Research Article

On the Discrete Orlicz Electrostatic q -Capacitary Minkowski Problem

Yibin Feng¹ and Yanping Zhou^{2,3}

¹School of Mathematics and Statistics, Hexi University, Zhangye 734000, China

²Department of Mathematics, China Three Gorges University, Yichang 443002, China

³Three Gorges Mathematical Research Center, CTGU, Yichang 443002, China

Correspondence should be addressed to Yanping Zhou; zhouyp@i.shu.edu.cn

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We establish the existence of solutions to the Orlicz electrostatic q -capacitary Minkowski problem for polytopes. This contains a new result of the discrete L_p electrostatic q -capacitary Minkowski problem for $p < 0$ and $1 < q < n$.

1. Introduction

The Orlicz Brunn-Minkowski theory was originated from the works of Ludwig [1], Ludwig and Reitzner [2], and Lutwak et al. [3, 4]. Hereafter, the new theory has quickly become an important branch of convex geometry (see, e.g., [5–10]). A special case of the theory is the L_p Brunn-Minkowski theory which is credited to Lutwak [11, 12] and attracted increasing interest in recent years (see, e.g., [13–20]).

It is well known that the L_1 Brunn-Minkowski theory is the classical Brunn-Minkowski theory. One of the cornerstones of the classical Brunn-Minkowski theory is the Minkowski problem. More than a century ago, Minkowski himself solved the Minkowski problem for discrete measures [21]. The complete solution for arbitrary measures was given by Aleksandrov [22] and Fenchel and Jessen [23]. The regularity was studied by, e.g., Lewy [24], Nirenberg [25], Pogorelov [26], Cheng and Yau [27], and Caffarelli et al. [28].

A generalization of the Minkowski problem is the L_p Minkowski problem in the L_p Brunn-Minkowski theory, which has been extensively studied (see, e.g., [29–49]). Naturally, the corresponding Minkowski problem in the Orlicz Brunn-Minkowski theory is called the Orlicz Minkowski problem which was first investigated by Haberl et al. [50] for even measures. Today, great progress has been made on

it (see, e.g., [51–60]). The present paper is aimed at dealing with the Orlicz capacitary Minkowski problem.

The electrostatic q -capacitary measure $\mu_q(\Omega, \cdot)$ (see [61]) of a bounded open convex set Ω in $\mathbb{R}^n$ is the measure on the unit sphere S^{n-1} defined for $\omega \subset S^{n-1}$ and $1 < q < n$ by

$$\mu_q(\Omega, \omega) = \int_{g^{-1}(\omega)} |\nabla U|^q d\mathcal{H}^{n-1}, \quad (1)$$

where $g^{-1} : S^{n-1} \rightarrow \partial\Omega$ (the boundary of Ω) denotes the inverse Gauss map, $\mathcal{H}^{n-1}$ the $(n-1)$ -dimensional Hausdorff measure, and U the q -equilibrium potential of Ω .

A convex body K is a compact convex set with nonempty interior in the n -dimensional Euclidean space $\mathbb{R}^n$. Let $\mathcal{K}^n$ denote the set of convex bodies in $\mathbb{R}^n$, and let $\mathcal{K}_o^n$ denote the set of convex bodies with the origin in their interiors. The support function (see [62, 63]) of $K \in \mathcal{K}^n$ is defined for $u \in S^{n-1}$ by

$$h_K(u) = h(K, u) = \max \{x \cdot u : x \in K\}, \quad (2)$$

where $x \cdot u$ denotes the standard inner product of x and u . Note that $h(cK, u) = ch(K, u)$ for $c > 0$.

Let $\varphi : (0, \infty) \rightarrow (0, \infty)$ be a given continuous function. For $1 < q < n$ and $K \in \mathcal{K}_o^n$, the Orlicz electrostatic q

-capacitary measure, $\mu_{\varphi,q}(K, \cdot)$, of K is defined by

$$d\mu_{\varphi,q}(K, \cdot) = \varphi(h_K) d\mu_q(K, \cdot). \quad (3)$$

When $\varphi(s) = s^{1-p}$ with $p \in \mathbb{R}$, the Orlicz electrostatic q -capacitary measure becomes the following L_p electrostatic q -capacitary measure introduced by Zou and Xiong [64]:

$$d\mu_{p,q}(K, \cdot) = h_K^{1-p} d\mu_q(K, \cdot). \quad (4)$$

The Minkowski problem characterizing the Orlicz electrostatic q -capacitary measure, proposed in [65], is the following.

1.1. The Orlicz Electrostatic q -Capacitary Minkowski Problem. Let $1 < q < n$. Given a continuous function $\varphi : (0, \infty) \rightarrow (0, \infty)$ and a finite Borel measure μ on S^{n-1} , what are the necessary and sufficient conditions so that $\mu = c\mu_{\varphi,q}(K, \cdot)$ for some convex body K and constant $c > 0$?

Let φ be a constant function. When $q = 2$, the Orlicz Minkowski-type problem is the classical electrostatic capacity Minkowski problem. In the paper [66], Jerison established the existence of a solution to the electrostatic capacity Minkowski problem. In a subsequent paper [67], he gave a new proof of the existence using a variational approach. The uniqueness was proved by Caffarelli et al. [68], and the regularity was given in [66]. When $1 < q < n$, the Orlicz Minkowski-type problem is the electrostatic q -capacitary Minkowski problem posed in [61]. The existence and regularity for $1 < q < 2$ and the uniqueness for $1 < q < n$ of its solutions were proved in [61], and the existence for $2 < q < n$ was very recently solved in [69].

Let $\varphi(s) = s^{1-p}$ with $p \in \mathbb{R}$. Then, the Orlicz Minkowski-type problem is the L_p electrostatic q -capacitary Minkowski problem introduced by Zou and Xiong [64]. In [64], they completely solved the L_p electrostatic q -capacitary Minkowski problem for the case $p > 1$ and $1 < q < n$. It is generally known that when $p < 1$, the L_p Minkowski problem becomes much harder. Actually, the L_p electrostatic q -capacitary Minkowski problem for the case $p < 1$ and $1 < q < n$ is also very difficult. Therefore, it is worth mentioning that an important breakthrough of the problem for the case $0 < p < 1$ and $1 < q < 2$ was made by Xiong et al. [70] for discrete measures.

The existence of the Orlicz electrostatic q -capacitary Minkowski problem was first investigated by Hong et al. [65]. As a consequence, in [65], they obtained a complete solution (including both existence and uniqueness) to the L_p electrostatic q -capacitary Minkowski problem for the case $p > 1$ and $1 < q < n$, which was independently solved by Zou and Xiong [64].

We observe the statement above. At present, there is no result about the L_p electrostatic q -capacitary Minkowski problem for the case $p < 0$ and $1 < q < n$. In this paper, we study the Orlicz electrostatic q -capacitary Minkowski problem including it.

A finite set E of S^{n-1} is said to be in general position if E is not contained in a closed hemisphere of S^{n-1} and any n elements of E are linearly independent.

A polytope in $\mathbb{R}^n$ is the convex hull of a finite set of points in $\mathbb{R}^n$ provided that it has positive n -dimensional volume. The convex hull of a subset of these points is called a facet of the polytope if it lies entirely on the boundary of the polytope and has positive $(n-1)$ -dimensional volume.

Our main theorem is stated as follows.

Theorem 1. Suppose $\varphi : (0, \infty) \rightarrow (0, \infty)$ is continuously differentiable and strictly increasing with $\varphi(s) \rightarrow \infty$ as $s \rightarrow \infty$ such that $\phi(t) = \int_t^\infty (1/\varphi(s)) ds$ exists for $t > 0$ and $\lim_{t \rightarrow 0^+} \phi(t) = \infty$. Let $\mu = \sum_{i=1}^N \alpha_i \delta_{u_i}$, where $\alpha_1, \dots, \alpha_N > 0$, the unit vectors $u_1, \dots, u_N \in S^{n-1}$ are in general position, and δ_{u_i} is the Dirac delta. Then, for $1 < q < n$, there exist a polytope P and constant $c > 0$ such that

$$\mu = c\mu_{\varphi,q}(P, \cdot). \quad (5)$$

When $\varphi(s) = s^{1-p}$ with $p < 0$, and $\phi(t) = -(1/p)t^p$, which satisfy the assumptions of Theorem 1, we obtain the following.

Corollary 2. Let $p < 0$ and $1 < q < n$. Suppose μ is a discrete measure on S^{n-1} , and its supports are in general position. If $p + q \neq n$, then there exists a polytope P_0 such that $\mu = \mu_{p,q}(P_0, \cdot)$; if $p + q = n$, then there exist a polytope P and constant $c > 0$ such that $\mu = c\mu_{p,q}(P, \cdot)$.

Obviously, this corollary makes up for the existing results for the L_p electrostatic q -capacitary Minkowski problem, to some extent.

The rest of this paper is organized as follows. In Section 2, some of the necessary facts about convex bodies and capacity are presented. In Section 3, a maximizing problem related to the Orlicz electrostatic q -capacitary Minkowski problem is considered and its corresponding solution is given. In Section 4, we give the proofs of Theorem 1 and Corollary 2.

2. Preliminaries

2.1. Basics regarding Convex Bodies. For quick later reference, we list some basic facts about convex bodies. Good general references are the books of Gardner [62] and Schneider [63].

The boundary and interior of $K \in \mathcal{K}^n$ will be denoted by ∂K and $\text{int } K$, respectively. $B = \{x \in \mathbb{R}^n : \sqrt{x \cdot x} \leq 1\}$ denotes the unit ball. The volume, the n -dimensional Lebesgue measure, of a convex body $K \in \mathcal{K}^n$ is denoted by $V(K)$, and the volume of B is denoted by ω_n . We will write $C(S^{n-1})$ for the set of continuous functions on S^{n-1} and $C^+(S^{n-1})$ for the set of positive functions in $C(S^{n-1})$.

For $x \in \partial K$ with $K \in \mathcal{K}^n$, $g_K(x)$ is the Gauss map of K which is the family of all unit exterior normal vectors at x . In particular, $g_K(x)$ consists of a unique vector for H^{n-1}

-almost all $x \in \partial K$. The surface area measure of K is a Borel measure on S^{n-1} defined for a Borel set $\omega \subset S^{n-1}$ by

$$S(K, \omega) = \int_{x \in g_K^{-1}(\omega)} dH^{n-1}(x). \quad (6)$$

For $f \in C^+(S^{n-1})$, the Aleksandrov body associated with f , denoted by $[f]$, is the convex body defined by

$$[f] = \bigcap_{u \in S^{n-1}} \{\xi \in \mathbb{R}^n : \xi \cdot u \leq f(u)\}. \quad (7)$$

It is easy to see that $h_{[f]} \leq f$ and $[h_K] = K$ for $K \in K^n$.

The Hausdorff distance of two convex bodies $K, L \in K^n$ is defined by

$$\delta(K, L) = \max_{u \in S^{n-1}} |h_K(u) - h_L(u)|. \quad (8)$$

For a sequence of convex bodies $K_i \in K^n$ and a convex body $K \in K^n$, we have $\lim_{i \rightarrow \infty} K_i = K$ provided that

$$\delta(K_i, K) \rightarrow 0, \quad (9)$$

as $i \rightarrow \infty$.

For $K \in K^n$ and $u \in S^{n-1}$, the support hyperplane $H(K, u)$ of K at u is defined by

$$H(K, u) = \{x \in \mathbb{R}^n : x \cdot u = h(K, u)\}, \quad (10)$$

the half-space $H^-(K, u)$ at u is defined by

$$H^-(K, u) = \{x \in \mathbb{R}^n : x \cdot u \leq h(K, u)\}, \quad (11)$$

and the support set $F(K, u)$ at u is defined by

$$F(K, u) = K \cap H(K, u). \quad (12)$$

Suppose that P is the set of polytopes in $\mathbb{R}^n$ and the unit vectors $u_1, \dots, u_N$ are in general position. Let $P(u_1, \dots, u_N)$ be the subset of P . If $P \in P$ with

$$P = \bigcap_{k=1}^N H^-(P, u_k), \quad (13)$$

then $P \in P(u_1, \dots, u_N)$. Obviously, if $P_i \in P(u_1, \dots, u_N)$ and P_i converges to a polytope P , then $P \in P(u_1, \dots, u_N)$. Let $P_N(u_1, \dots, u_N)$ be the subset of $P(u_1, \dots, u_N)$ that any polytope in $P_N(u_1, \dots, u_N)$ has exactly N facets.

2.2. Electrostatic q -Capacity and q -Capacitary Measure. Here, we collect some notion and basic facts on electrostatic q -capacity and q -capacitary measure (see [61, 64, 70]).

Let E be a compact set in n -dimensional Euclidean space $\mathbb{R}^n$. For $1 < q < n$, the electrostatic q -capacity, $C_q(E)$, of E is defined (see [61]) by

$$C_q(E) = \inf \left\{ \int_{\mathbb{R}^n} |\nabla u|^q dx : u \in C_c^\infty(\mathbb{R}^n) \text{ and } u \geq 1 \text{ on } E \right\}, \quad (14)$$

where $C_c^\infty(\mathbb{R}^n)$ is the set of smooth functions with compact supports. When $q = 2$, the electrostatic q -capacity becomes the classical electrostatic capacity $C_2(E)$.

For $K \in K^n$ and $1 < q < n$, we need the following isocapacitary inequality which is due to Mařya [71]:

$$V(K)^{(n-q)/n} \leq \left(\frac{q-1}{n-q} \right)^{q-1} (n\omega_n^{q/n})^{-1} C_q(K). \quad (15)$$

The following lemma (see [64, 70]) gives some basic properties of the electrostatic q -capacity.

Lemma 3. *Let E and F be two compact sets in $\mathbb{R}^n$ and $1 < q < n$.*

(i) *If $E \subset F$, then*

$$C_q(E) \leq C_q(F) \quad (16)$$

(ii) *For $\lambda > 0$,*

$$C_q(\lambda E) = \lambda^{n-q} C_q(E) \quad (17)$$

(iii) *For $x_0 \in \mathbb{R}^n$,*

$$C_q(E + x_0) = C_q(E) \quad (18)$$

(iv) *The functional $C_q(\cdot)$ is continuous on K^n with respect to the Hausdorff metric*

The following lemma is some basic properties of the electrostatic q -capacitary measure (compare [64, 70]).

Lemma 4. *Let $K \in K^n$ and $1 < q < n$.*

(i) *For $\lambda > 0$,*

$$\mu_q(\lambda K, \cdot) = \lambda^{n-q-1} \mu_q(K, \cdot) \quad (19)$$

(ii) *For $x_0 \in \mathbb{R}^n$,*

$$\mu_q(K + x_0, \cdot) = \mu_q(K, \cdot) \quad (20)$$

(iii) For $K_j, K \in \mathcal{K}^n$, if $K_j \rightarrow K$, then

$$\mu_q(K_j, \cdot) \rightarrow \mu_q(K, \cdot) \quad (21)$$

weakly as $j \rightarrow +\infty$

(iv) The measure $\mu_q(K, \cdot)$ is absolutely continuous with respect to the surface area measure $S(K, \cdot)$

The following variational formula given in [61] of electrostatic q -capacity is critical.

Lemma 5. Let $I \subset \mathbb{R}$ be an interval containing 0 in its interior, and let $h_t(u) = h(t, u): I \times S^{n-1} \rightarrow (0, \infty)$ be continuous such that the convergence in

$$h'(0, u) = \lim_{t \rightarrow 0} \frac{h(t, u) - h(0, u)}{t} \quad (22)$$

is uniform on S^{n-1} . Then,

$$\left. \frac{dC_q([h_t])}{dt} \right|_{t=0} = (q-1) \int_{S^{n-1}} h'(0, u) d\mu_q([h_0], u). \quad (23)$$

3. An Associated Maximization Problem

In this section, we solve a maximization problem, and its solution is exactly the solution in Theorem 1.

Suppose ϕ satisfies the assumptions of Theorem 1 and the unit vectors $u_1, \dots, u_N$ are in general position. For $\alpha_1, \dots, \alpha_N > 0$ and $P \in P(u_1, \dots, u_N)$, define the function, $\Phi_P : \text{int } P \rightarrow \mathbb{R}$, by

$$\Phi_P(\xi) = \sum_{k=1}^N \alpha_k \phi(h(P, u_k) - \xi \cdot u_k). \quad (24)$$

Let $1 < q < n$. We consider the following maximization problem:

$$\sup \{ \min_{\xi \in \text{int } Q} \Phi_Q(\xi) : C_q(Q) = 1, Q \in P(u_1, \dots, u_N) \}. \quad (25)$$

The solution to problem (25) is given in Theorem 9. Its proof requires the following three lemmas which are similar to those in [58].

Lemma 6. Suppose $\phi : (0, \infty) \rightarrow (0, \infty)$ is continuously differentiable and strictly increasing with $\phi(s) \rightarrow \infty$ as $s \rightarrow \infty$ such that $\phi(t) = \int_t^\infty (1/\phi(s)) ds$ exists for $t > 0$ and $\lim_{t \rightarrow 0} \phi(t) = \infty$.

For $\alpha_1, \dots, \alpha_N > 0$, if the unit vectors $u_1, \dots, u_N \in S^{n-1}$ are in general position, then there exists a unique $\xi_\phi(P) \in \text{int } P$ such that

$$\Phi_P(\xi_\phi(P)) = \min_{\xi \in \text{int } P} \Phi_P(\xi). \quad (26)$$

Proof. Since $\phi : (0, \infty) \rightarrow (0, \infty)$ is continuously differentiable and strictly increasing, we have for $t > 0$,

$$\phi''(t) = \frac{\phi'(t)}{\phi^2(t)} > 0. \quad (27)$$

Therefore, ϕ is strictly convex on $(0, \infty)$.

Let $0 < \lambda < 1$ and $\xi_1, \xi_2 \in \text{int } P$. Then,

$$\begin{aligned} \lambda \Phi_P(\xi_1) + (1-\lambda) \Phi_P(\xi_2) &= \lambda \sum_{k=1}^N \alpha_k \phi(h(P, u_k) - \xi_1 \cdot u_k) \\ &\quad + (1-\lambda) \sum_{k=1}^N \alpha_k \phi(h(P, u_k) - \xi_2 \cdot u_k) \\ &= \sum_{k=1}^N \alpha_k [\lambda \phi(h(P, u_k) - \xi_1 \cdot u_k) \\ &\quad + (1-\lambda) \phi(h(P, u_k) - \xi_2 \cdot u_k)] \geq \sum_{k=1}^N \alpha_k \phi(h(P, u_k) \\ &\quad - (\lambda \xi_1 + (1-\lambda) \xi_2) \cdot u_k) = \Phi_P(\lambda \xi_1 + (1-\lambda) \xi_2). \end{aligned} \quad (28)$$

Equality holds if and only if $\xi_1 \cdot u_k = \xi_2 \cdot u_k$ for all $k = 1, \dots, N$. Since $u_1, \dots, u_N$ are in general position, $\mathbb{R}^n = \text{lin}\{u_1, \dots, u_N\}$ which is the smallest linear subspace of $\mathbb{R}^n$ containing $\{u_1, \dots, u_N\}$. Thus, $\xi_1 = \xi_2$. Namely, Φ_P is strictly convex on $\text{int } P$.

Since $P \in P(u_1, \dots, u_N)$, it follows that for any $x \in \partial P$, there exists a $u_{i_0} \in \{u_1, \dots, u_N\}$ such that

$$h(P, u_{i_0}) = x \cdot u_{i_0}. \quad (29)$$

Note that ϕ is strictly decreasing on $(0, \infty)$ and $\lim_{t \rightarrow 0} \phi(t) = \infty$. Then, $\Phi_P(\xi) \rightarrow \infty$ whenever $\xi \in \text{int } P$ and $\xi \rightarrow x$. This together with the strict convexity of Φ_P means that there exists a unique interior point $\xi_\phi(P)$ of P such that

$$\Phi_P(\xi_\phi(P)) = \min_{\xi \in \text{int } P} \Phi_P(\xi). \quad (30)$$

Lemma 7. Suppose $\alpha_1, \dots, \alpha_N > 0$, the unit vectors $u_1, \dots, u_N \in S^{n-1}$ are in general position, and $\phi : (0, \infty) \rightarrow (0, \infty)$ is continuously differentiable and strictly increasing with $\phi(s) \rightarrow \infty$ as $s \rightarrow \infty$ such that $\phi(t) = \int_t^\infty (1/\phi(s)) ds$ exists for $t > 0$ and $\lim_{t \rightarrow 0} \phi(t) = \infty$. If $P_i \in P(u_1, \dots, u_N)$ converges to a polytope P , then $\lim_{i \rightarrow \infty} \xi_\phi(P_i) = \xi_\phi(P)$ and

$$\lim_{i \rightarrow \infty} \Phi_{P_i}(\xi_\phi(P_i)) = \Phi_P(\xi_\phi(P)). \quad (31)$$

Proof. Since $P_i \rightarrow P$ and $\xi_\phi(P_i) \in \text{int } P_i$, it follows that $\xi_\phi(P_i)$ is bounded. Let $\xi_\phi(P_{i_j})$ be a subsequence of $\xi_\phi(P_i)$ with $\lim_{j \rightarrow \infty} \xi_\phi(P_{i_j}) = \xi_0$. We first show that $\xi_0 \in \text{int } P$ by contradiction.

Assume $\xi_0 \in \partial P$. Then, $\lim_{j \rightarrow \infty} \Phi_{P_{i_j}}(\xi_\phi(P_{i_j})) = \infty$, which contradicts the fact that

$$\lim_{j \rightarrow \infty} \Phi_{P_{i_j}}(\xi_\phi(P_{i_j})) \leq \lim_{j \rightarrow \infty} \Phi_{P_{i_j}}(\xi_\phi(P)) = \Phi_P(\xi_\phi(P)) < \infty. \quad (32)$$

We next show that $\xi_0 = \xi_\phi(P)$. Let $\xi_0 \neq \xi_\phi(P)$. Then,

$$\lim_{j \rightarrow \infty} \Phi_{P_{i_j}}(\xi_\phi(P_{i_j})) = \Phi_P(\xi_0) > \Phi_P(\xi_\phi(P)) = \lim_{j \rightarrow \infty} \Phi_{P_{i_j}}(\xi_\phi(P)). \quad (33)$$

This contradicts the fact that

$$\lim_{j \rightarrow \infty} \Phi_{P_{i_j}}(\xi_\phi(P_{i_j})) \leq \lim_{j \rightarrow \infty} \Phi_{P_{i_j}}(\xi_\phi(P)). \quad (34)$$

This means that $\lim_{i \rightarrow \infty} \xi_\phi(P_i) = \xi_\phi(P)$ and

$$\lim_{i \rightarrow \infty} \Phi_{P_i}(\xi_\phi(P_i)) = \Phi_P(\xi_\phi(P)). \quad (35)$$

Lemma 8. Suppose $\alpha_1, \dots, \alpha_N > 0$, the unit vectors $u_1, \dots, u_N \in S^{n-1}$ are in general position, and $\varphi : (0, \infty) \rightarrow (0, \infty)$ is continuously differentiable and strictly increasing with $\varphi(s) \rightarrow \infty$ as $s \rightarrow \infty$ such that $\phi(t) = \int_t^\infty (1/\varphi(s))ds$ exists for $t > 0$ and $\lim_{t \rightarrow 0} \phi(t) = \infty$. Let $P \in P(u_1, \dots, u_N)$ and $\delta \geq 0$ be small enough such that for $k_0 \in \{1, \dots, N\}$,

$$P_\delta = P \cap \{x : x \cdot u_{k_0} \leq h(P, u_{k_0}) - \delta\} \in P(u_1, \dots, u_N). \quad (36)$$

If the continuous function $\lambda : [0, \infty) \rightarrow (0, \infty)$ is continuously differentiable on $(0, \infty)$ and $\lim_{\delta \rightarrow 0} \lambda'(\delta)$ exists, then $\xi(\delta) := \xi_\phi(\lambda(\delta)P_\delta)$ has a right derivative, denoted by $\xi'_+(0)$, at 0.

Proof. The proof is based on the ideas of Wu et al. [58]. Let $\delta \geq 0$ be small enough and

$$\begin{aligned} \Phi(\delta) &= \min_{\xi \in \text{int}(\lambda(\delta)P_\delta)} \sum_{k=1}^N \alpha_k \phi(\lambda(\delta)h(P_\delta, u_k) - \xi \cdot u_k) \\ &= \sum_{k=1}^N \alpha_k \phi(\lambda(\delta)h(P_\delta, u_k) - \xi(\delta) \cdot u_k). \end{aligned} \quad (37)$$

From this and the fact that $\xi(\delta)$ is an interior point of $\lambda(\delta)P_\delta$, it follows that for $i = 1, \dots, n$,

$$\sum_{k=1}^N \alpha_k \phi'(\lambda(\delta)h(P_\delta, u_k) - \xi(\delta) \cdot u_k) u_{k,i} = 0, \quad (38)$$

where $u_k = (u_{k,1}, \dots, u_{k,n})^T$.

Let

$$F_i(\delta, \xi_1, \dots, \xi_n) = \sum_{k=1}^N \alpha_k \phi'(\lambda(\delta)h(P_\delta, u_k) - \xi \cdot u_k) u_{k,i} \quad (39)$$

for $i = 1, \dots, n$, where $\xi = (\xi_1, \dots, \xi_n)$. Then,

$$\frac{\partial F_i}{\partial \xi_j} = - \sum_{k=1}^N \alpha_k \phi''(\lambda(\delta)h(P_\delta, u_k) - \xi \cdot u_k) u_{k,i} u_{k,j}, \quad (40)$$

$$\begin{aligned} \frac{\partial F_i}{\partial \delta} &= \sum_{k=1}^N \alpha_k \phi'(\lambda(\delta)h(P_\delta, u_k) - \xi \cdot u_k) u_{k,i} \lambda'(\delta) h(P_\delta, u_k) \\ &\quad - \alpha_{k_0} \phi''(\lambda(\delta)h(P, u_{k_0}) - \delta \lambda(\delta) - \xi \cdot u_{k_0}) u_{k_0,i} \lambda(\delta). \end{aligned} \quad (41)$$

Let $F = (F_1, \dots, F_n)$. Then,

$$\left(\frac{\partial F}{\partial \xi} \Big|_{(\delta, \xi_1(\delta), \dots, \xi_n(\delta))} \right)_{n \times n} = - \sum_{k=1}^N \alpha_k \phi''(\lambda(\delta)h(P_\delta, u_k) - \xi(\delta) \cdot u_k) u_k u_k^T, \quad (42)$$

where $u_k u_k^T$ is an $n \times n$ matrix.

Since $u_1, \dots, u_N$ are in general position, $\mathbb{R}^n = \text{lin}\{u_1, \dots, u_N\}$. Thus, for any $x \in \mathbb{R}^n$ with $x \neq 0$, there exists a $u_{i_0} \in \{u_1, \dots, u_N\}$ such that $u_{i_0} \cdot x \neq 0$. Note that $\phi'' > 0$. Then, we have

$$\begin{aligned} x^T &\left(- \sum_{k=1}^N \alpha_k \phi''(\lambda(\delta)h(P_\delta, u_k) - \xi(\delta) \cdot u_k) u_k u_k^T \right) x \\ &= - \sum_{k=1}^N \alpha_k \phi''(\lambda(\delta)h(P_\delta, u_k) - \xi(\delta) \cdot u_k) (x \cdot u_k)^2 \\ &\leq - \alpha_{i_0} \phi''(\lambda(\delta)h(P_\delta, u_{i_0}) - \xi(\delta) \cdot u_{i_0}) (x \cdot u_{i_0})^2 < 0. \end{aligned} \quad (43)$$

Therefore, $(\partial F / \partial \xi|_{\delta, \xi_1(\delta), \dots, \xi_n(\delta)})$ is negative definite. Thus,

$$\det \left(\frac{\partial F}{\partial \xi} \Big|_{\delta, \xi_1(\delta), \dots, \xi_n(\delta)} \right) \neq 0. \quad (44)$$

From this, the fact that for $i = 1, \dots, n$, $F_i(\delta, \xi_1(\delta), \dots, \xi_n(\delta)) = 0$ follows by (38), the fact that $\partial F_i / \partial \xi_j$ is continuous on ξ and δ for all $1 \leq i, j \leq n$, and the implicit function theorem, it follows that $\xi(\delta) = \xi_\phi(\lambda(\delta)P_\delta)$ is continuously differentiable on a neighbourhood of δ small enough. Thus, $\xi(\delta)$ is continuously differentiable for small enough $\delta > 0$, and

$$\begin{pmatrix} \frac{d\xi_1}{d\delta} \\ \frac{d\xi_2}{d\delta} \\ \vdots \\ \frac{d\xi_n}{d\delta} \end{pmatrix} = - \begin{pmatrix} \frac{\partial F_1}{\partial \xi_1} & \frac{\partial F_1}{\partial \xi_2} & \cdots & \frac{\partial F_1}{\partial \xi_n} \\ \frac{\partial F_2}{\partial \xi_1} & \frac{\partial F_2}{\partial \xi_2} & \cdots & \frac{\partial F_2}{\partial \xi_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial \xi_1} & \frac{\partial F_n}{\partial \xi_2} & \cdots & \frac{\partial F_n}{\partial \xi_n} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial F_1}{\partial \delta} \\ \frac{\partial F_2}{\partial \delta} \\ \vdots \\ \frac{\partial F_n}{\partial \delta} \end{pmatrix}. \quad (45)$$

This together with (40) and (41) implies $\lim_{\delta \rightarrow 0^+} \xi'(\delta)$ exists.

From the Lagrange mean value theorem, we obtain that for $\delta > 0$ and $i = 1, \dots, n$, there exists a $\delta_i(\delta)$ with $0 < \delta_i(\delta) < \delta$ such that

$$\frac{\xi_i(\delta) - \xi_i(0)}{\delta} = \xi'_i(\delta_i(\delta)). \quad (46)$$

Thus,

$$\left. \frac{d\xi_i(\delta)}{d\delta} \right|_{\delta=0^+} = \lim_{\delta \rightarrow 0^+} \frac{\xi_i(\delta) - \xi_i(0)}{\delta} = \lim_{\delta \rightarrow 0^+} \xi'_i(\delta_i(\delta)) \quad (47)$$

exists. Namely, $\xi'_+(0)$ exists.

We are ready to show the existence of a maximizer to problem (25).

Theorem 9. Suppose $\alpha_1, \dots, \alpha_N > 0$ and the unit vectors $u_1, \dots, u_N \in S^{n-1}$ are in general position. Let $\varphi : (0, \infty) \rightarrow (0, \infty)$ be continuously differentiable and strictly increasing with $\varphi(s) \rightarrow \infty$ as $s \rightarrow \infty$ such that $\phi(t) = \int_t^\infty (1/\varphi(s)) ds$ exists for $t > 0$ and $\lim_{t \rightarrow 0} \phi(t) = \infty$. Then, there exists a polytope $P \in P_N(u_1, \dots, u_N)$ such that $\xi_\phi(P) = o$, $C_q(P) = 1$, and

$$\Phi_P(o) = \sup \{ \min_{\xi \in \text{int } Q} \Phi_Q(\xi) : C_q(Q) = 1, Q \in P(u_1, \dots, u_N) \}. \quad (48)$$

Proof. For $x \in \mathbb{R}^n$ and $P \in P(u_1, \dots, u_N)$, we first show

$$\Phi_{P+x}(\xi_\phi(P+x)) = \Phi_P(\xi_\phi(P)). \quad (49)$$

From Lemma 6 and definition (24), we have

$$\begin{aligned} \Phi_{P+x}(\xi_\phi(P+x)) &= \min_{\xi \in \text{int } (P+x)} \Phi_{P+x}(\xi) \\ &= \min_{\xi \in \text{int } (P+x)} \sum_{k=1}^N \alpha_k \phi(h(P+x, u_k) - \xi \cdot u_k) \\ &= \min_{\xi-x \in \text{int } P} \sum_{k=1}^N \alpha_k \phi(h(P, u_k) - (\xi-x) \cdot u_k) \\ &= \min_{\xi' \in \text{int } P} \sum_{k=1}^N \alpha_k \phi(h(P, u_k) - \xi' \cdot u_k) \\ &= \Phi_P(\xi_\phi(P)). \end{aligned} \quad (50)$$

Therefore, by (49) and (iii) of Lemma 2.1, we can choose a sequence $P_i \in P(u_1, \dots, u_N)$ with $\xi_\phi(P_i) = o$ and $C_q(P_i) = 1$ such that

$$\lim_{i \rightarrow \infty} \Phi_{P_i}(o) = \sup \{ \min_{\xi \in \text{int } Q} \Phi_Q(\xi) : C_q(Q) = 1, Q \in P(u_1, \dots, u_N) \}. \quad (51)$$

We next prove that P_i is bounded. Assume that P_i is not bounded. Since the unit vectors $u_1, \dots, u_N$ are in general position, from the proof of ([45], Theorem 4.3), we see $V(P_i)$ is not bounded. However, from (15), and noting that $C_q(P_i) = 1$, we have

$$V(P_i) \leq \left(\frac{q-1}{n-q} \right)^{n(q-1)/(n-q)} (n\omega_n^{q/n})^{-n/(n-q)}, \quad (52)$$

which is a contradiction. Therefore, P_i is bounded.

By the Blaschke selection theorem, we can assume that a subsequence of P_i converges to a polytope $P \in P(u_1, \dots, u_N)$. Thus, from (iv) of Lemma 3 and Lemma 7, it follows that $C_q(P) = 1$, $\xi_\phi(P) = o$, and

$$\Phi_P(o) = \sup \{ \min_{\xi \in \text{int } Q} \Phi_Q(\xi) : C_q(Q) = 1, Q \in P(u_1, \dots, u_N) \}. \quad (53)$$

We now prove that $P \in P_N(u_1, \dots, u_N)$, i.e., $F(P, u_i)$ are facets for all $i = 1, \dots, N$. If not, there exists an $i_0 \in \{1, \dots, N\}$ such that $F(P, u_{i_0})$ is not a facet of P . Choose $\delta \geq 0$ small enough so that the polytope

$$P_\delta = P \cap \{x : x \cdot u_{i_0} \leq h(P, u_{i_0}) - \delta\} \in P(u_1, \dots, u_N). \quad (54)$$

Let $\lambda(\delta) = C_q(P_\delta)^{-(1/(n-q))}$. Then, $\lambda(\delta)P_\delta \in P(u_1, \dots, u_N)$, $C_q(\lambda(\delta)P_\delta) = 1$ follows by (ii) of Lemma 3, and $\lambda(\delta)$ is continuous in $[0, \infty)$. Since for any $\delta_i \rightarrow 0$, there is that $\lambda(\delta_i)P_{\delta_i} \rightarrow P$, it follows from Lemma 7 that $\xi_\phi(\lambda(\delta_i)P_{\delta_i}) \rightarrow \xi_\phi(P) = o$. This implies

$$\lim_{\delta \rightarrow 0} \xi_\phi(\lambda(\delta)P_\delta) = o. \quad (55)$$

Let

$$r_{i_0}(u) = \begin{cases} 1, & u = u_{i_0}, \\ 0, & u \neq u_{i_0} \end{cases} \quad (56)$$

for $u \in S^{n-1}$. Then, from Lemma 5, we have for small enough $\delta \geq 0$,

$$\begin{aligned} \frac{dC_q(P_\delta)}{d\delta} &= \frac{dC_q([h_{P_\delta}])}{d\delta} = \lim_{t \rightarrow 0} \frac{C_q([h_{P_\delta} + tr_{i_0}]) - C_q([h_{P_\delta}])}{t} \\ &= (q-1) \int_{S^{n-1}} r_{i_0}(u) d\mu_q(P_\delta, u) = (q-1) \mu_q(P_\delta, u_{i_0}). \end{aligned} \quad (57)$$

Thus, from (iii) and (iv) of Lemma 4, it follows that $C_q(P_\delta)$ is continuously differentiable for every $\delta > 0$, and

$$\lim_{\delta \rightarrow 0^+} \frac{dC_q(P_\delta)}{d\delta} = 0. \quad (58)$$

These imply that

$$\lambda'(\delta) = -\frac{1}{n-q} C_q(P_\delta)^{-(1/(n-q))-1} \frac{dC_q(P_\delta)}{d\delta} \quad (59)$$

is continuous for every $\delta > 0$, and

$$\lim_{\delta \rightarrow 0^+} \lambda'(\delta) = 0. \quad (60)$$

Therefore, $\lambda(\delta) = C_q(P_\delta)^{-(1/(n-q))}$ satisfies the conditions of Lemma 8. Noting that $\xi(\delta) = \xi_\phi(\lambda(\delta)P_\delta)$, we see $\xi'_+(0)$ exists.

Recall

$$\Phi(\delta) = \sum_{k=1}^N \alpha_k \phi(\lambda(\delta)h(P_\delta, u_k) - \xi(\delta) \cdot u_k). \quad (61)$$

From this and (38), we have

$$\sum_{k=1}^N \alpha_k \phi'(h(P, u_k)) u_k = 0. \quad (62)$$

Thus, it follows from (60), (61), and (62) that

$$\begin{aligned} \left. \frac{d\Phi(\delta)}{d\delta} \right|_{\delta=0^+} &= -\alpha_{i_0} \phi'(h(P, u_{i_0})) \\ &\quad - \sum_{k=1}^N \alpha_k \phi'(h(P, u_k)) (\xi'_+(0) \cdot u_k) \\ &= -\alpha_{i_0} \phi'(h(P, u_{i_0})) - \xi'_+(0) \cdot \sum_{k=1}^N \alpha_k \phi'(h(P, u_k)) u_k \\ &= -\alpha_{i_0} \phi'(h(P, u_{i_0})) > 0. \end{aligned} \quad (63)$$

This means

$$\lim_{\delta \rightarrow 0^+} \frac{\Phi(\delta) - \Phi(0)}{\delta} > 0. \quad (64)$$

Therefore, there exists a $\delta_0 > 0$ small enough such that

$$\Phi(\delta_0) > \Phi(0). \quad (65)$$

This together with (61) has

$$\Phi_{\lambda(\delta_0)P_{\delta_0}}(\xi_\phi(\lambda(\delta_0)P_{\delta_0})) > \Phi_P(\xi_\phi(P)) = \Phi_P(o). \quad (66)$$

Note that $\lambda(\delta_0) = C_q(P_{\delta_0})^{-(1/(n-q))}$. Let $P_0 = \lambda(\delta_0)P_{\delta_0} - \xi_\phi(\lambda(\delta_0)P_{\delta_0})$. Then, $P_0 \in P(u_1, \dots, u_N)$, $C_q(P_0) = 1$, $\xi_\phi(P_0) = o$, and

$$\Phi_{P_0}(o) > \Phi_P(o). \quad (67)$$

This contradicts (53). Thus, $P \in P_N(u_1, \dots, u_N)$.

4. Solving the Orlicz Electrostatic q -Capacitary Minkowski Problem

Proof of Theorem 1. By Theorem 9, there exists a polytope $P \in P_N(u_1, \dots, u_N)$ with $\xi_\phi(P) = o$ and $C_q(P) = 1$ such that

$$\Phi_P(o) = \sup \{ \min_{\xi \in \text{int } Q} \Phi_Q(\xi) : C_q(Q) = 1, Q \in P(u_1, \dots, u_N) \}. \quad (68)$$

For $\gamma_1, \dots, \gamma_N \in \mathbb{R}$, choose $|t|$ small enough so that the polytope P_t defined by

$$P_t = \bigcap_{i=1}^N \{x : x \cdot u_i \leq h(P, u_i) + t\gamma_i\} \quad (69)$$

has exactly N facets. Then, $h(P_t, u_i) = h(P, u_i) + t\gamma_i$ for $i = 1, \dots, N$. Let

$$\beta(t) = C_q(P_t)^{-(1/(n-q))}. \quad (70)$$

Then, $\beta(t)P_t \in P_N(u_1, \dots, u_N)$ and $C_q(\beta(t)P_t) = 1$. By Lemma 5 and (iv) of Lemma 4, we obtain

$$\beta'(0) = -\frac{1}{n-q} \frac{dC_q(P_t)}{dt} \Big|_{t=0} = -\frac{q-1}{n-q} \sum_{i=1}^N \gamma_i u_{iq}(P, u_i). \quad (71)$$

Define $\xi(t) = \xi_\phi(\beta(t)P_t)$ and

$$\begin{aligned} \Phi(t) &= \min_{\xi \in \text{int } (\beta(t)P_t)} \sum_{k=1}^N \alpha_k \phi(\beta(t)h(P_t, u_k) - \xi \cdot u_k) \\ &= \sum_{k=1}^N \alpha_k \phi(\beta(t)h(P_t, u_k) - \xi(t) \cdot u_k). \end{aligned} \quad (72)$$

Since $\xi(t)$ is an interior point of $\beta(t)P_t$, this has

$$\sum_{k=1}^N \alpha_k \phi'(\beta(t)h(P_t, u_k) - \xi(t) \cdot u_k) u_{k,i} = 0, \quad (73)$$

for $i = 1, \dots, n$, where $u_k = (u_{k,1}, \dots, u_{k,n})^T$. Note that $\xi(0)$ is the origin. Then, letting $t = 0$ in (73), we have

$$\sum_{k=1}^N \alpha_k \phi'(h(P, u_k)) u_{k,i} = 0, \quad (74)$$

for $i = 1, \dots, n$. Hence,

$$\sum_{k=1}^N \alpha_k \phi'(h(P, u_k)) u_k = 0. \quad (75)$$

Let

$$F_i(t, \xi_1, \dots, \xi_n) = \sum_{k=1}^N \alpha_k \phi'(\beta(t)h(P, u_k) - (\xi_1 u_{k,1} + \dots + \xi_n u_{k,n})) u_{k,i} \quad (76)$$

for $i = 1, \dots, n$. Then,

$$\begin{aligned} \frac{\partial F_i}{\partial t} \Big|_{(t, \xi_1, \dots, \xi_n)} &= \sum_{k=1}^N \alpha_k \phi''(\beta(t)h(P, u_k) - (\xi_1 u_{k,1} + \dots + \xi_n u_{k,n})) \left[\beta'(t)h(P, u_k) + \beta(t)\gamma_k \right] u_{k,i}, \\ &\quad \frac{\partial F_i}{\partial \xi_j} \Big|_{(t, \xi_1, \dots, \xi_n)} \\ &= - \sum_{k=1}^N \alpha_k \phi''(\beta(t)h(P, u_k) - (\xi_1 u_{k,1} + \dots + \xi_n u_{k,n})) u_{k,i} u_{k,j}. \end{aligned} \quad (77)$$

Thus,

$$\left(\frac{\partial F}{\partial \xi} \Big|_{(0, \dots, 0)} \right)_{n \times n} = - \sum_{k=1}^N \alpha_k \phi''(h(P, u_k)) u_k u_k^T, \quad (78)$$

where $u_k u_k^T$ is an $n \times n$ matrix.

Since $u_1, \dots, u_N$ are in general position, $\mathbb{R}^n = \text{lin}\{u_1, \dots, u_N\}$. Thus, for any $x \in \mathbb{R}^n$ with $x \neq 0$, there exists a $u_{i_0} \in \{u_1, \dots, u_N\}$ such that $u_{i_0} \cdot x \neq 0$. Note that $\phi'' > 0$. Then, we have

$$\begin{aligned} x^T \left(- \sum_{k=1}^N \alpha_k \phi''(h(P, u_k)) u_k u_k^T \right) x \\ = - \sum_{k=1}^N \alpha_k \phi''(h(P, u_k)) (x \cdot u_k)^2 \\ \leq - \alpha_{i_0} \phi''(h(P, u_{i_0})) (x \cdot u_{i_0})^2 < 0. \end{aligned} \quad (79)$$

Hence, $(\partial F / \partial \xi)|_{(0, \dots, 0)}$ is negative definite. This implies $\det(\partial F / \partial \xi)|_{(0, \dots, 0)} \neq 0$. By this, the facts that for all $i = 1, \dots, n$, $F_i(0, \dots, 0) = 0$ follows by (74) and $\partial F_i / \partial \xi_j$ is continuous on t and ξ for all $1 \leq i, j \leq n$, and for the implicit function theorem, it follows that

$$\xi'(0) = (\xi'_1(0), \dots, \xi'_n(0)) \quad (80)$$

exists.

Since $\Phi(0)$ is a maximizer of $\Phi(t)$, from (71), (72), and (75), we get

$$\begin{aligned} 0 = \Phi'(0) &= \sum_{k=1}^N \alpha_k \phi'(h(P, u_k)) (\beta'(0)h(P, u_k) + \gamma_k - \xi'(0) \cdot u_k) \\ &= \sum_{k=1}^N \alpha_k \phi'(h(P, u_k)) \left[-\frac{q-1}{n-q} \left(\sum_{i=1}^N \gamma_i \mu_q(P, u_i) \right) h(P, u_k) + \gamma_k \right] \\ &\quad - \xi'(0) \cdot \sum_{k=1}^N \alpha_k \phi'(h(P, u_k)) u_k = \sum_{k=1}^N \alpha_k \phi'(h(P, u_k)) \gamma_k \\ &\quad - \frac{q-1}{n-q} \left(\sum_{i=1}^N \gamma_i \mu_q(P, u_i) \right) \left(\sum_{k=1}^N \alpha_k \phi'(h(P, u_k)) h(P, u_k) \right) \\ &= \sum_{k=1}^N \left[\alpha_k \phi'(h(P, u_k)) - \frac{q-1}{n-q} \left(\sum_{j=1}^N \alpha_j \phi'(h(P, u_j)) h(P, u_j) \right) \mu_q(P, u_k) \right] \gamma_k. \end{aligned} \quad (81)$$

Since $\gamma_1, \dots, \gamma_N$ are arbitrary,

$$\alpha_k = -\frac{q-1}{n-q} \left(\sum_{j=1}^N \alpha_j \phi'(h(P, u_j)) h(P, u_j) \right) \frac{1}{\phi'(h(P, u_k))} \mu_q(P, u_k), \quad (82)$$

for $k = 1, \dots, N$. Let

$$c = \frac{q-1}{n-q} \left(\sum_{j=1}^N \alpha_j \phi'(h(P, u_j)) h(P, u_j) \right). \quad (83)$$

Then, for $k = 1, \dots, N$,

$$\alpha_k = c \phi(h(P, u_k)) \mu_q(P, u_k), \quad (84)$$

i.e.,

$$\mu = c \mu_{\phi, q}(P, \cdot). \quad (85)$$

This completes the proof.

Proof of Corollary 2. Let $\varphi(s) = s^{1-p}$ with $p < 0$ in Theorem 1. Then, $\phi(t) = -(1/p)t^p$ for $t > 0$ and $\lim_{t \rightarrow 0^+} \phi(t) = \infty$. Therefore, we see φ and ϕ satisfy the conditions of Theorem 1. Thus, from the theorem, (3), and (4), we obtain

$$\mu = c h_P^{1-p}(\cdot) \mu_q(P, \cdot) = c \mu_{p, q}(P, \cdot). \quad (86)$$

If $p + q \neq n$, then from (i) of Lemma 4, we have

$$\mu = \mu_{p, q} \left(c^{1/(n-p-q)} P, \cdot \right). \quad (87)$$

Let $P_0 = c^{1/(n-p-q)} P$. Then, our desired result is given. If $p + q = n$, then (86) is just the desired result.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

All authors contributed equally to this work. All authors have read and approved the final manuscript.

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