on the solution that $\|H_3 u^3\|_{L^\infty(0,T;L^p(\mathbb{R}^3))}$ is bounded for $1 \leq p \leq \infty$ to three-dimensional incompressible Navier-Stokes equations in orthogonal curvilinear coordinate systems. These extent greatly the corresponding results on axisymmetric cylindrical flow.

1. Introduction

In this paper, we investigate the regularity problem on the following three-dimensional (3D) incompressible Navier-Stokes equations

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P = \Delta \mathbf{u}, \mathbf{x} \in \mathbb{R}^3, t > 0, \\ \operatorname{div} \mathbf{u} = 0, \mathbf{x} \in \mathbb{R}^3, t > 0, \\ \mathbf{u}(0, \mathbf{x}) = \mathbf{u}_0(\mathbf{x}), \mathbf{x} \in \mathbb{R}^3 \end{cases} \quad (1)$$

in general orthogonal curvilinear coordinate systems. Here, $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$, $\mathbf{u} = (u_1, u_2, u_3)(t, \mathbf{x})$ denotes the velocity fields, $P = P(t, \mathbf{x})$ is the scalar pressure, and $\mathbf{u}_0$ is a given initial velocity with $\operatorname{div} \mathbf{u}_0 = 0$.

The existence of global weak solutions to (1) is known since the famous work of Leray [1] (see also Hopf [2] for the bounded domain case) for initial data $\mathbf{u}_0 \in L^2(\mathbb{R}^3)$ with $\operatorname{div} \mathbf{u}_0 = 0$. The uniqueness and global regularity of Leray-Hopf weak solutions is still one of the most challenging open problems in the mathematical fluid dynamics [1–6]. Many researchers are devoted to looking for certain sufficient con-

ditions to ensure the smoothness of solutions, called the regularity criterion or Serrin-type criterion. Thanks to the pioneering work by [6–8], we have known that the weak solution $\mathbf{u}$ will be smooth as long as

$$\mathbf{u} \in L^p(0, T; L^q(\mathbb{R}^3)), \frac{2}{p} + \frac{3}{q} \leq 1, 2 \leq p \leq \infty, 3 \leq q \leq \infty. \quad (2)$$

Afterwards, there are many progresses on the regularity criteria involving only one component of the velocity fields, one can refer to [9–12] for details.

An interesting problem is to study the globally stabilizing effects of the geometry structures of the domain or/and solutions on the evolution of solution in time to the 3D incompressible Navier-Stokes equations. For example, the axisymmetric flow makes the 3D flow close to 2D flow, that is, all velocity components (radial, angular (or swirl) and x_3 -component) and the pressure are independent of the angular variable in the cylindrical coordinates. It is well known that the 3D incompressible axisymmetric Navier-Stokes equations without swirl have the unique global smooth solution [13–16]. However, it is still open for the global regularity with swirl ([17–20] and therein). These

results indicate that the swirl of the fluid plays a crucial role in the issue of global regularity. Subsequently, to understand this problem better, many efforts have been devoted to looking for suitable regularity criteria, see [21–26] for details.

The paper is motivated by the studies on the axisymmetric flow (see [5, 13, 14, 16]) and the helical flow (see [27] and references therein) of the 3D incompressible flows and on the absence of simple hyperbolic blow-up for the 3D incompressible Euler and quasigeostrophic equations [28], we investigate the regularity criteria of the weak solutions to the 3D incompressible Navier-Stokes equations in general orthogonal curvilinear coordinate systems. Recently, global well-posedness results on the smooth solution for 3D incompressible Navier-Stokes equations in spherical coordinates are obtained in [29–31]. The main purpose of this paper is to establish the a priori estimate and the regularity criteria for the 3D incompressible Navier-Stokes equations in general orthogonal curvilinear coordinate systems.

To state our main results in this paper, let us begin with some notations, see [32]. A point in $\mathbb{R}^3$ is denoted by $\mathbf{x} = (\xi_1, \xi_2, \xi_3)$, where (ξ_1, ξ_2, ξ_3) is general orthogonal curvilinear coordinates with a line element ds given by

$$ds = H_1 d\xi_1 \mathbf{e}_{\xi_1} + H_2 d\xi_2 \mathbf{e}_{\xi_2} + H_3 d\xi_3 \mathbf{e}_{\xi_3}. \quad (3)$$

Here, $\mathbf{e}_{\xi_i} = \mathbf{e}_{\xi_i}(\xi)(i = 1, 2, 3)$ are orthogonal, of unit length and parallel to the coordinate lines with ξ_i increasing; the nonnegative functions $H_i (i = 1, 2, 3)$ are the G.Lamé coefficients corresponding to the vectors $\mathbf{e}_{\xi_1}, \mathbf{e}_{\xi_2}, \mathbf{e}_{\xi_3}$, respectively. Throughout this paper, we assume $H_i (i = 1, 2, 3)$ are independent of ξ_3 , i.e., $H_i = H_i(\xi_1, \xi_2) (i = 1, 2, 3)$, and the measure of the set $\{(\xi_1, \xi_2) \in D : (H_1 H_2 H_3)(\xi_1, \xi_2) = 0\}$ is zero in the sense of Lebesgue measure in $\mathbb{R}^2$.

In this paper, we consider the solution (u, P) to the 3D incompressible Navier-Stokes (1) of having the form

$$\begin{aligned} \mathbf{u}(t, \mathbf{x}) &= u^1(t, \xi_1, \xi_2) \mathbf{e}_{\xi_1} + u^2(t, \xi_1, \xi_2) \mathbf{e}_{\xi_2} \\ &\quad + u^3(t, \xi_1, \xi_2) \mathbf{e}_{\xi_3}, \quad P(t, \mathbf{x}) = P(t, \xi_1, \xi_2), \end{aligned} \quad (4)$$

with the initial data

$$\mathbf{u}_0(\mathbf{x}) = u_0^1(\xi_1, \xi_2) \mathbf{e}_{\xi_1} + u_0^2(\xi_1, \xi_2) \mathbf{e}_{\xi_2} + u_0^3(\xi_1, \xi_2) \mathbf{e}_{\xi_3}. \quad (5)$$

Our main results are the a priori estimates and the regularity criteria involving only in a vorticity component ω^3 on 3D incompressible Navier-Stokes equations in general orthogonal curvilinear coordinate systems.

Theorem 1 (the a priori estimate of $\|H_3 u^3\|_{L^\infty(0, T; L^p(\mathbb{R}^3))}$). Suppose that $\mathbf{u}$ be a smooth solution of system (1) with the form (4) and the initial data (5) satisfying $\operatorname{div} \mathbf{u}_0 = 0$ and $H_3 u_0^3 \in L^p(\mathbb{R}^3)$ for $p \in [1, \infty]$. Then, we have for any $T > 0$,

$$H_3 u^3 \in L^\infty(0, T; L^2(\mathbb{R}^3)) \text{ and } |H_3 u^3| \in L^2(0, T; H^1(\mathbb{R}^3)), \quad (6)$$

and, moreover, if assume that the G.Lamé coefficient $H_3(\xi_1, \xi_2)$ satisfies

$$|\Delta(\ln H_3)| \leq C < \infty, \quad (7)$$

it holds, for any $T > 0$ and any $p \in [1, 2) \cup (2, \infty]$, that

$$H_3 u^3 \in L^\infty(0, T; L^p(\mathbb{R}^3)) \text{ and } |H_3 u^3|^{p/2} \in L^2(0, T; H^1(\mathbb{R}^3)). \quad (8)$$

Theorem 2 (the regularity criteria involving ω^3). Let $\mathbf{u}$ be a weak solution of system (1) with the form (4) and the initial data (5) satisfying $\mathbf{u}_0 \in H^2(\mathbb{R}^3)$ and $\operatorname{div} \mathbf{u}_0 = 0$. Then, the solution $\mathbf{u}$ is smooth in $(0, T] \times \mathbb{R}^3$, if

$$\omega^3 \in L^p(0, T; L^q(\mathbb{R}^3)), \quad (9)$$

where $(2/p) + (3/q) \leq 2$, $1 \leq p \leq \infty$, $3/2 \leq q \leq \infty$.

Remark 3. The assumption (7) comes from the geometry on the harmonic mapping in some sense. It is easy to see that

$$\Delta(\ln H_3) = \tilde{\Delta}(\ln H_3) \quad (10)$$

in (7), based on the notation $\tilde{\Delta}$ introduced in Section 2, because H_3 is independent of ξ_3 . Thus, if H_3 is a radial function in $\mathbb{R}^2$, i.e.,

$$H_3 = H_3(r) \quad (11)$$

with

$$r = \sqrt{x_1^2 + x_2^2} \text{ or } r = \sqrt{x_2^2 + x_3^2} \text{ or } r = \sqrt{x_1^2 + x_3^2}, \quad (12)$$

then

$$\Delta(\ln H_3) \equiv 0. \quad (13)$$

In this case, the assumption (7) is naturally true because the function $\ln H_3$ is the harmonic one.

Remark 4. The assumption (7) of Theorem 1 is satisfied in the cases of cylindrical coordinates. More precisely, we have the known results on 3D problem in the case of cylindrical coordinate system in $\mathbb{R}^3$ are covered in Theorem 1 and Theorem 2, i.e., let

$$\xi_1 = r = \sqrt{x_1^2 + x_2^2}, \xi_2 = x_2, \xi_3 = \theta = \arctan \frac{x_3}{x_1};$$

$$\begin{aligned} u^1(t, \xi_1, \xi_2) &= u^r(t, r, x_2), u^2(t, \xi_1, \xi_2) \\ &= u_3(t, r, x_2), u^3(t, \xi_1, \xi_2) = u^\theta(t, r, x_2); \end{aligned}$$

$$\begin{aligned} \mathbf{e}_{\xi_1} &= \mathbf{e}_r = \left(\frac{x_1}{r}, \frac{x_2}{r}, 0 \right), \mathbf{e}_{\xi_2} = \mathbf{e}_3 = (0, 0, 1), \mathbf{e}_{\xi_3} \\ &= \mathbf{e}_\theta = \left(-\frac{x_2}{r}, \frac{x_1}{r}, 0 \right); \end{aligned} \quad (14)$$

we consider an axisymmetric solution of the Navier-Stokes equations of the form (4) in the cylindrical coordinate system, where the mapping is taken as $x = x(\xi_1, \xi_2, \xi_3) = (\xi_1 \cos \xi_3, \xi_2, \xi_1 \sin \xi_3)$, and G.Lamé coefficients are $H_1 = H_2 = 1, H_3 = \xi_1$, satisfying the assumption in Theorem 1. Then, Theorem 1 is equivalent to Proposition 1 in [9], Theorem 2 is equivalent to Theorem 1.3 in [21].

Remark 5. This difference from the case of curvilinear cylindrical coordinates may be imply that one should care about the advantage or overcome the difficulty brought by the choice of curvilinear coordinates, including nonorthogonal curvilinear coordinates, which will be discussed in the future.

The remaining of this paper is organized as follows. In Section 2, we will derive the Navier-Stokes equations in orthogonal curvilinear coordinate systems. In Section 3, we introduce some basic lemmas and one estimate used for the proof of main theorems. In Section 4 and Section 5, we prove Theorem 1 and Theorem 2 separately.

2. Navier-Stokes Equations in Orthogonal Curvilinear Coordinate Systems

In this section, we will first derive the incompressible Navier-Stokes equations in orthogonal curvilinear coordinate systems ξ_1, ξ_2, ξ_3 , given by Section 1.

We assume that $\mathbf{x} = \mathbf{x}(\xi)$, being one-to-one and onto mapping, transforms $\xi \in \tilde{\Omega} \subset \mathbb{R}^3$ into $\mathbf{x} \in \Omega \subset \mathbb{R}^3$ with $\tilde{\Omega} = D \times [\alpha, \beta] \subset \mathbb{R}^3$ and

$$\Omega = \{(x_1, x_2, x_3) = (x_1, x_2, x_3)(\xi) \in \mathbb{R}^3 : (\xi_1, \xi_2) \in D \subset \mathbb{R}^2, -\infty < \alpha \leq \xi_3 \leq \beta < \infty\}, \quad (15)$$

where the domain D is the bounded or unbounded domain of $\mathbb{R}^2$ with the smooth boundary ∂D if D is bounded, and the constants α and β satisfy $-\infty < \alpha \leq \beta < \infty$.

Since $H_i = H_i(\xi_1, \xi_2)$ ($i = 1, 2, 3$), by the derivatives of the unit vectors $\mathbf{e}_{\xi_1}, \mathbf{e}_{\xi_2}$ and $\mathbf{e}_{\xi_3}$, we have

$$\begin{aligned} \frac{\partial \mathbf{e}_{\xi_1}}{\partial \xi_1} &= -\frac{\mathbf{e}_{\xi_2}}{H_2} \frac{\partial H_1}{\partial \xi_2}, \quad \frac{\partial \mathbf{e}_{\xi_1}}{\partial \xi_2} = \frac{\mathbf{e}_{\xi_2}}{H_1} \frac{\partial H_2}{\partial \xi_1}, \quad \frac{\partial \mathbf{e}_{\xi_1}}{\partial \xi_3} = \frac{\mathbf{e}_{\xi_3}}{H_1} \frac{\partial H_3}{\partial \xi_1}, \\ \frac{\partial \mathbf{e}_{\xi_2}}{\partial \xi_1} &= \frac{\mathbf{e}_{\xi_1}}{H_2} \frac{\partial H_1}{\partial \xi_2}, \quad \frac{\partial \mathbf{e}_{\xi_2}}{\partial \xi_2} = -\frac{\mathbf{e}_{\xi_1}}{H_1} \frac{\partial H_2}{\partial \xi_1}, \quad \frac{\partial \mathbf{e}_{\xi_2}}{\partial \xi_3} = \frac{\mathbf{e}_{\xi_3}}{H_2} \frac{\partial H_3}{\partial \xi_2}, \\ \frac{\partial \mathbf{e}_{\xi_3}}{\partial \xi_1} &= \frac{\partial \mathbf{e}_{\xi_3}}{\partial \xi_2} = 0, \quad \frac{\partial \mathbf{e}_{\xi_3}}{\partial \xi_3} = -\frac{\mathbf{e}_{\xi_1}}{H_1} \frac{\partial H_3}{\partial \xi_1} - \frac{\mathbf{e}_{\xi_2}}{H_2} \frac{\partial H_3}{\partial \xi_2}. \end{aligned} \quad (16)$$

Using the definition of gradient, we get the expression of the gradient operator ∇ in orthogonal curvilinear coordinate systems:

$$\nabla = \mathbf{e}_{\xi_1} \frac{1}{H_1} \frac{\partial}{\partial \xi_1} + \mathbf{e}_{\xi_2} \frac{1}{H_2} \frac{\partial}{\partial \xi_2} + \mathbf{e}_{\xi_3} \frac{1}{H_3} \frac{\partial}{\partial \xi_3}, \quad (17)$$

we also obtain the expression of Laplacian operator Δ in orthogonal curvilinear coordinate systems:

$$\Delta = \frac{1}{H_1 H_2 H_3} \left[\frac{\partial}{\partial \xi_1} \left(\frac{H_2 H_3}{H_1} \frac{\partial}{\partial \xi_1} \right) + \frac{\partial}{\partial \xi_2} \left(\frac{H_1 H_3}{H_2} \frac{\partial}{\partial \xi_2} \right) + \frac{\partial}{\partial \xi_3} \left(\frac{H_1 H_2}{H_3} \frac{\partial}{\partial \xi_3} \right) \right]. \quad (18)$$

Furthermore, for a vector field $\mathbf{V} = \mathbf{V}(\xi_1, \xi_2, \xi_3) = V_1 \mathbf{e}_{\xi_1} + V_2 \mathbf{e}_{\xi_2} + V_3 \mathbf{e}_{\xi_3}$, we get the expressions of $\text{div } \mathbf{V}$ and $\text{rot } \mathbf{V}$ in orthogonal curvilinear coordinate systems:

$$\text{div } \mathbf{V} = \frac{1}{H_1 H_2 H_3} \left[\frac{\partial}{\partial \xi_1} (H_2 H_3 V_1) + \frac{\partial}{\partial \xi_2} (H_1 H_3 V_2) + \frac{\partial}{\partial \xi_3} (H_1 H_2 V_3) \right], \quad (19)$$

$$\text{rot } \mathbf{V} = \frac{1}{H_1 H_2 H_3} \begin{vmatrix} H_1 \mathbf{e}_{\xi_1} & H_2 \mathbf{e}_{\xi_2} & H_3 \mathbf{e}_{\xi_3} \\ \frac{\partial}{\partial \xi_1} & \frac{\partial}{\partial \xi_2} & \frac{\partial}{\partial \xi_3} \\ H_1 V_1 & H_2 V_2 & H_3 V_3 \end{vmatrix}. \quad (20)$$

By the above expressions (17)–(19), then taking the inner product of equation (1)₁ with $\mathbf{e}_{\xi_1}, \mathbf{e}_{\xi_2}, \mathbf{e}_{\xi_3}$, respectively, we can derive the Navier-Stokes equations in orthogonal curvilinear coordinate systems as follows:

$$\begin{cases} \frac{\partial u^1}{\partial t} + \frac{u^1}{H_1} \frac{\partial u^1}{\partial \xi_1} + \frac{u^2}{H_2} \frac{\partial u^1}{\partial \xi_2} + \frac{1}{H_1} \frac{\partial P}{\partial \xi_1} = \left[L_1 u^1 - \frac{1}{(H_1 H_3)^2} \left(\frac{\partial H_3}{\partial \xi_1} \right)^2 u^1 + L_2 u^2 - \frac{1}{H_1 H_2 (H_3)^2} \frac{\partial H_3}{\partial \xi_1} \frac{\partial H_3}{\partial \xi_2} u^2 \right] - \frac{u^1 u^2}{H_1 H_2} \frac{\partial H_1}{\partial \xi_2} + \frac{(u^2)^2}{H_1 H_2} \frac{\partial H_2}{\partial \xi_1} + \frac{(u^3)^2}{H_1 H_3} \frac{\partial H_3}{\partial \xi_2}, \\ \frac{\partial u^2}{\partial t} + \frac{u^1}{H_1} \frac{\partial u^2}{\partial \xi_1} + \frac{u^2}{H_2} \frac{\partial u^2}{\partial \xi_2} + \frac{1}{H_2} \frac{\partial P}{\partial \xi_2} = \left[L_1 u^2 - \frac{1}{(H_2 H_3)^2} \left(\frac{\partial H_3}{\partial \xi_2} \right)^2 u^2 - L_2 u^1 - \frac{1}{H_1 H_2 (H_3)^2} \frac{\partial H_3}{\partial \xi_1} \frac{\partial H_3}{\partial \xi_2} u^1 \right] - \frac{u^1 u^2}{H_1 H_2} \frac{\partial H_2}{\partial \xi_1} + \frac{(u^1)^2}{H_1 H_2} \frac{\partial H_1}{\partial \xi_2} + \frac{(u^3)^2}{H_1 H_3} \frac{\partial H_3}{\partial \xi_1}, \\ \frac{\partial u^3}{\partial t} + \frac{u^1}{H_1} \frac{\partial u^3}{\partial \xi_1} + \frac{u^2}{H_2} \frac{\partial u^3}{\partial \xi_2} = L_3 u^3 - \frac{u^3}{H_3} \left(\frac{u^1}{H_1} \frac{\partial H_3}{\partial \xi_1} + \frac{u^2}{H_2} \frac{\partial H_3}{\partial \xi_2} \right), \end{cases} \quad (21)$$

where $(\xi_1, \xi_2) \in D \subset \mathbb{R}^2$, $t > 0$ and

$$\begin{aligned} \nabla_{\xi_1, \xi_2} &= \mathbf{e}_{\xi_1} \frac{1}{H_1} \frac{\partial}{\partial \xi_1} + \mathbf{e}_{\xi_2} \frac{1}{H_2} \frac{\partial}{\partial \xi_2}, \\ \tilde{\Delta} &= \Delta_{\xi_1, \xi_2} = \frac{1}{H_1 H_2 H_3} \left[\frac{\partial}{\partial \xi_1} \left(\frac{H_2 H_3}{H_1} \frac{\partial}{\partial \xi_1} \right) + \frac{\partial}{\partial \xi_2} \left(\frac{H_1 H_3}{H_2} \frac{\partial}{\partial \xi_2} \right) \right], \\ L_1 &= \Delta_{\xi_1, \xi_2} - \frac{1}{(H_1 H_2)^2} \left(\left(\frac{\partial H_1}{\partial \xi_2} \right)^2 + \left(\frac{\partial H_2}{\partial \xi_1} \right)^2 \right), \\ L_2 &= \frac{1}{H_1 H_2 H_3} \left(\frac{2 H_3}{H_1} \frac{\partial H_1}{\partial \xi_2} \frac{\partial}{\partial \xi_1} + \frac{\partial}{\partial \xi_1} \left(\frac{H_3}{H_1} \frac{\partial H_1}{\partial \xi_2} \right) \right. \\ &\quad \left. - \frac{2 H_3}{H_2} \frac{\partial H_2}{\partial \xi_1} \frac{\partial}{\partial \xi_2} - \frac{\partial}{\partial \xi_2} \left(\frac{H_3}{H_2} \frac{\partial H_2}{\partial \xi_1} \right) \right), \\ L_3 &= \Delta_{\xi_1, \xi_2} - \frac{1}{(H_3)^2} \left(\left(\frac{1}{H_1} \frac{\partial H_3}{\partial \xi_1} \right)^2 + \left(\frac{1}{H_2} \frac{\partial H_3}{\partial \xi_2} \right)^2 \right). \end{aligned} \quad (22)$$

The incompressible constraint is

$$\frac{\partial}{\partial \xi_1} (H_2 H_3 u^1) + \frac{\partial}{\partial \xi_2} (H_1 H_3 u^2) = 0. \quad (23)$$

It is clear that equations (21) and (23) completely determine the evolution of 3D Navier-Stokes equations in orthogonal curvilinear coordinate systems, respectively, once the initial value and/or boundary conditions are given.

We take initial condition for the system (21) as follows:

$$(u^1, u^2, u^3)(t=0, \xi_1, \xi_2) = (u_0^1, u_0^2, u_0^3)(\xi_1, \xi_2). \quad (24)$$

Moreover, the boundary condition $\mathbf{u} \rightarrow 0$ as $|\mathbf{x}| \rightarrow \infty, t \geq 0$ is equivalent to the following condition

$$(u^1, u^2, u^3)|_{\partial D} = 0, t \geq 0, \quad (25)$$

if the domain D is bounded or of having partially bounded boundary.

By the expressions (4) and (20), using the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$, in orthogonal curvilinear coordinate systems, we have

$$\boldsymbol{\omega}(t, \mathbf{x}) = \omega^1(t, \xi_1, \xi_2) \mathbf{e}_{\xi_1} + \omega^2(t, \xi_1, \xi_2) \mathbf{e}_{\xi_2} + \omega^3(t, \xi_1, \xi_2) \mathbf{e}_{\xi_3}, \quad (26)$$

with the initial vorticity

$$\boldsymbol{\omega}_0 = \boldsymbol{\omega}(t=0, \mathbf{x}) = \omega_0^1(\xi_1, \xi_2) \mathbf{e}_{\xi_1} + \omega_0^2(\xi_1, \xi_2) \mathbf{e}_{\xi_2} + \omega_0^3(\xi_1, \xi_2) \mathbf{e}_{\xi_3}, \quad (27)$$

Where

$$\begin{aligned} \omega^1 &= \frac{1}{H_2 H_3} \frac{\partial}{\partial \xi_2} (H_3 u^3), \omega^2 = -\frac{1}{H_1 H_3} \frac{\partial}{\partial \xi_1} (H_3 u^3), \omega^3 \\ &= \frac{1}{H_1 H_2} \left(\frac{\partial}{\partial \xi_1} (H_2 u^2) - \frac{\partial}{\partial \xi_2} (H_1 u^1) \right). \end{aligned} \quad (28)$$

Moreover, with the help of (28), we can get the equation of ω^3 from (21) as follows:

$$\begin{aligned} \frac{\partial \omega^3}{\partial t} + \frac{u^1}{H_1} \frac{\partial \omega^3}{\partial \xi_1} + \frac{u^2}{H_2} \frac{\partial \omega^3}{\partial \xi_2} &= L_3 \omega^3 + \frac{\omega^3}{H_3} \left(\frac{u^1}{H_1} \frac{\partial H_3}{\partial \xi_1} + \frac{u^2}{H_2} \frac{\partial H_3}{\partial \xi_2} \right) \\ &+ \frac{1}{H_1 H_2 (H_3)^2} \left[\frac{\partial H_3}{\partial \xi_2} \frac{\partial (H_3 u^3)^2}{\partial \xi_1} - \frac{\partial H_3}{\partial \xi_1} \frac{\partial (H_3 u^3)^2}{\partial \xi_2} \right]. \end{aligned} \quad (29)$$

3. Some Useful Estimates

To study the main estimates of Theorem 1 and Theorem 2, we need to introduce two basic lemmas and one estimate relates to ω^3 in orthogonal curvilinear coordinate systems.

Lemma 6 (see [6–8]). *Suppose that the initial data $\mathbf{u}_0 \in H^2(\mathbb{R}^3)$ in (1), then any Leray-Hopf weak solution $\mathbf{u}$ of 3D incompressible Navier-Stokes equations (1) is also a smooth solution in $(0, T] \times \mathbb{R}^3$, if there holds that*

$$\mathbf{u} \in L^p(0, T; L^q(\mathbb{R}^3)), \quad (30)$$

in which p and q satisfy the conditions

$$\frac{2}{p} + \frac{3}{q} \leq 1 \text{ with } 3 < q \leq \infty, 2 \leq p < \infty \text{ or } 3 \leq q < \infty, p = \infty. \quad (31)$$

And we would like to recall the well-known relation between the velocity and vorticity of 3D flow.

Lemma 7 (see [33]). *Let $\mathbf{u} \in W^{1,p}(\mathbb{R}^3)$ be a velocity field with its vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$, then the inequality*

$$\|\nabla \mathbf{u}\|_{L^p(\mathbb{R}^3)} \leq C_p \left(\|\boldsymbol{\omega}\|_{L^p(\mathbb{R}^3)} + \|\operatorname{div} \mathbf{u}\|_{L^p(\mathbb{R}^3)} \right) \quad (32)$$

holds for any $p \in (1, \infty)$, where the constant C_p depends only on p .

As one kind of fluid with the special geometry structure of form (4), the incompressible 3D flow also has one particular property, which is shown as follows.

Proposition 8. *Suppose that $\mathbf{u} \in W^{1,p}(\mathbb{R}^3)$ with the form (4) be a field with zero divergence, then the estimate*

$$\|\nabla \tilde{\mathbf{u}}\|_{L^p(\mathbb{R}^3)} \leq \tilde{C}_p \|\omega^3\|_{L^p(\mathbb{R}^3)} \quad (33)$$

holds for any $p \in (1, \infty)$, where

$$\tilde{\mathbf{u}} = u^1(t, \xi_1, \xi_2) \mathbf{e}_{\xi_1} + u^2(t, \xi_1, \xi_2) \mathbf{e}_{\xi_2} \quad (34)$$

and the constant $\tilde{C}_p$ depends only on p .

Proof. Since $\operatorname{div} \mathbf{u} = 0$ and we have

$$\begin{aligned} 0 &= \nabla \cdot \mathbf{u} = \frac{1}{H_1 H_2 H_3} \left[\frac{\partial}{\partial \xi_1} (H_2 H_3 u^1) + \frac{\partial}{\partial \xi_2} (H_1 H_3 u^2) + \frac{\partial}{\partial \xi_3} (H_1 H_2 u^3) \right] \\ &= \frac{1}{H_1 H_2 H_3} \left[\frac{\partial}{\partial \xi_1} (H_2 H_3 u^1) + \frac{\partial}{\partial \xi_2} (H_1 H_3 u^2) \right]. \end{aligned} \quad (35)$$

Thus, we obtain

$$\operatorname{div} \tilde{\mathbf{u}} = \frac{1}{H_1 H_2 H_3} \left[\frac{\partial}{\partial \xi_1} (H_2 H_3 u^1) + \frac{\partial}{\partial \xi_2} (H_1 H_3 u^2) \right] = 0. \quad (36)$$

On the other hand, by the expressions (20), (34), and (28), we get

$$\begin{aligned} \nabla \times \tilde{\mathbf{u}} &= \frac{1}{H_2 H_3} \left[-\frac{\partial}{\partial \xi_3} (H_2 u^2) \right] \mathbf{e}_{\xi_1} + \frac{1}{H_1 H_3} \left[\frac{\partial}{\partial \xi_3} (H_1 u^1) \right] \mathbf{e}_{\xi_2} \\ &\quad + \frac{1}{H_1 H_2} \left[\frac{\partial}{\partial \xi_1} (H_2 u^2) - \frac{\partial}{\partial \xi_2} (H_1 u^1) \right] \mathbf{e}_{\xi_3} \\ &= \left[\frac{1}{H_1 H_2} \left(\frac{\partial}{\partial \xi_1} (H_2 u^2) - \frac{\partial}{\partial \xi_2} (H_1 u^1) \right) \right] \mathbf{e}_{\xi_3} = \omega^3 \mathbf{e}_{\xi_3}. \end{aligned} \quad (37)$$

Consequently, by Lemma 7 and using (36) and (37), one has

$$\|\nabla \tilde{\mathbf{u}}\|_{L^p(\mathbb{R}^3)} \leq \tilde{C}_p \|\omega^3 \mathbf{e}_{\xi_3}\|_{L^p(\mathbb{R}^3)} = \tilde{C}_p \|\omega^3\|_{L^p(\mathbb{R}^3)}. \quad (38)$$

This finishes the proof of the proposition.

4. Proof of Theorem 1

In this section, we prove Theorem 1.

Proof of Theorem 1. Let $F(t, \xi_1, \xi_2) = H_3(\xi_1, \xi_2) u^3(t, \xi_1, \xi_2)$, then putting $u^3(t, \xi_1, \xi_2) = F(t, \xi_1, \xi_2)/H_3(\xi_1, \xi_2)$ into (21)₃, we can obtain the equation for F

$$\begin{aligned} \partial_t F &+ \frac{u^1}{H_1} \frac{\partial F}{\partial \xi_1} + \frac{u^2}{H_2} \frac{\partial F}{\partial \xi_2} \\ &= \Delta_{\xi_1, \xi_2} F - \frac{1}{H_1 H_2} \left[\frac{\partial}{\partial \xi_1} \left(\frac{H_2}{H_1 H_3} \frac{\partial H_3}{\partial \xi_1} \right) F + \frac{2H_2}{H_1 H_3} \frac{\partial F}{\partial \xi_1} \frac{\partial H_3}{\partial \xi_1} \right. \\ &\quad \left. + \frac{\partial}{\partial \xi_2} \left(\frac{H_1}{H_2 H_3} \frac{\partial H_3}{\partial \xi_2} \right) F + \frac{2H_1}{H_2 H_3} \frac{\partial F}{\partial \xi_2} \frac{\partial H_3}{\partial \xi_2} \right] \\ &\quad - \frac{F}{H_3^2} \left[\left(\frac{1}{H_1} \frac{\partial H_3}{\partial \xi_1} \right)^2 + \left(\frac{1}{H_2} \frac{\partial H_3}{\partial \xi_2} \right)^2 \right], \end{aligned} \quad (39)$$

with the help of the following calculations

$$\begin{aligned} \partial_t u^3 &= \frac{1}{H_3} \partial_t F, \\ \frac{u^1}{H_1} \frac{\partial u^3}{\partial \xi_1} &+ \frac{u^2}{H_2} \frac{\partial u^3}{\partial \xi_2} \\ &= \frac{u^1}{H_1} \left(\frac{1}{H_3} \frac{\partial F}{\partial \xi_1} - \frac{F}{H_3^2} \frac{\partial H_3}{\partial \xi_1} \right) + \frac{u^2}{H_2} \left(\frac{1}{H_3} \frac{\partial F}{\partial \xi_2} - \frac{F}{H_3^2} \frac{\partial H_3}{\partial \xi_2} \right) \\ &= \frac{1}{H_3} \left(\frac{u^1}{H_1} \frac{\partial F}{\partial \xi_1} + \frac{u^2}{H_2} \frac{\partial F}{\partial \xi_2} \right) - \frac{F}{H_3^2} \left(\frac{u^1}{H_1} \frac{\partial H_3}{\partial \xi_1} + \frac{u^2}{H_2} \frac{\partial H_3}{\partial \xi_2} \right), \\ L_3 u^3 &= \frac{1}{H_3} \left\{ \frac{1}{H_1 H_2 H_3} \left[\frac{\partial}{\partial \xi_1} \left(\frac{H_2 H_3}{H_1} \frac{\partial F}{\partial \xi_1} \right) + \frac{\partial}{\partial \xi_2} \left(\frac{H_1 H_3}{H_2} \frac{\partial F}{\partial \xi_2} \right) \right. \right. \\ &\quad \left. \left. + \left(\frac{H_2 H_3}{H_1} \frac{\partial^2 F}{\partial \xi_1^2} \right) + \left(\frac{H_1 H_3}{H_2} \frac{\partial^2 F}{\partial \xi_2^2} \right) \right] \right. \\ &\quad \left. - \frac{F}{H_3^2} \left[\left(\frac{1}{H_1} \frac{\partial H_3}{\partial \xi_1} \right)^2 + \left(\frac{1}{H_2} \frac{\partial H_3}{\partial \xi_2} \right)^2 \right] \right\} \\ &\quad + \frac{1}{H_1 H_2 H_3} \left[F \frac{\partial}{\partial \xi_1} \left(\frac{H_2 H_3}{H_1} \frac{\partial}{\partial \xi_1} \right) + \frac{2H_2 H_3}{H_1} \frac{\partial F}{\partial \xi_1} \frac{\partial}{\partial \xi_1} \right. \\ &\quad \left. + F \frac{\partial}{\partial \xi_2} \left(\frac{H_1 H_3}{H_2} \frac{\partial}{\partial \xi_2} \right) + \frac{2H_1 H_3}{H_2} \frac{\partial F}{\partial \xi_2} \frac{\partial}{\partial \xi_2} \right] \frac{1}{H_3} \\ &= \frac{1}{H_3} \Delta_{\xi_1, \xi_2} F - \frac{1}{H_3} \frac{1}{H_1 H_2 H_3} \\ &\quad \cdot \left[\frac{\partial}{\partial \xi_1} \left(\frac{H_2}{H_1 H_3} \frac{\partial H_3}{\partial \xi_1} \right) F + \frac{2H_2}{H_1 H_3} \frac{\partial F}{\partial \xi_1} \frac{\partial H_3}{\partial \xi_1} \right. \\ &\quad \left. + \frac{\partial}{\partial \xi_2} \left(\frac{H_1}{H_2 H_3} \frac{\partial H_3}{\partial \xi_2} \right) F + \frac{2H_1}{H_2 H_3} \frac{\partial F}{\partial \xi_2} \frac{\partial H_3}{\partial \xi_2} \right] \\ &\quad - \frac{F}{H_3^2} \left[\left(\frac{1}{H_1} \frac{\partial H_3}{\partial \xi_1} \right)^2 + \left(\frac{1}{H_2} \frac{\partial H_3}{\partial \xi_2} \right)^2 \right]. \end{aligned} \quad (40)$$

Multiplying the both sides of equation (39) by $|F|^{p-2} F$ and integrating over $\mathbb{R}^3$, we have

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \|F\|_{L^p(\mathbb{R}^3)}^p &+ \frac{4(p-1)}{p^2} \|\nabla^- |F|^p\|_{L^2(\mathbb{R}^3)}^2 \\ &+ \int_{\mathbb{R}^3} |F|^{p-1} \left(\frac{u^1}{H_1} \frac{\partial F}{\partial \xi_1} + \frac{u^2}{H_2} \frac{\partial F}{\partial \xi_2} \right) dx \\ &= - \int_{\mathbb{R}^3} \frac{1}{H_1 H_2} \left[\frac{\partial}{\partial \xi_1} \left(\frac{H_2}{H_1 H_3} \frac{\partial H_3}{\partial \xi_1} \right) F^p + \frac{2}{p} \frac{H_2}{H_1 H_3} \frac{\partial H_3}{\partial \xi_1} \frac{\partial F^p}{\partial \xi_1} \right. \\ &\quad \left. + \frac{\partial}{\partial \xi_2} \left(\frac{H_1}{H_2 H_3} \frac{\partial H_3}{\partial \xi_2} \right) F^p + \frac{2}{p} \frac{H_1}{H_2 H_3} \frac{\partial H_3}{\partial \xi_2} \frac{\partial F^p}{\partial \xi_2} \right] dx \\ &\quad - \int_{\mathbb{R}^3} \frac{F^p}{H_3^2} \left[\left(\frac{1}{H_1} \frac{\partial H_3}{\partial \xi_1} \right)^2 + \left(\frac{1}{H_2} \frac{\partial H_3}{\partial \xi_2} \right)^2 \right] dx \\ &= - \int_{\mathbb{R}^3} \frac{1}{H_1 H_2} \left[\frac{\partial}{\partial \xi_1} \left(\frac{H_2}{H_1 H_3} \frac{\partial H_3}{\partial \xi_1} \right) F^p + \frac{H_2}{H_1 H_3} \frac{\partial F^p}{\partial \xi_1} \frac{\partial H_3}{\partial \xi_1} \right. \\ &\quad \left. + \frac{\partial}{\partial \xi_2} \left(\frac{H_1}{H_2 H_3} \frac{\partial H_3}{\partial \xi_2} \right) F^p + \frac{H_1}{H_2 H_3} \frac{\partial F^p}{\partial \xi_2} \frac{\partial H_3}{\partial \xi_2} \right] dx \\ &\quad + \frac{F^p}{H_3^2} \left[\left(\frac{1}{H_1} \frac{\partial H_3}{\partial \xi_1} \right)^2 + \left(\frac{1}{H_2} \frac{\partial H_3}{\partial \xi_2} \right)^2 \right] dx \\ &\quad - \int_{\mathbb{R}^3} \left(\frac{2}{p} - 1 \right) \frac{1}{H_1 H_2} \left[\frac{H_2}{H_1 H_3} \frac{\partial H_3}{\partial \xi_1} \frac{\partial F^p}{\partial \xi_1} + \frac{H_1}{H_2 H_3} \frac{\partial H_3}{\partial \xi_2} \frac{\partial F^p}{\partial \xi_2} \right] dx = I_1 + I_2. \end{aligned} \quad (41)$$

Here, thanks to (15) and (25) and the incompressibility condition (23), by the fact of $d\mathbf{x} = H_1 H_2 H_3 d\xi_1 d\xi_2 d\xi_3$, we have

$$\begin{aligned}
& \int_{\mathbb{R}^3} |F|^{p-1} \left(\frac{u^1}{H_1} \frac{\partial F}{\partial \xi_1} + \frac{u^2}{H_2} \frac{\partial F}{\partial \xi_2} \right) d\mathbf{x} \\
&= \frac{1}{p} (\beta - \alpha) \int_D \left(\frac{u^1}{H_1} \frac{\partial}{\partial \xi_1} + \frac{u^2}{H_2} \frac{\partial}{\partial \xi_2} \right) |F|^p(t, \xi_1, \xi_2) H_1 H_2 H_3 d\xi_1 d\xi_2 \\
&= \frac{1}{p} (\beta - \alpha) \int_D \left[(H_2 H_3 u^1) \frac{\partial}{\partial \xi_1} + (H_1 H_3 u^2) \frac{\partial}{\partial \xi_2} \right] |F|^p(t, \xi_1, \xi_2) d\xi_1 d\xi_2 \\
&= -\frac{1}{p} (\beta - \alpha) \int_D \left[\frac{\partial}{\partial \xi_1} (H_2 H_3 u^1) + \frac{\partial}{\partial \xi_2} (H_1 H_3 u^2) \right] |F|^p(t, \xi_1, \xi_2) d\xi_1 d\xi_2 \equiv 0
\end{aligned} \tag{42}$$

For the I_1 and I_2 on the right of (41), by simple integration, one has

$$\begin{aligned}
I_1 &= - \int_{\mathbb{R}^3} \frac{1}{H_1 H_2} \left[F^p \frac{\partial}{\partial \xi_1} \left(\frac{H_2}{H_1 H_3} \frac{\partial H_3}{\partial \xi_1} \right) + \frac{H_2}{H_1 H_3} \frac{\partial F^p}{\partial \xi_1} \frac{\partial H_3}{\partial \xi_1} \right. \\
&\quad \left. + F^p \frac{\partial}{\partial \xi_2} \left(\frac{H_1}{H_2 H_3} \frac{\partial H_3}{\partial \xi_2} \right) + \frac{H_1}{H_2 H_3} \frac{\partial H_3}{\partial \xi_2} \frac{\partial F^p}{\partial \xi_2} \right] \\
&\quad + \frac{F^p}{H_3^2} \left[\left(\frac{1}{H_1} \frac{\partial H_3}{\partial \xi_1} \right)^2 + \left(\frac{1}{H_2} \frac{\partial H_3}{\partial \xi_2} \right)^2 \right] d\mathbf{x} \\
&= -(\beta - \alpha) \int_D H_3 \left[\frac{\partial}{\partial \xi_1} \left(\frac{H_2}{H_1 H_3} \frac{\partial H_3}{\partial \xi_1} \right) F^p + \frac{H_2}{H_1 H_3} \frac{\partial F^p}{\partial \xi_1} \frac{\partial H_3}{\partial \xi_1} \right. \\
&\quad \left. + \frac{\partial}{\partial \xi_2} \left(\frac{H_1}{H_2 H_3} \frac{\partial H_3}{\partial \xi_2} \right) F^p + \frac{H_1}{H_2 H_3} \frac{\partial H_3}{\partial \xi_2} \frac{\partial F^p}{\partial \xi_2} \right] d\xi_1 d\xi_2 \\
&\quad - (\beta - \alpha) \int_D \frac{H_1 H_2}{H_3} \left[\left(\frac{1}{H_1} \frac{\partial H_3}{\partial \xi_1} \right)^2 + \left(\frac{1}{H_2} \frac{\partial H_3}{\partial \xi_2} \right)^2 \right] F^p d\xi_1 d\xi_2 \\
&= -(\beta - \alpha) \int_D H_3 \left[\frac{\partial}{\partial \xi_1} \left(\frac{H_2}{H_1 H_3} \frac{\partial H_3}{\partial \xi_1} F^p \right) \right. \\
&\quad \left. + \frac{\partial}{\partial \xi_2} \left(\frac{H_1}{H_2 H_3} \frac{\partial H_3}{\partial \xi_2} F^p \right) \right] d\xi_1 d\xi_2 \\
&\quad - (\beta - \alpha) \int_D \frac{H_1 H_2}{H_3} \left[\left(\frac{1}{H_1} \frac{\partial H_3}{\partial \xi_1} \right)^2 + \left(\frac{1}{H_2} \frac{\partial H_3}{\partial \xi_2} \right)^2 \right] F^p d\xi_1 d\xi_2 \\
&= (\beta - \alpha) \int_D \left[\frac{H_2}{H_1 H_3} \left(\frac{\partial H_3}{\partial \xi_1} \right)^2 + \frac{H_1}{H_2 H_3} \left(\frac{\partial H_3}{\partial \xi_2} \right)^2 \right] F^p d\xi_1 d\xi_2 \\
&\quad - (\beta - \alpha) \int_D \left[\frac{H_2}{H_1 H_3} \left(\frac{\partial H_3}{\partial \xi_1} \right)^2 + \frac{H_1}{H_2 H_3} \left(\frac{\partial H_3}{\partial \xi_2} \right)^2 \right] F^p d\xi_1 d\xi_2 \equiv 0
\end{aligned} \tag{43}$$

$$\begin{aligned}
I_2 &= - \left(\frac{2}{p} - 1 \right) \int_{\mathbb{R}^3} \frac{1}{H_1 H_2} \left[\frac{H_2}{H_1 H_3} \frac{\partial H_3}{\partial \xi_1} \frac{\partial F^p}{\partial \xi_1} + \frac{H_1}{H_2 H_3} \frac{\partial H_3}{\partial \xi_2} \frac{\partial F^p}{\partial \xi_2} \right] d\mathbf{x} \\
&= - \left(\frac{2}{p} - 1 \right) (\beta - \alpha) \int_D \left[\frac{H_2}{H_1} \frac{\partial H_3}{\partial \xi_1} \frac{\partial F^p}{\partial \xi_1} + \frac{H_1}{H_2} \frac{\partial H_3}{\partial \xi_2} \frac{\partial F^p}{\partial \xi_2} \right] d\xi_1 d\xi_2 \\
&= \left(\frac{2}{p} - 1 \right) (\beta - \alpha) \int_D F^p \left[\frac{\partial}{\partial \xi_1} \left(\frac{H_2}{H_1} \frac{\partial H_3}{\partial \xi_1} \right) + \frac{\partial}{\partial \xi_2} \left(\frac{H_1}{H_2} \frac{\partial H_3}{\partial \xi_2} \right) \right] d\xi_1 d\xi_2 \\
&= \left(\frac{2}{p} - 1 \right) \int_{\mathbb{R}^3} \frac{F^p}{H_1 H_2 H_3} \left[\frac{\partial}{\partial \xi_1} \left(\frac{H_2}{H_1} \frac{\partial H_3}{\partial \xi_1} \right) + \frac{\partial}{\partial \xi_2} \left(\frac{H_1}{H_2} \frac{\partial H_3}{\partial \xi_2} \right) \right] d\mathbf{x} \\
&= \left(\frac{2}{p} - 1 \right) \int_{\mathbb{R}^3} \frac{F^p}{H_1 H_2 H_3} \left[\frac{\partial}{\partial \xi_1} \left(\frac{H_2 H_3}{H_1} \frac{1}{H_3} \frac{\partial H_3}{\partial \xi_1} \right) \right. \\
&\quad \left. + \frac{\partial}{\partial \xi_2} \left(\frac{H_1 H_3}{H_2} \frac{1}{H_3} \frac{\partial H_3}{\partial \xi_2} \right) \right] d\mathbf{x} \\
&= \left(\frac{2}{p} - 1 \right) \int_{\mathbb{R}^3} \frac{F^p}{H_1 H_2 H_3} \left[\frac{\partial}{\partial \xi_1} \left(\frac{H_2 H_3}{H_1} \frac{\partial(\ln H_3)}{\partial \xi_1} \right) \right. \\
&\quad \left. + \frac{\partial}{\partial \xi_2} \left(\frac{H_1 H_3}{H_2} \frac{\partial(\ln H_3)}{\partial \xi_2} \right) \right] d\mathbf{x} \\
&= \left(\frac{2}{p} - 1 \right) \int_{\mathbb{R}^3} F^p \Delta(\ln H_3) d\mathbf{x}.
\end{aligned} \tag{44}$$

Combining (42)–(44) with (41), we have

$$\begin{aligned}
& \frac{1}{p} \frac{d}{dt} \|F\|_{L^p(\mathbb{R}^3)}^p + \frac{4(p-1)}{p^2} \left\| \nabla_{\xi_1, \xi_2} |F|^{\frac{p}{2}} \right\|_{L^2(\mathbb{R}^3)}^2 \\
&= \left(\frac{2}{p} - 1 \right) \int_{\mathbb{R}^3} F^p \Delta(\ln H_3) d\mathbf{x}.
\end{aligned} \tag{45}$$

Note that $\|\nabla_{\xi_1, \xi_2} |F|^{\frac{p}{2}}\|_{L^2(\mathbb{R}^3)} = \|\nabla |F|^{\frac{p}{2}}\|_{L^2(\mathbb{R}^3)}$.

If $p = 2$, it has, immediately, by Gronwall's inequality,

$$\begin{aligned}
& \sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} |H_3 u^3|^p d\mathbf{x} + C \int_0^T \int_{\mathbb{R}^3} |\nabla |H_3 u^3|^{\frac{p}{2}}|^2 d\mathbf{x} dt \\
&\leq |H_3 u_0^3|_{L^p(\mathbb{R}^3)}^p.
\end{aligned} \tag{46}$$

If $p \in [1, 2) \cup (2, \infty]$, from (45) and (7), the right hand of (45) can be estimated by

$$\begin{aligned}
& \left(\frac{2}{p} - 1 \right) \int_{\mathbb{R}^3} F^p \Delta(\ln H_3) d\mathbf{x} \\
&\leq C \int_{\mathbb{R}^3} F^p |\Delta(\ln H_3)| d\mathbf{x} \leq C \|F\|_{L^p(\mathbb{R}^3)}.
\end{aligned} \tag{47}$$

and, then, by Gronwall's inequality again, one also has

$$\begin{aligned}
& \sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} |H_3 u^3|^p d\mathbf{x} + C \int_0^T \int_{\mathbb{R}^3} |\nabla |H_3 u^3|^{\frac{p}{2}}|^2 d\mathbf{x} dt \\
&\leq C \left(T, \|H_3 u_0^3\|_{L^p(\mathbb{R}^3)}^p \right).
\end{aligned} \tag{48}$$

The case $p = \infty$ is immediate if we let $p \rightarrow \infty$ in (48). Thus, we finish the proof of Theorem 1.

5. Proof of Theorem 2

In this section, we prove Theorem 2.

Proof of Theorem 2. From (1:1)₁, we have the basic energy inequality, for any $T > 0$,

$$\sup_{0 \leq t \leq T} \|u\|_{L^2(\mathbb{R}^3)}^2 + 2 \int_0^T \|\nabla u\|_{L^2(\mathbb{R}^3)}^2 \leq \|u_0\|_{L^2(\mathbb{R}^3)}^2. \tag{49}$$

It is known that the vorticity equation for the vorticity $\omega = \nabla \times \mathbf{u}$ for 3D incompressible Navier-Stokes equation is the following:

$$\partial_t \omega + (\mathbf{u} \cdot \nabla) \omega - \Delta \omega = (\omega \cdot \nabla) \mathbf{u}. \quad (50)$$

Using the expressions (4), (17), and (26), we have

$$\begin{aligned} (\omega \cdot \nabla) \mathbf{u} &= (\omega^1 \mathbf{e}_{\xi_1} + \omega^2 \mathbf{e}_{\xi_2} + \omega^3 \mathbf{e}_{\xi_3}) \\ &\quad \cdot \left(\mathbf{e}_{\xi_1} \frac{1}{H_1} \frac{\partial}{\partial \xi_1} + \mathbf{e}_{\xi_2} \frac{1}{H_2} \frac{\partial}{\partial \xi_2} + \mathbf{e}_{\xi_3} \frac{1}{H_3} \frac{\partial}{\partial \xi_3} \right) (u^1 \mathbf{e}_{\xi_1} + u^2 \mathbf{e}_{\xi_2} + u^3 \mathbf{e}_{\xi_3}) \\ &= \left(\omega^1 \frac{1}{H_1} \frac{\partial}{\partial \xi_1} + \omega^2 \frac{1}{H_2} \frac{\partial}{\partial \xi_2} + \omega^3 \frac{1}{H_3} \frac{\partial}{\partial \xi_3} \right) (u^1 \mathbf{e}_{\xi_1} + u^2 \mathbf{e}_{\xi_2} + u^3 \mathbf{e}_{\xi_3}) \\ &= \omega^1 \frac{1}{H_1} \left[\left(u^1 \frac{\partial \mathbf{e}_{\xi_1}}{\partial \xi_1} + \mathbf{e}_{\xi_1} \frac{\partial u^1}{\partial \xi_1} \right) + \left(u^2 \frac{\partial \mathbf{e}_{\xi_2}}{\partial \xi_1} + \mathbf{e}_{\xi_2} \frac{\partial u^2}{\partial \xi_1} \right) + \left(u^3 \frac{\partial \mathbf{e}_{\xi_3}}{\partial \xi_1} + \mathbf{e}_{\xi_3} \frac{\partial u^3}{\partial \xi_1} \right) \right] \\ &\quad + \omega^2 \frac{1}{H_2} \left[\left(u^1 \frac{\partial \mathbf{e}_{\xi_1}}{\partial \xi_2} + \mathbf{e}_{\xi_1} \frac{\partial u^1}{\partial \xi_2} \right) + \left(u^2 \frac{\partial \mathbf{e}_{\xi_2}}{\partial \xi_2} + \mathbf{e}_{\xi_2} \frac{\partial u^2}{\partial \xi_2} \right) + \left(u^3 \frac{\partial \mathbf{e}_{\xi_3}}{\partial \xi_2} + \mathbf{e}_{\xi_3} \frac{\partial u^3}{\partial \xi_2} \right) \right] \\ &\quad + \omega^3 \frac{1}{H_3} \left[\left(u^1 \frac{\partial \mathbf{e}_{\xi_1}}{\partial \xi_3} + \mathbf{e}_{\xi_1} \frac{\partial u^1}{\partial \xi_3} \right) + \left(u^2 \frac{\partial \mathbf{e}_{\xi_2}}{\partial \xi_3} + \mathbf{e}_{\xi_2} \frac{\partial u^2}{\partial \xi_3} \right) + \left(u^3 \frac{\partial \mathbf{e}_{\xi_3}}{\partial \xi_3} + \mathbf{e}_{\xi_3} \frac{\partial u^3}{\partial \xi_3} \right) \right] \\ &= \left(\frac{\omega^1}{H_1} \frac{\partial u^1}{\partial \xi_1} + \frac{\omega^2}{H_2} \frac{\partial u^1}{\partial \xi_2} + \frac{\omega^1 u^2}{H_1 H_2} \frac{\partial H_2}{\partial \xi_1} - \frac{\omega^2 u^2}{H_1 H_2} \frac{\partial H_2}{\partial \xi_1} - \frac{\omega^3 u^3}{H_1 H_3} \frac{\partial H_3}{\partial \xi_1} \right) \mathbf{e}_{\xi_1} \\ &\quad + \left(\frac{\omega^1}{H_1} \frac{\partial u^2}{\partial \xi_1} + \frac{\omega^2}{H_2} \frac{\partial u^2}{\partial \xi_2} - \frac{\omega^1 u^1}{H_1 H_2} \frac{\partial H_1}{\partial \xi_2} - \frac{\omega^2 u^1}{H_1 H_2} \frac{\partial H_2}{\partial \xi_1} - \frac{\omega^3 u^3}{H_2 H_3} \frac{\partial H_3}{\partial \xi_2} \right) \mathbf{e}_{\xi_2} \\ &\quad + \left(\frac{\omega^1}{H_1} \frac{\partial u^3}{\partial \xi_1} + \frac{\omega^2}{H_2} \frac{\partial u^3}{\partial \xi_2} + \frac{\omega^3 u^1}{H_1 H_3} \frac{\partial H_3}{\partial \xi_1} + \frac{\omega^3 u^2}{H_2 H_3} \frac{\partial H_3}{\partial \xi_2} \right) \mathbf{e}_{\xi_3}. \end{aligned} \quad (51)$$

Then, by multiplying ω on the both sides of equation (50) and integrating over $\mathbb{R}^3$, we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\omega\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla \omega\|_{L^2(\mathbb{R}^3)}^2 &= \int_{\mathbb{R}^3} (\omega \cdot \nabla) \mathbf{u} \cdot \omega \, dx \\ &= \int_{\mathbb{R}^3} \left[\left(\frac{\omega^1}{H_1} \frac{\partial u^1}{\partial \xi_1} + \frac{\omega^2}{H_2} \frac{\partial u^1}{\partial \xi_2} + \frac{\omega^1 u^2}{H_1 H_2} \frac{\partial H_2}{\partial \xi_1} \right. \right. \\ &\quad \left. \left. - \frac{\omega^2 u^2}{H_1 H_2} \frac{\partial H_2}{\partial \xi_1} - \frac{\omega^3 u^3}{H_1 H_3} \frac{\partial H_3}{\partial \xi_1} \right) \omega^1 \right. \\ &\quad \left. + \left(\frac{\omega^1}{H_1} \frac{\partial u^2}{\partial \xi_1} + \frac{\omega^2}{H_2} \frac{\partial u^2}{\partial \xi_2} - \frac{\omega^1 u^1}{H_1 H_2} \frac{\partial H_1}{\partial \xi_2} \right. \right. \\ &\quad \left. \left. - \frac{\omega^2 u^1}{H_1 H_2} \frac{\partial H_2}{\partial \xi_1} - \frac{\omega^3 u^3}{H_2 H_3} \frac{\partial H_3}{\partial \xi_2} \right) \omega^2 \right. \\ &\quad \left. + \left(\frac{\omega^1}{H_1} \frac{\partial u^3}{\partial \xi_1} + \frac{\omega^2}{H_2} \frac{\partial u^3}{\partial \xi_2} + \frac{\omega^3 u^1}{H_1 H_3} \frac{\partial H_3}{\partial \xi_1} + \frac{\omega^3 u^2}{H_2 H_3} \frac{\partial H_3}{\partial \xi_2} \right) \omega^3 \right] dx \\ &= \int_{\mathbb{R}^3} \frac{\omega^1 \omega^1}{H_1} \left(\frac{\partial u^1}{\partial \xi_1} + \frac{u^2}{H_2} \frac{\partial H_1}{\partial \xi_2} \right) dx + \int_{\mathbb{R}^3} \frac{\omega^1 \omega^2}{H_1} \left(\frac{\partial u^2}{\partial \xi_1} - \frac{u^1}{H_2} \frac{\partial H_1}{\partial \xi_2} \right) dx \\ &\quad + \int_{\mathbb{R}^3} \frac{\omega^1 \omega^3}{H_1} \frac{\partial u^3}{\partial \xi_1} dx + \int_{\mathbb{R}^3} \frac{\omega^1 \omega^2}{H_2} \left(\frac{\partial u^1}{\partial \xi_2} - \frac{u^2}{H_1} \frac{\partial H_2}{\partial \xi_1} \right) dx \\ &\quad + \int_{\mathbb{R}^3} \frac{\omega^2 \omega^2}{H_2} \left(\frac{\partial u^2}{\partial \xi_2} + \frac{u^1}{H_1} \frac{\partial H_2}{\partial \xi_1} \right) dx + \int_{\mathbb{R}^3} \frac{\omega^2 \omega^3}{H_2} \frac{\partial u^3}{\partial \xi_2} dx \\ &\quad - \int_{\mathbb{R}^3} \frac{\omega^2 \omega^3 u^3}{H_2 H_3} \frac{\partial H_3}{\partial \xi_2} dx - \int_{\mathbb{R}^3} \frac{\omega^1 \omega^3 u^3}{H_1 H_3} \frac{\partial H_3}{\partial \xi_1} dx \\ &\quad + \int_{\mathbb{R}^3} \frac{\omega^3 \omega^3}{H_3} \left(\frac{u^1}{H_1} \frac{\partial H_3}{\partial \xi_1} + \frac{u^2}{H_2} \frac{\partial H_3}{\partial \xi_2} \right) dx = \sum_{i=1}^3 J_i. \end{aligned} \quad (52)$$

Just for the sake of clarity, we give the expressions as follows:

$$\begin{aligned} \nabla \mathbf{u} &= \left(\mathbf{e}_{\xi_1} \frac{1}{H_1} \frac{\partial}{\partial \xi_1} + \mathbf{e}_{\xi_2} \frac{1}{H_2} \frac{\partial}{\partial \xi_2} + \mathbf{e}_{\xi_3} \frac{1}{H_3} \frac{\partial}{\partial \xi_3} \right) (u^1 \mathbf{e}_{\xi_1} + u^2 \mathbf{e}_{\xi_2} + u^3 \mathbf{e}_{\xi_3}) \\ &= \frac{\mathbf{e}_{\xi_1} \otimes \mathbf{e}_{\xi_1}}{H_1} \left(\frac{\partial u^1}{\partial \xi_1} + \frac{u^2}{H_2} \frac{\partial H_1}{\partial \xi_2} \right) + \frac{\mathbf{e}_{\xi_1} \otimes \mathbf{e}_{\xi_2}}{H_1} \left(\frac{\partial u^2}{\partial \xi_1} - \frac{u^1}{H_2} \frac{\partial H_1}{\partial \xi_2} \right) \\ &\quad + \frac{\mathbf{e}_{\xi_1} \otimes \mathbf{e}_{\xi_3}}{H_1} \frac{\partial u^3}{\partial \xi_1} + \frac{\mathbf{e}_{\xi_2} \otimes \mathbf{e}_{\xi_1}}{H_2} \left(\frac{\partial u^1}{\partial \xi_2} - \frac{u^2}{H_1} \frac{\partial H_2}{\partial \xi_1} \right) \\ &\quad + \frac{\mathbf{e}_{\xi_2} \otimes \mathbf{e}_{\xi_2}}{H_2} \left(\frac{\partial u^2}{\partial \xi_2} + \frac{u^1}{H_1} \frac{\partial H_2}{\partial \xi_1} \right) + \frac{\mathbf{e}_{\xi_2} \otimes \mathbf{e}_{\xi_3}}{H_2} \frac{\partial u^3}{\partial \xi_2} \\ &\quad + \frac{\mathbf{e}_{\xi_3} \otimes \mathbf{e}_{\xi_1}}{H_3} \left(-\frac{u^3}{H_1} \frac{\partial H_3}{\partial \xi_1} \right) + \frac{\mathbf{e}_{\xi_3} \otimes \mathbf{e}_{\xi_2}}{H_3} \left(-\frac{u^3}{H_2} \frac{\partial H_3}{\partial \xi_2} \right) \\ &\quad + \frac{\mathbf{e}_{\xi_3} \otimes \mathbf{e}_{\xi_3}}{H_3} \left(\frac{u^1}{H_1} \frac{\partial H_3}{\partial \xi_1} + \frac{u^2}{H_2} \frac{\partial H_3}{\partial \xi_2} \right) \end{aligned} \quad (53)$$

$$\begin{aligned} \nabla \tilde{\mathbf{u}} &= \left(\mathbf{e}_{\xi_1} \frac{1}{H_1} \frac{\partial}{\partial \xi_1} + \mathbf{e}_{\xi_2} \frac{1}{H_2} \frac{\partial}{\partial \xi_2} + \mathbf{e}_{\xi_3} \frac{1}{H_3} \frac{\partial}{\partial \xi_3} \right) (u^1 \mathbf{e}_{\xi_1} + u^2 \mathbf{e}_{\xi_2}) \\ &= \frac{\mathbf{e}_{\xi_1} \otimes \mathbf{e}_{\xi_1}}{H_1} \left(\frac{\partial u^1}{\partial \xi_1} + \frac{u^2}{H_2} \frac{\partial H_1}{\partial \xi_2} \right) + \frac{\mathbf{e}_{\xi_1} \otimes \mathbf{e}_{\xi_2}}{H_1} \left(\frac{\partial u^2}{\partial \xi_1} - \frac{u^1}{H_2} \frac{\partial H_1}{\partial \xi_2} \right) \\ &\quad + \frac{\mathbf{e}_{\xi_2} \otimes \mathbf{e}_{\xi_1}}{H_2} \left(\frac{\partial u^1}{\partial \xi_2} - \frac{u^2}{H_1} \frac{\partial H_2}{\partial \xi_1} \right) + \frac{\mathbf{e}_{\xi_2} \otimes \mathbf{e}_{\xi_2}}{H_2} \left(\frac{\partial u^2}{\partial \xi_2} + \frac{u^1}{H_1} \frac{\partial H_2}{\partial \xi_1} \right). \end{aligned} \quad (54)$$

Then, we estimate each integral $J_i (i = 1, \dots, 9)$ in the right-hand side of (52) by using the relation (53) and the relation (54), respectively.

On the one hand, by applying Proposition 8, Hölder inequality, Sobolev's imbedding inequality, and Young inequality, it follows, with the help of the relation (54), that

$$\begin{aligned} J_1 &= \int_{\mathbb{R}^3} \frac{\omega^1 \omega^1}{H_1} \left(\frac{\partial u^1}{\partial \xi_1} + \frac{u^2}{H_2} \frac{\partial H_1}{\partial \xi_2} \right) dx \leq \int_{\mathbb{R}^3} |\omega| |\omega| |\nabla \tilde{\mathbf{u}}| dx \\ &\leq \|\omega\|_{L^{\frac{2q}{q-1}}(\mathbb{R}^3)}^2 \|\nabla \tilde{\mathbf{u}}\|_{L^q(\mathbb{R}^3)} \leq C \|\omega\|_{L^{\frac{2q}{q-1}}(\mathbb{R}^3)}^2 \|\omega^3\|_{L^q(\mathbb{R}^3)} \\ &\leq C \|\omega\|_{L^2(\mathbb{R}^3)}^{\frac{2q-3}{q}} \|\nabla \omega\|_{L^2(\mathbb{R}^3)}^{\frac{3}{q}} \|\omega^3\|_{L^q(\mathbb{R}^3)} \\ &\leq C \|\omega^3\|_{L^q(\mathbb{R}^3)}^{\frac{2q}{2q-3}} \|\omega\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{18} \|\nabla \omega\|_{L^2(\mathbb{R}^3)}^2. \end{aligned} \quad (55)$$

The term J_2 can be estimated similarly to J_1 as

$$\begin{aligned} J_2 &= \int_{\mathbb{R}^3} \frac{\omega^1 \omega^2}{H_1} \left(\frac{\partial u^2}{\partial \xi_1} - \frac{u^1}{H_2} \frac{\partial H_1}{\partial \xi_2} \right) dx \\ &\leq \int_{\mathbb{R}^3} |\omega| |\omega| |\nabla \tilde{\mathbf{u}}| dx \leq C \|\omega^3\|_{L^q(\mathbb{R}^3)}^{\frac{2q}{2q-3}} \|\omega\|_{L^2(\mathbb{R}^3)}^2 \\ &\quad + \frac{1}{18} \|\nabla \omega\|_{L^2(\mathbb{R}^3)}^2. \end{aligned} \quad (56)$$

Similarly,

$$\begin{aligned}
 J_4 &= \int_{\mathbb{R}^3} \frac{\omega^1 \omega^2}{H_2} \left(\frac{\partial u^1}{\partial \xi_2} - \frac{u^2}{H_1} \frac{\partial H_2}{\partial \xi_1} \right) d\mathbf{x} \leq \int_{\mathbb{R}^3} |\omega| |\omega| |\nabla \tilde{u}| d\mathbf{x} \\
 &\leq C \|\omega^3\|_{L^q(\mathbb{R}^3)}^{\frac{2q}{2q-3}} \|\omega\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{18} \|\nabla \omega\|_{L^2(\mathbb{R}^3)}^2 \\
 J_5 &= \int_{\mathbb{R}^3} \frac{\omega^2 \omega^2}{H_2} \left(\frac{\partial u^2}{\partial \xi_2} + \frac{u^1}{H_1} \frac{\partial H_2}{\partial \xi_1} \right) d\mathbf{x} \leq C \int_{\mathbb{R}^3} |\omega| |\omega| |\nabla \tilde{u}| d\mathbf{x} \\
 &\leq C \|\omega^3\|_{L^q(\mathbb{R}^3)}^{\frac{2q}{2q-3}} \|\omega\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{18} \|\nabla \omega\|_{L^2(\mathbb{R}^3)}^2.
 \end{aligned} \tag{57}$$

On the other hand, by using Lemma 7, Hölder inequality, Sobolev imbedding inequality, and Young inequality, it yields, with the help of the relation (53), that

$$\begin{aligned}
 J_3 &= \int_{\mathbb{R}^3} \frac{\omega^1 \omega^3}{H_1} \frac{\partial u^3}{\partial \xi_1} d\mathbf{x} \leq \int_{\mathbb{R}^3} |\omega| |\nabla u| |\omega^3| d\mathbf{x} \\
 &\leq \|\omega\|_{L^q(\mathbb{R}^3)}^{\frac{2q}{q-1}} \|\nabla u\|_{L^q(\mathbb{R}^3)}^{\frac{2q}{q-1}} \|\omega^3\|_{L^q(\mathbb{R}^3)} \\
 &\leq C \|\omega\|_{L^q(\mathbb{R}^3)}^{\frac{2q}{q-1}} \|\omega^3\|_{L^q(\mathbb{R}^3)} \\
 &\leq C \|\omega\|_{L^2(\mathbb{R}^3)}^{\frac{2q-3}{q}} \|\nabla \omega\|_{L^2(\mathbb{R}^3)}^{\frac{3}{q}} \|\omega^3\|_{L^q(\mathbb{R}^3)} \\
 &\leq C \|\omega^3\|_{L^q(\mathbb{R}^3)}^{\frac{2q}{2q-3}} \|\omega\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{18} \|\nabla \omega\|_{L^2(\mathbb{R}^3)}^2.
 \end{aligned} \tag{58}$$

The terms J_6, J_7, J_8 and J_9 can be estimated similarly to J_3 as

$$\begin{aligned}
 J_6 + J_7 + J_8 + J_9 &\leq C \int_{\mathbb{R}^3} |\omega| |\nabla u| |\omega^3| d\mathbf{x} \\
 &\leq C \|\omega^3\|_{L^q(\mathbb{R}^3)}^{\frac{2q}{2q-3}} \|\omega\|_{L^2(\mathbb{R}^3)}^2 + \frac{2}{9} \|\nabla \omega\|_{L^2(\mathbb{R}^3)}^2.
 \end{aligned} \tag{59}$$

Combining all above estimates about $J_i (i=1, \dots, 9)$, from (52), we obtain

$$\frac{d}{dt} \|\omega\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla \omega\|_{L^2(\mathbb{R}^3)}^2 \leq C \|\omega^3\|_{L^q(\mathbb{R}^3)}^{\frac{2q}{2q-3}} \|\omega\|_{L^2(\mathbb{R}^3)}^2. \tag{60}$$

Applying Gronwall's inequality, it yields, for any $0 \leq t \leq T$

$$\omega \in L^\infty(0, T; L^2(\mathbb{R}^3)). \tag{61}$$

Together with Lemma 7 and (49), it implies that, for any $0 \leq t \leq T$

$$u \in L^\infty(0, T; H^1(\mathbb{R}^3)). \tag{62}$$

By Sobolev's imbedding Theorem, we obtain that, for any $0 \leq t \leq T$

$$u \in L^\infty(0, T; L^6(\mathbb{R}^3)). \tag{63}$$

Hence, with the help of Lemma 6, we finish the proof of Theorem 2.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All authors contributed equally to writing of this paper. All authors read and approved the final manuscript.

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