Research Article

Functional Geominimal Surface Area and Its Related Affine Isoperimetric Inequality

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The first variation of the total mass of log-concave functions was studied by Colesanti and Fragalà, which includes the L_p mixed volume of convex bodies. Using Colesanti and Fragalà's first variation formula, we define the geominimal surface area for log-concave functions, and its related affine isoperimetric inequality is also established.

1. Introduction

As we have known, Minkowski addition (the vector addition of convex bodies) is the cornerstone in the classical Brunn-Minkowski theory. Combining with volume, it leads to the Brunn-Minkowski inequality that is one of the most important results in convex geometry. The first variation of volume with respect to Minkowski addition is named the first mixed volume, and its related inequality is the Minkowski inequality. For more history and developments of the Brunn-Minkowski inequality, one may refer to the excellent survey [1]. For instance, the Prékopa-Leindler inequality [2–8] is known as the functional version of the Brunn-Minkowski inequality. In recent years, finding the functional counterparts of existing geometric results, especially for log-concave functions, has been receiving intensive attentions (see, e.g., [9–34]).

In 2013, Colesanti and Fragalà [35] introduced the “Minkowski addition” and “scalar multiplication,” $\alpha \cdot f \oplus \beta \cdot g$ (where $\alpha, \beta > 0$), of log-concave functions f and g as

$$\alpha \cdot f \oplus \beta \cdot g(x) = \sup_{y \in \mathbb{R}^n} f\left(\frac{x-y}{\alpha}\right)^\alpha g\left(\frac{y}{\beta}\right)^\beta. \quad (1)$$

We remark that a function $f: \mathbb{R}^n \rightarrow [0, \infty)$ is log-concave if it has the form $f(x) = e^{-u(x)}$, where $u: \mathbb{R}^n \rightarrow \mathbb{R}$

$\cup \{+\infty\}$ is convex. The total mass of f is defined as

$$J(f) = \int_{\mathbb{R}^n} f(x) dx. \quad (2)$$

Similar to the case of convex bodies, Colesanti and Fragalà [35] considered the following variational

$$\delta J(f, g) = \lim_{t \rightarrow 0^+} \frac{J(f \oplus t \cdot g) - J(f)}{t}, \quad (3)$$

and it is called the first variation of J at f along g . The first variation, $\delta J(f, g)$, includes the L_p mixed volume when it restricted f and g to the subclass of log-concave functions (see [35], Proposition 3.12).

Colesanti and Fragalà's work inspired us a natural way to extend the L_p geominimal surface area for convex bodies to the class of log-concave functions. For convenience, we recall the definition of L_p geominimal surface area. For a convex body K containing the origin in its interior, its L_p geominimal surface area, $G_p(K)$, is defined as (the case $p = 1$, see Petty [36], and $p > 1$, see Lutwak [37])

$$\omega_n^{p/n} G_p(K) = \inf \{ n V_p(K, Q) V(Q^\circ)^{p/n} : Q \in \mathcal{K}_o^n \}, \quad (4)$$

where ω_n is the volume of the unit ball in n -dimensional Euclidean space $\mathbb{R}^n$, Q° is the polar body of Q defined by $Q^\circ = \{x \in \mathbb{R}^n : \langle x, y \rangle \leq 1, \forall y \in Q\}$, $\mathcal{K}_o^n$ denotes the class of convex bodies in $\mathbb{R}^n$ that contain the origin in their interiors, and $V_p(K, Q)$ is the L_p mixed volume (for detailed definition, see Section 2). The fundamental inequality for L_p geominimal surface area is the following affine isoperimetric inequality (see, e.g., [37], Theorem 3.12):

$$G_p(K)^n \leq n^n \omega_n^p V(K)^{n-p}, \quad (5)$$

with equality if and only if K is an ellipsoid.

The L_p geominimal surface area, $G_p(K)$, is an important notation in the L_p Brunn-Minkowski theory, which serves as a bridge connecting affine differential geometry, relative differential geometry, and Minkowski geometry. In the past three decades, the L_p geominimal surface area has developed rapidly (see [25, 38–42] for some of the pertinent results).

Since $\delta J(f, g)$ includes the L_p mixed volume, we extend the L_p geominimal surface area to the functional version as follows.

Definition 1. Let $f : \mathbb{R}^n \rightarrow [0, \infty)$ be an integral log-concave function and $p > 0$. The L_p geominimal surface area of f is defined as

$$c_n^{p/n} G_p(f) = \inf \{ \delta J(f, g) J(g^\circ)^{p/n} : g \text{ is a log-concave function} \}, \quad (6)$$

where $c_n = (2\pi)^{n/2}$, and $g^\circ(x) = \inf_{y \in \mathbb{R}^n} e^{-\langle x, y \rangle} / g(y)$ is the polar function of g .

In Lemma 5, we prove that the above definition includes the L_p geominimal surface area (4) when $p \geq 1$ and restricted f, g to the subclass of log-concave functions.

In order to study the functional geominimal surface area, we need the integral formula of $\delta J(\cdot, \cdot)$. Hence, we need some notations. We write $\langle x, y \rangle$ for the usual inner product of $x, y \in \mathbb{R}^n$, and $\|x\|$ denotes the Euclidean norm of $x \in \mathbb{R}^n$. We say that $g = e^{-v}$ is an admissible perturbation for $f = e^{-u}$ if there exists a constant $c > 0$ such that $u^* - cv^*$ is convex, where $u^*(y) = \sup_{x \in \mathbb{R}^n} \{\langle x, y \rangle - u(x)\}$ is the Legendre conjugate of u . Let $\mathcal{A}'$ denote the set of log-concave functions given by function f such that $u = -\log f$ belongs to

$$\mathcal{L}' = \left\{ u \in \mathcal{L} : \text{dom}(u) = \mathbb{R}^n, u \in \mathcal{C}_+^2(\mathbb{R}^n), \lim_{\|x\| \rightarrow +\infty} \frac{u(x)}{\|x\|} = +\infty \right\}. \quad (7)$$

Here, $\text{dom}(u) = \{x \in \mathbb{R}^n : u(x) < +\infty\}$ and

$$\mathcal{L} = \left\{ u : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\} \mid u \text{ is convex, } \text{dom}(u) \neq \emptyset, \lim_{\|x\| \rightarrow +\infty} u(x) = +\infty \right\}. \quad (8)$$

Colesanti and Fragalà ([35], Theorem 4.5) provided an integral formula for the first variation $\delta J(f, g)$ when $f, g \in \mathcal{A}'$ and g is an admissible perturbation for f . For our aims, we consider the following optimization problem:

$$\inf \left\{ \delta J(f, g) J(g^\circ)^{p/n} : f, g \in \mathcal{A}', p > 0 \text{ and } g \text{ is an admissible perturbation for } f \right\}. \quad (9)$$

If the extremum in (9) exists, then it is denoted by $c_n^{p/n} G_p^{(1)}(f)$.

In Section 3, we prove that for $p > 0$ and $f \in \mathcal{A}'$, if $J(f)$ is finite, then there exists a unique log-concave function $\bar{f} \in \mathcal{A}'$ such that

$$G_p^{(1)}(f) = \delta J(f, \bar{f}) \text{ and } J(\bar{f}^\circ) = c_n. \quad (10)$$

Similar to the geometric case, the unique log-concave function $\bar{f}$ is called p -Petty functions of f and denoted by $T_p f$.

Using p -Petty functions, we obtain the following analytic inequality with equality conditions involving $G_p^{(1)}(f)$.

Theorem 2. Suppose $f \in \mathcal{A}'$ and $p > 0$. If f has its barycenter at 0 (i.e., $\int_{\mathbb{R}^n} x f(x) dx = 0$), then

$$J(f)^{p/n} G_p^{(1)}(f) \leq c_n^{p/n} \left(n J(f) + \int_{\mathbb{R}^n} f \log f dx \right), \quad (11)$$

with equality if $T_p f(x) = f(x)$ and $f(x) = ce^{-\langle Ax \rangle^2/2}$ for $A \in SL(n)$ and $c > 0$.

2. Background

2.1. Functional Setting. Let $u : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ if for every $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$ it satisfies

$$u((1-\lambda)x + \lambda y) \leq (1-\lambda)u(x) + \lambda u(y), \quad (12)$$

we say u is a convex function; let

$$\text{dom}(u) = \{x \in \mathbb{R}^n : u(x) \in \mathbb{R}\}. \quad (13)$$

By the convexity of u , $\text{dom}(u)$ is a convex set. We say that u is proper if $\text{dom}(u) \neq \emptyset$. The Legendre conjugate of u is the convex function defined by

$$u^*(y) = \sup_{x \in \mathbb{R}^n} \{\langle x, y \rangle - u(x)\} \quad \forall y \in \mathbb{R}^n. \quad (14)$$

Clearly, $u(x) + u^*(y) \geq \langle x, y \rangle$ for all $x, y \in \mathbb{R}^n$; there is an equality if and only if $x \in \text{dom}(u)$ and y is in the subdifferential of u at x . Hence, it can be checked that

$$u^*(\nabla u(x)) + u(x) = \langle x, \nabla u(x) \rangle. \quad (15)$$

On the class of convex functions from $\mathbb{R}^n$ to $\mathbb{R} \cup \{+\infty\}$,

the infimal convolution is defined by

$$u \square v(x) = \inf_{y \in \mathbb{R}^n} \{u(x-y) + v(y)\} \quad \forall x \in \mathbb{R}^n, \quad (16)$$

and the right scalar multiplication by a nonnegative real number $\alpha > 0$,

$$(u\alpha)(x) = \alpha u\left(\frac{x}{\alpha}\right). \quad (17)$$

It was proved in [21] (Proposition 2.1) that if $u, v : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ are convex functions and $\alpha > 0$, then

$$\begin{aligned} (u \square v)^* &= u^* + v^*, \\ (u\alpha)^* &= \alpha u^*. \end{aligned} \quad (18)$$

The following result will be used later.

Theorem 3 ([43], Theorem 10.9). *Let C be a relatively open convex set, and let $f_1, f_2, \dots$ be a sequence of finite convex functions on C . Suppose that the real number $f_1(x), f_2(x), \dots$ is bounded for each $x \in C$. It is then possible to select a subsequence of $f_1, f_2, \dots$, which converges uniformly on closed bounded subsets of C to some finite convex function f .*

The functional Blaschke-Santaló inequality states that let f, g be nonnegative integrable functions on $\mathbb{R}^n$ satisfying

$$f(x)g(y) \leq e^{-\langle x, y \rangle}, \quad \forall x, y \in \mathbb{R}^n. \quad (19)$$

If f has its barycenter at 0, which means that $\int_{\mathbb{R}^n} xf(x) dx = 0$, then

$$\left(\int_{\mathbb{R}^n} f(x) dx \right) \left(\int_{\mathbb{R}^n} g(x) dx \right) \leq c_n^2, \quad (20)$$

with equality if and only if there exists a positive definite matrix A and $C > 0$ such that, a.e. in $\mathbb{R}^n$,

$$f(x) = Ce^{-\frac{\langle Ax, x \rangle}{2}}, \quad g(y) = C^{-1}e^{-\frac{\langle A^{-1}y, y \rangle}{2}}. \quad (21)$$

2.2. The First Variation of the Total Mass of Log-Concave Functions. In this paper, we set

$$\begin{aligned} \mathcal{L} &= \left\{ u : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\} \mid u \text{ proper, convex, } \lim_{\|x\| \rightarrow +\infty} u(x) = +\infty \right\}, \\ \mathcal{A} &= \{ f : \mathbb{R}^n \rightarrow \mathbb{R} \mid f = e^{-u}, u \in \mathcal{L} \}. \end{aligned} \quad (22)$$

The total mass functional of f is defined as

$$J(f) = \int_{\mathbb{R}^n} f(x) dx. \quad (23)$$

The Gaussian function

$$\gamma(x) = e^{-\frac{\|x\|^2}{2}} \quad (24)$$

plays within class $\mathcal{A}$ the role of the ball in the set of convex bodies, and $J(\gamma) = (2\pi)^{n/2} = c_n$. For every $A \in \text{GL}(n)$, we write

$$\gamma_A(x) = e^{-\frac{\|Ax\|^2}{2}}. \quad (25)$$

From the definition of polar function and Legendre conjugate of function, we note that if $f \in \mathcal{A}$, then

$$f^\circ = e^{-\varphi^*}. \quad (26)$$

The support function of log-concave function $f = e^{-\varphi}$ is (see [44])

$$h_f(x) = \varphi^*(x). \quad (27)$$

This is a proper generalization, in the sense that $h_{\chi_K} = h_K$. Let $f = e^{-u}$, $g = e^{-v}$, and let $\alpha, \beta > 0$, then

$$\alpha \cdot f \oplus \beta \cdot g = e^{-[(u\alpha) \square (v\beta)]}, \quad (28)$$

which in explicit form reads

$$(\alpha \cdot f \oplus \beta \cdot g)(x) = \sup_{y \in \mathbb{R}^n} f\left(\frac{x-y}{\alpha}\right)^\alpha g\left(\frac{y}{\beta}\right)^\beta. \quad (29)$$

The support function of $\alpha \cdot f \oplus \beta \cdot g$ satisfies

$$h_{\alpha \cdot f \oplus \beta \cdot g}(x) = \alpha h_f(x) + \beta h_g(x). \quad (30)$$

In particular,

$$h_{\alpha \cdot f}(x) = \alpha h_f(x). \quad (31)$$

Let $f, g \in \mathcal{A}$. The first variation of J at f along g is defined as

$$\delta J(f, g) = \lim_{t \rightarrow 0^+} \frac{J(f \oplus t \cdot g) - J(f)}{t}. \quad (32)$$

The existence of the above limit was proved by Colesanti and Fragalà [35], and $\delta J(f, g) \in [-k, +\infty]$ with $k = \max \{ \inf (-\log g), 0 \} J(f)$. In particular, for every $f \in \mathcal{A}$ with $J(f) > 0$, then

$$\delta J(f, f) = nJ(f) + \int_{\mathbb{R}^n} f \log f dx. \quad (33)$$

The functional version of Minkowski first inequality reads as follows (see, e.g., [35], Theorem 5.1): let $f, g \in \mathcal{A}$ and assume that $J(f) > 0$. Then,

$$\delta J(f, g) \geq J(f) \left[\log \frac{J(g)}{J(f)} + n \right] + \int_{\mathbb{R}^n} f \log f dx, \quad (34)$$

with equality if and only if there exists $x_0 \in \mathbb{R}^n$ such that $g(x) = f(x - x_0)$ for $\forall x \in \mathbb{R}^n$.

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in the Euclidean space $\mathbb{R}^n$. We write $\mathcal{K}_o^n$ for the set of convex bodies that contain the origin in their interiors. Let $V(K)$ denote the n -dimensional volume of convex body K . The volume of the standard unit ball in $\mathbb{R}^n$ is denoted by $\omega_n = \pi^{n/2}/\Gamma((n/2) + 1)$. A convex body $K \in \mathcal{K}^n$ is uniquely determined by its support function, which is defined as $h_K(x) = \max \{ \langle x, y \rangle : y \in K \}$, where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^n$. The polar body of K is defined by $K^\circ = \{ x \in \mathbb{R}^n : \langle x, y \rangle \leq 1, \forall y \in K \}$.

For real $p \geq 1$, $K, L \in \mathcal{K}^n$, and real $\varepsilon > 0$, the Minkowski-Firey L_p combination $K +_p \varepsilon \cdot L$ is a convex body whose support function is given by

$$h(K +_p \varepsilon \cdot L, \cdot)^p = h(K, \cdot)^p + \varepsilon h(L, \cdot)^p. \quad (35)$$

The L_p mixed volume $V_p(K, L)$ of convex bodies K and L is defined by

$$V_p(K, L) = \frac{p}{n} \lim_{\varepsilon \rightarrow 0^+} \frac{V(K +_p \varepsilon \cdot L) - V(K)}{\varepsilon}. \quad (36)$$

The existence of this limit is showed in [45].

The following result show that $\delta J(f, g)$ includes the L_p mixed volume for convex bodies.

Proposition 4 ([35], Proposition 3.12). *Let $q \in (1, +\infty)$, $p = q/(q-1)$ and $K, L \in \mathcal{K}_o^n$. Let $u = (h_{K^\circ}(x)^q)/q$, $v(x) = (h_L(x)^q)/q$, and $f = e^{-u}$, $g = e^{-v}$. Then, there exists a positive constant $c = c(n, q)$ such that*

$$J(f) = c(n, q) V(K), \quad (37)$$

with $c(n, q) = q^{n/q} \Gamma((n+q)/q)$, and

$$\frac{p}{n} \delta J(f, g) = c(n, q) V_p(K, L). \quad (38)$$

We set $\mathcal{A}'$ as the subclasses of $\mathcal{A}$ given by the function f such that $u = \log f$ belongs to

$$\mathcal{A}' = \left\{ u \in \mathcal{A} : \text{dom}(u) = \mathbb{R}^n, \quad u \in \mathcal{C}_+^2(\mathbb{R}^n), \quad \lim_{\|x\| \rightarrow +\infty} \frac{u(x)}{\|x\|} = +\infty \right\}. \quad (39)$$

For log-concave function $f = e^{-u}$, the Borel measure μ_f on $\mathbb{R}^n$ is defined by (see [35])

$$\mu_f = (\nabla u)_\#(f \mathcal{H}^n). \quad (40)$$

Here, $\mathcal{H}^n$ is the n -dimensional Hausdorff measure. We need the fact that the barycenter of μ_f is the origin; i.e.,

$$\int_{\mathbb{R}^n} x \mu_f(x) = 0. \quad (41)$$

We recall that the log-concave function $g = e^{-v}$ is an

admissible perturbation for log-concave function $f = e^{-u}$ if

$$\exists a > 0 : u^* - av^* \quad (42)$$

is convex.

Colesanti and Fragalà [35] provided an integral representation of the first variation $\delta J(\cdot, \cdot)$ (see, e.g., [35], Theorem 4.5): let $f = e^{-u}$ and $g = e^{-v} \in \mathcal{A}'$ and assume that g is an admissible perturbation for f . Then, $\delta J(f, g)$ is finite and is given by

$$\delta J(f, g) = \int_{\text{dom}(u)} h_g(x) d\mu_f(x). \quad (43)$$

3. Functional L_p Geominimal Surface Areas

Analogy to convex bodies, for $f \in \mathcal{A}$ and $p \in \mathbb{R}$, we consider the following optimization problem:

$$c_n^{p/n} G_\lambda(f) = \inf \{ \delta J(f, g) J(g^\circ)^{p/n} : g \in \mathcal{A} \}. \quad (44)$$

The following result shows that the above optimization problem includes Lutwak's L_p geominimal surface areas for convex bodies (4) when $p > 1$ (up to a constant which is dependent on n and p). This is one of the reasons why $G_p(f)$ is called the L_p geominimal surface area for log-concave function f .

Lemma 5. *Let $p > 1$, $q = p/(p-1)$, and $K \in \mathcal{K}_o^n$. If $f = e^{-h_{K^\circ}^q(x)/q}$, then*

$$G_p(f) = \alpha(n, p) G_p(K), \quad (45)$$

with $\alpha(n, p) = (1/p) c(n, q) c(n, p)^{p/n} \omega_n^{p/n} c_n^{-(p/n)}$ for $c(n, q) = q^{n/q} \Gamma((n+q)/q)$.

Proof. Let $K, L \in \mathcal{K}_o^n$, $u(x) = h_{K^\circ}(x)^q/q$, $v(x) = h_L(x)^q/q$, and $f = e^{-u}$, $g = e^{-v}$. It is not hard to see that

$$v^*(x) = \frac{h_L(x)^p}{p}. \quad (46)$$

Then, Proposition 4 tells us that

$$\begin{aligned} J(g^\circ) &= c(n, p) V(L^\circ), \\ \frac{p}{n} \delta J(f, g) &= c(n, q) V_p(K, L), \end{aligned} \quad (47)$$

with $c(n, q) = q^{n/q} \Gamma((n+q)/q)$.

From the definitions of L_p geominimal surface area of convex bodies (4) and log-concave functions (44), we have

$$\begin{aligned} c_n^{p/n} G_p(f) &= \inf \{ \delta J(f, g) J(g^\circ)^{p/n} : g \in \mathcal{A} \} \\ &= \inf \left\{ \frac{n}{p} c(n, q) c(n, p)^{p/n} V_p(K, L) V(L^\circ)^{p/n} : K, L \in \mathcal{K}_o^n \right\} \\ &= \frac{1}{p} c(n, q) c(n, p)^{p/n} \omega_n^{p/n} G_p(K). \end{aligned} \quad (48)$$

Since we need the integral representation of the first variation $\delta J(f, g)$ in dealing the problem (44), we focus on

$$c_n^{p/n} G_p^{(1)}(f) = \inf \{ \delta J(f, g) J(g^\circ)^{p/n} : g \in \mathcal{A}' \text{ and } g \text{ is an admissible perturbation for } f \} \quad (49)$$

for $f \in \mathcal{A}'$ and $p \in \mathbb{R}$. Trivially, $G_p(f) \leq G_p^{(1)}(f)$.

We need the next lemma.

Lemma 6. *Let $f, g \in \mathcal{A}'$ and assume that g is an admissible perturbation for f . If $A \in SL(n)$, then*

$$\delta J(f \circ A, g \circ A) = \delta J(f, g). \quad (50)$$

Proof. Let $f(x) = e^{-u(x)}$ and $g(x) = e^{-v(x)}$. We note that

$$\begin{aligned} (\nu \circ A)^*(x) &= \sup_{y \in \mathbb{R}^n} \{ \langle x, y \rangle - \nu(Ay) \} \\ &= \sup_{y \in \mathbb{R}^n} \{ \langle A^{-t}x, Ay \rangle - \nu(Ay) \} = \nu^*(A^{-t}x). \end{aligned} \quad (51)$$

Since $\nabla_x(u \circ A) = A^t \nabla_{Ax} u$, we have

$$\begin{aligned} \delta J(f \circ A, g \circ A) &= \int_{\mathbb{R}^n} (\nu \circ A)^*(\nabla(u \circ A)(x)) f \circ A(x) dx \\ &= \int_{\mathbb{R}^n} (\nu \circ A)^*(A^t \nabla_{Ax} u(Ax)) f(Ax) dx \\ &= \int_{\mathbb{R}^n} \nu^*(\nabla_{Ax} u(Ax)) f(Ax) dx \\ &= \int_{\mathbb{R}^n} \nu^*(\nabla u(z)) f(z) dz = \delta J(f, g). \end{aligned} \quad (52)$$

The following result shows that the functional geominimal surface area is affine invariant.

Lemma 7. *Suppose $f \in \mathcal{A}'$ and $p > 0$. If $A \in SL(n)$, then*

$$G_p^{(1)}(f \circ A) = G_p^{(1)}(f). \quad (53)$$

Proof. By (51) and the definition of polar function (26), we have

$$J((g \circ A)^\circ) = J(g^\circ \circ A^{-t}) = J(g^\circ), \quad (54)$$

for $A \in SL(n)$. Combing with Lemma 6, we have

$$\delta J(f \circ A, g) J(g^\circ)^{p/n} = \delta J(f, g \circ A^{-1}) J(((g \circ A^{-1})^\circ))^{p/n}. \quad (55)$$

Therefore, we obtain

$$G_p^{(1)}(f \circ A) = G_p^{(1)}(f), \quad (56)$$

for $A \in SL(n)$.

The following lemma was proved by Cordero-Erausquin and Klartag ([46], Lemma 16).

Lemma 8. *Let μ be a finite Borel measure in $\mathbb{R}^n$, and let K be the interior of $\text{conv}(\text{Supp}(\mu))$. If $x_0 \in K$ and the barycenter of μ lies at the origin, then there exists a constant $C_{\mu, x_0} > 0$ with the following property: for any nonnegative, μ -integrable, convex function $\varphi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$,*

$$\varphi(x_0) \leq C_{\mu, x_0} \int_{\mathbb{R}^n} \varphi d\mu. \quad (57)$$

The next proposition shows that the infimum in the definition of the p -geominimal surface area of log-concave function is a minimum.

Proposition 9. *Let $p > 0$ and $f \in \mathcal{A}'$. If $J(f)$ is finite, then there exists a unique log-concave function $\bar{f} \in \mathcal{A}$ such that*

$$G_p^{(1)}(f) = \delta J(f, \bar{f}) \text{ and } J(\bar{f}^\circ) = c_n. \quad (58)$$

Proof. From the definition of $G_p^{(1)}(f)$, there exists a sequence $g_i \in \mathcal{A}'$ such that $J(g_i^\circ) = c_n$, with $\delta J(f, \gamma) \geq \delta J(f, g_i)$ for all i , and

$$\delta J(f, g_i) \rightarrow G_p^{(1)}(f). \quad (59)$$

Let $g_i(x) = e^{-v_i(x)}$, then

$$\delta J(f, \gamma) \geq \delta J(f, g_i) = \int_{\mathbb{R}^n} v_i^*(x) d\mu_f(x). \quad (60)$$

First, we assume that v_i are nonnegative and $v_i(0) = 0$ for all i . In this case, from (14), we have

$$v_i^*(x) = \sup_{y \in \mathbb{R}^n} \{ \langle y, x \rangle - v_i(y) \} \geq \langle 0, x \rangle - v_i(0) = 0, \quad (61)$$

and

$$v_i^*(0) = \sup_{y \in \mathbb{R}^n} \{ \langle y, 0 \rangle - v_i(y) \} = - \inf_{y \in \mathbb{R}^n} v_i(y) = 0. \quad (62)$$

Let K be the interior of $\text{conv}(\text{Supp}(\mu_f))$. By Lemma 8 and ((59)), we conclude that v_i^* are uniformly upper bound which

is dependent only on f . According to Theorem 3, there exists a subsequence $\{v_{i_j}^*\}_{j=1,2,\dots}$ that converges pointwise in K to a convex function $v^* : K \rightarrow \mathbb{R}$. We extend the definition of v^* by setting $v^*(x) = +\infty$ for $x \in \bar{K}$ and for $x \in \partial K$,

$$v^*(x) = \lim_{\lambda \rightarrow 1^-} v^*(\lambda x). \quad (63)$$

This limit always exists in $[0, +\infty]$, since the function $\lambda \mapsto v^*(\lambda x)$ is nondecreasing for $\lambda \in (0, 1)$ following from the convexity of v^* and $v^*(0) = 0$. Moreover, we have that $v^*(\lambda x)Zv^*(x)$ as $\lambda \rightarrow 1^-$ for any $x \in \bar{K}$. Because $v_i^* \rightarrow v^*$ is equivalent to $v_i \rightarrow v$ (here, $v = (v^*)^*$), hence, there exists a log-concave function $\bar{f} = e^{-v}$ which satisfies the claim.

In the general case, there exist $x_0^{(i)} \in \mathbb{R}^n$ and $\inf_{x \in \mathbb{R}^n} v_i(x) = d_i \in \mathbb{R}$ such that $\bar{v}_i(x) = v_i(x - x_0^{(i)}) - d_i$ are nonnegative and $\bar{v}_i(0) = 0$ for all $i = 1, 2, \dots$. The convexity of v_i and $e^{-v_i} \in \mathcal{A}'$ ensures the finiteness of d_i ; i.e., $|d_i| < k$ for some $k > 0$. Similar to the first case, we have

$$\delta J(f, \gamma) \geq \delta J(f, \bar{g}_i) = \int_{\mathbb{R}^n} \bar{v}_i^*(x) d\mu_f(x), \quad (64)$$

where $\bar{g}_i = e^{-\bar{v}_i}$. Lemma 8 deduces that

$$\delta J(f, \gamma) \geq \delta J(f, \bar{g}_i) = \int_{\mathbb{R}^n} \bar{v}_i^*(x) d\mu_f(x) \geq \frac{1}{C_{\mu_f, x_0}} \bar{v}_i^*(x_0) \quad (65)$$

holds for $x_0 \in K$. Moreover,

$$\begin{aligned} \bar{v}_i^*(x) &= \sup_{y \in \mathbb{R}^n} \{ \langle y, x \rangle - \bar{v}_i(y) \} \\ &= \sup_{y \in \mathbb{R}^n} \left\{ \langle y, x \rangle - v_i(y - x_0^{(i)}) \right\} + d_i \\ &= \sup_{y \in \mathbb{R}^n} \left\{ \langle y + x_0^{(i)}, x \rangle - v_i(y) \right\} + d_i \\ &= v_i^*(x) + \langle x, x_0^{(i)} \rangle + d_i. \end{aligned} \quad (66)$$

Therefore,

$$C_{\mu_f, x_0} \delta J(f, \gamma) \geq \bar{v}_i^*(x_0) = v_i^*(x_0) + \langle x_0, x_0^{(i)} \rangle + d_i, \quad (67)$$

i.e.,

$$v_i^*(x_0) \leq C_{\mu_f, x_0} \delta J(f, \gamma) - \langle x_0, x_0^{(i)} \rangle - d_i, \quad (68)$$

for any $x_0 \in K$. Then, along the same line of the first case, we conclude that the claim of this proposition holds.

The uniqueness of the minimizing function is demonstrated as follows. Suppose $h_1, h_2 \in \mathcal{A}$, such that $J(h_1) = J(h_2) = c_n$, and

$$\begin{aligned} \delta J(f, h_1) J(h_1^\circ)^{p/n} &= \inf \left\{ \delta J(f, g) J(g^\circ)^{p/n} : g \in \mathcal{A}' \right\} \\ &= \delta J(f, h_2) J(h_2^\circ)^{p/n}, \end{aligned} \quad (69)$$

i.e.,

$$\delta J(f, h_1) = \delta J(f, h_2). \quad (70)$$

Let $h_1 = e^{-v_1}$ and $h_2 = e^{-v_2}$. Define $h \in \mathcal{A}'$, by

$$h = \frac{1}{2} \cdot h_1 \oplus \frac{1}{2} \cdot h_2 = e^{-v_1(1/2) \square v_2(1/2)}. \quad (71)$$

Then, from (18) and (70), we have

$$\begin{aligned} \delta J(f, h) &= \int_{\mathbb{R}^n} \left(v_1 \frac{1}{2} \square v_2 \frac{1}{2} \right)^*(x) d\mu_f(x) \\ &= \frac{1}{2} \int_{\mathbb{R}^n} v_1^*(x) d\mu_f(x) + \frac{1}{2} \int_{\mathbb{R}^n} v_2^*(x) d\mu_f(x) \\ &= \frac{1}{2} \delta J(f, h_1) + \frac{1}{2} \delta J(f, h_2) = \delta J(f, h_1) = \delta J(f, h_2), \end{aligned} \quad (72)$$

and by the basic inequality $\sqrt{ab} \leq (a+b)/2$ for $a, b > 0$ and (18), we have

$$\begin{aligned} J(h^\circ) &= \int_{\mathbb{R}^n} e^{-[(v_1/2) \square (v_2/2)]^*} dx = \int_{\mathbb{R}^n} e^{-[(1/2)v_1^*(x) + (1/2)v_2^*(x)]} \\ &\cdot dx \leq \frac{1}{2} J(h_1^\circ) + \frac{1}{2} J(h_2^\circ), \end{aligned} \quad (73)$$

with equality if and only if $h_1^\circ = h_2^\circ$. Therefore,

$$\delta J(f, h) J(h^\circ)^{p/n} \leq \delta J(f, h_1) J(h_1^\circ)^{p/n} \quad (74)$$

is the contradiction that would arise if it was the case that $h_1 \neq h_2$.

The unique function whose existence is guaranteed by Proposition 9 will be denoted by $T_p f$, and will be called the p -Petty body of log-concave function f (or the λ -Petty function). The polar function of $T_p f$ will be denoted by $T_p^\circ f$, rather than $(T_p f)^\circ$. For $f \in \mathcal{A}$ and $p > 0$, the log-concave function $T_p f$ is defined by

$$\begin{aligned} G_p^{(1)}(f) &= \delta J(f, T_p f), \\ J(T_p^\circ f) &= c_n. \end{aligned} \quad (75)$$

Lemma 10. If $p > 0$ and $f \in \mathcal{A}$, then for $A \in SL(n)$,

$$T_p(f \circ A) = T_p f \circ A. \quad (76)$$

Proof. From the definition of T_p and Lemma 7,

$$\delta J(f, T_p f) = G_p^{(1)}(f) = G_p^{(1)}(f \circ A) = \delta J(f \circ A, T_p(f \circ A)), \quad (77)$$

Lemma 6 deduces

$$\delta J(f, T_p f) = \delta J(f \circ A, T_p(f \circ A)) = \delta J(f, T_p(f \circ A) \circ A^{-1}). \quad (78)$$

The uniqueness of Proposition 9 ensures that $T_p f = T_p(f \circ A) \circ A^{-1}$.

By the Blaschke-Santaló inequality, we obtain the following affine isoperimetric inequality for the functional geominimal surface area.

Theorem 11. *Let $f \in \mathcal{A}'$ and $p > 0$. If f has its barycenter at 0, then*

$$J(f)^{p/n} G_p^{(1)}(f) \leq c_n^{p/n} \left(nJ(f) + \int_{\mathbb{R}^n} f \log f dx \right), \quad (79)$$

with equality if $T_p f(x) = f(x)$ and $f(x) = ce^{-(\|Ax\|^2/2)}$ for $A \in SL(n)$ and $c > 0$.

Proof. Taking $g = f$ in (49), together with (33), we have,

$$c_n^{p/n} G_p^{(1)}(f) \leq \delta J(f, f) J(f^c)^{p/n} = \left(nJ(f) + \int_{\mathbb{R}^n} f \log f dx \right) J(f^c)^{p/n}, \quad (80)$$

i.e.,

$$c_n \left(\frac{G_p^{(1)}(f)^n}{(n + (1/J(f)) \int_{\mathbb{R}^n} f \log f dx)^n J(f)^{n-p}} \right)^{1/p} \leq J(f) J(f^c). \quad (81)$$

By Blaschke-Santaló inequality (20) and the above inequality, we have

$$\left(\frac{G_p^{(1)}(f)^n}{(n + (1/J(f)) \int_{\mathbb{R}^n} f \log f dx)^n J(f)^{n-p}} \right)^{1/p} \leq c_n. \quad (82)$$

This is the desired inequality.

To obtain the equality condition, first assume that $T_p f = f$. Formula (77) tells us that

$$G_p^{(1)}(f) = \delta J(f, f) \text{ and } J(f^c) = c_n. \quad (83)$$

This shows that there is equality in (81). From the condition of Blaschke-Santaló inequality, we know that there exists a positive definite matrix A and $c > 0$ such

that, a.e. in $\mathbb{R}^n$,

$$f(x) = ce^{-\frac{\|Ax\|^2}{2}}. \quad (84)$$

Therefore, we obtain the equality condition, namely, $T_p f = f$ and $f(x) = ce^{-(\|Ax\|^2/2)}$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

All authors contributed equally to this work. All authors have read and agreed to the published version of this manuscript. The second author is the corresponding author.

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