Research Article

Optimal Controls for a Class of Impulsive Katugampola Fractional Differential Equations with Nonlocal Conditions

Xingru Chen,¹ Haibo Gu ,¹ and Yu Sun²

¹*School of Mathematics Sciences, Xinjiang Normal University, Urumqi, Xinjiang 830017, China*

²*School of Statistics and Data Sciences, Xinjiang University of Finance and Economics, Urumqi 830012, China*

Correspondence should be addressed to Haibo Gu; hbgu_math@163.com

Received 10 April 2020; Accepted 15 June 2020; Published 1 December 2020

Guest Editor: Abdon Atangana

Copyright © 2020 Xingru Chen et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In this paper, we investigate a class of impulsive Katugampola fractional differential equations with nonlocal conditions in a Banach space. First, by using a fixed-point theorem, we obtain the existence results for a class of impulsive Katugampola fractional differential equations. Secondly, we derive the sufficient conditions for optimal controls by building approximating minimizing sequences of functions twice.

1. Introduction

In recent years, fractional calculus has received more and more attention in many fields because of the local limit definition of integral-order ordinary differential equation or partial differential equation which is not suitable to describe the history-dependent process and has received more and more attention in many fields. Leibnitz discovered fractional derivatives in 1695. Because fractional-order differential equation can describe objective laws and the essence of things more accurately than integral differential equation, many scholars have devoted themselves to the study of fractional calculus. Now, Riemann-Liouville calculus definition, Caputo differential definition, Grunwald-Letnikov differential definition, etc. are the most commonly used fractional calculus definition in basic mathematics research and engineering application research (see [1]). Fractional differential equations are widely used in practice and greatly enriched the content of mathematical theory and penetrated into many fields of natural science.

In 2011, Katugampola presented a new fractional integral, which generalizes the Riemann-Liouville and Hadamard integral into a single form. When a parameter was fixed at different values, it produces the above integrals as special cases; when $\rho \rightarrow 1$, we can get the Riemann-Liouville operators; when $\rho \rightarrow 0$, we can get the Hadamard

operators (see [2]). And he presented two representations of the generalized derivative called Katugampola derivative in [3]. In the same work, he obtained boundedness in an extended Lebesgue measurable space and gave illustrative examples of generalized fractional integral. In [4], he obtained existence and uniqueness results to the solution of initial value problem for a class of generalized fractional differential equations. Since then, many scholars have done a lot of research on Katugampola derivative (see [5–15]).

Since the end of the last century, many scholars have devoted themselves to the study of approximate controllability (see [15–20]). Impulsive differential equation, which provides a natural description of observed evolution processes, is an important mathematical tool to solve some practical problems. The theory of impulsive differential equations of integer order has been widely used in practical mathematical modeling and has become an important area of research in recent years, which steadily receives attention of many authors (see [21–34]). Sun et al. [35] considered a class of impulsive fractional differential equations with Riemann-Liouville fractional derivative, the existence of solution was proved by using Darbo-Sadovskii's fixed-point theorem, and the optimal control results were obtained.

On the basis of previous work, in this paper, we initiate to study the existence of the following impulsive Katugampola fractional differential equation with nonlocal condition:

$$\begin{cases} {}^\rho D_{a^+}^\alpha x(t) = f(t, x(t)) + B(t)u(t), & t \in J' = (a, T], t \neq t_k, \\ {}^\rho I_{a^+}^{1-\alpha} x(\alpha) + g(x) = x_0, & u \in U_{\text{ad}}, \\ {}^\rho D_{a^+}^\beta x(t_k^+) - {}^\rho D_{a^+}^\beta x(t_k^-) = J_k(x(t_k)), & k = 1, \dots, m, \end{cases} \quad (1)$$

where $0 < \beta < \alpha < 1$, $\rho > 0$, and $a \in \mathbf{R}$ and ${}^\rho I_{(a^+)}^{(1-\alpha)}$ and ${}^\rho D_{(a^+)}^\alpha$ are the Katugampola fractional integral and derivative, which will be given in next section. $x \in X$, and X is a real Banach space; f is a nonlinear perturbation; $g(x)$ is a given X -valued function; $J_k(x(t_k))$ is a nonlinear mapping; $a < t_1 < t_2 < \dots < t_k \leq T$; and U_{ad} is a control set, the control $u \in U_{\text{ad}}$.

The rest of the paper is organised as follows. In Section 2, we give some preliminary facts that we need in the subsequent sections. In Section 3, we obtain the existence of solution of system (1) by using Darbo-Sadovskii's fixed-point theorem. In Section 4, we establish the optimal controls for a Lagrange problem. In Section 5, an illustrative example is given to show the practical usefulness of the analytical results.

2. Preliminaries

In this section, we present some results which will be useful throughout the paper.

Let X be a real Banach space; $\mathcal{L}(X)$ is a kind of linear operators in X and not necessarily bounded.

$C([a, T]; X)$ denotes the Banach space of all continuous functions $u : [a, T] \rightarrow X$ with the norm

$$\|u\| = \sup \{ \|u(t)\|, t \in [a, T] \}. \quad (2)$$

$L^1([a, T]; X)$ denotes the Banach space of all X -valued Bochner integrable functions defined on $[a, T]$ with the norm

$$\|u\|_1 = \int_a^T \|u(t)\| dt. \quad (3)$$

The Banach space PC_δ is defined by $\{x : (a, T] \rightarrow X : x \in C(t_k, t_{k+1})\}$, and with the norm

$$\|x\|_\delta = \sup_{t \in (a, T]} \left\| \left(\frac{t^\rho - a^\rho}{\rho} \right)^\delta x(t) \right\|, \quad (4)$$

where $k = 1, \dots, m$; $\alpha + \delta - 1 \geq 0$; $x(t_k^-)$ and $x(t_k^+)$ exist; $x(t_k^-) = x(t_k)$; and $\lim_{t \rightarrow a^+} (t^\rho - a^\rho/\rho)^\delta x(t)$ exists.

Consider the space $X_c^p(a, b)$ ($c \in \mathbf{R}$, $1 \leq p \leq \infty$) of those functions f on $[a, b]$ for which $\|f\|_{X_c^p} < \infty$, with the norm

$$\|f\|_{X_c^p} = \left(\int_a^b |t^c f(t)|^p \frac{dt}{t} \right)^{1/p} < \infty. \quad (5)$$

Definition 1 (see [2]). The Katugampola left-sided fractional integral of order $\alpha > 0$, $\rho > 0$ of $f \in X_c^p(a, b)$ for $-\infty < a < t < \infty$, is defined by

$$({}^\rho I_{a^+}^\alpha f)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} f(\tau) d\tau. \quad (6)$$

Definition 2 (see [3]). The Katugampola fractional derivative, associated with the generalized fractional integrals (6), is defined, for $0 \leq a < t < \infty$, $n = [\alpha] + 1$, by

$$\begin{aligned} ({}^\rho D_{a^+}^\alpha f)(t) &= \left(t^{1-\rho} \frac{d}{dt} \right)^n ({}^\rho I_{a^+}^{n-\alpha} f)(t) \\ &= \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \left(t^{1-\rho} \frac{d}{dt} \right)^n \int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{\alpha-n+1}} f(\tau) d\tau. \end{aligned} \quad (7)$$

Lemma 3 (see [3]). Let $0 \leq \alpha < 1$, then for $a > 0$ and $\rho > 0$,

$$({}^\rho D_{a^+}^\alpha {}^\rho I_{a^+}^\alpha x)(t) = x(t). \quad (8)$$

Lemma 4 (see [36]). Let $\alpha > 0$, $0 \leq \gamma < 1$. If $x \in X$ and ${}^\rho I_{a^+}^{1-\alpha} x \in C_\gamma^1[a, b]$, then

$$({}^\rho I_{a^+}^\alpha {}^\rho D_{a^+}^\alpha x)(t) = x(t) - \frac{({}^\rho I_{a^+}^{1-\alpha} x)(a)}{\Gamma(\alpha)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\alpha-1}. \quad (9)$$

Lemma 5. Let $0 \leq \beta < \alpha < 1$ and $x \in X$, then

$$\left({}^\rho D_{a^+}^\beta {}^\rho I_{a^+}^\alpha x \right)(t) = {}^\rho I_{a^+}^{\alpha-\beta} x(t). \quad (10)$$

Hausdorff measure of noncompactness $\sigma(\cdot)$ defined on each bounded subset Ω of the Banach space X , and $\sigma(\Omega) = \inf \{ \varepsilon > 0 \mid (\Omega \text{ has a finite } \varepsilon\text{-net in } X) \}$.

Lemma 6 (see [37]). The measure of noncompactness $\sigma(\cdot)$ satisfies the following:

- (i) $\sigma(B) = 0$ if and only if B is relatively compact in X
- (ii) If $\sigma(\{x\} \cup B) = \sigma(B)$ for every $x \in X$ and every nonempty subset $B \subseteq X$
- (iii) $\sigma(\lambda B) \leq |\lambda| \sigma(B)$ for any $\lambda \in \mathbf{R}$
- (iv) $\sigma(B_1 + B_2) \leq \sigma(B_1) + \sigma(B_2)$, where $B_1 + B_2 = \{x + y : x \in B_1, y \in B_2\}$
- (v) $\sigma(B_1 \cup B_2) \leq \max \{ \sigma(B_1), \sigma(B_2) \}$

For any $D \subset C([a, T], X)$, we define

$$\int_a^t D(s) ds = \left\{ \int_a^t u(s) ds : u \in D \right\} \quad \text{for } t \in [a, T], \quad (11)$$

where $D(s) = \{u(s) \in X : u \in D\}$.

Lemma 7 (see [37]). If $G \subset X$ is a convex bounded and closed set, the continuous mapping $\Lambda : G \rightarrow G$ is a σ -contraction and then has at least one fixed point in G .

The set N which has controls u is another separable reflexive Banach space. Let $P_f(N) \subset N$ be nonempty closed and convex. We assume that mapping $w : (a, T] \rightarrow P_f(N)$ is

multivalued and measurable, $w(\cdot) \subset O$, $O \subset N$ is bounded, the control set U_{ad} is defined as $U_{ad} = V_w^p = \{u \in L^p(O) \mid u(t) \in w(t), \text{ a.e.}\}$, $p > 1$, and $U_{ad} \neq \emptyset$.

3. Existence of Solution

In this section, we obtain the existence of solution of system (1) by using Darbo-Sadovskii's fixed-point theorem.

Lemma 8. A solution $x \in PC_\delta((a, T]; X)$ of problem (1) can be expressed as

$$\begin{aligned} x(t) = & \frac{x_0 - g(x)}{\Gamma(\alpha)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\alpha-1} + \frac{\Gamma(\alpha - \beta)}{\Gamma(\alpha)} \\ & \cdot \left(\sum_{a < t_k < t} \left(\frac{t_k^\rho - a^\rho}{\rho} \right)^{1+\beta-\alpha} J_k(x(t_k)) \right) \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\alpha-1} \\ & + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} (f(\tau, x(\tau)) + B(\tau)u(\tau)) d\tau. \end{aligned} \quad (12)$$

Proof. Suppose that $x \in PC_\delta((a, T]; X)$ is a solution of problem (1). By (9), we can get

$$\begin{aligned} x(t) = & {}^\rho I_{a^+}^\alpha {}^\rho D_{a^+}^\alpha x(t) + \frac{{}^\rho I_{a^+}^{1-\alpha} x(0)}{\Gamma(\alpha)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\alpha-1} \\ = & \frac{x_0 - g(x)}{\Gamma(\alpha)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\alpha-1} + {}^\rho I_{a^+}^\alpha (f(\cdot, x(\cdot)) \\ & + B(\cdot)u(\cdot))(t), t \in (a, t_1]. \end{aligned} \quad (13)$$

By using formula (10), we can get

$$\begin{aligned} {}^\rho D_{a^+}^\beta x(t) = & \frac{x_0 - g(x)}{\Gamma(\alpha - \beta)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\alpha-\beta-1} \\ & + {}^\rho I_{a^+}^{\alpha-\beta} (f(\cdot, x(\cdot)) + B(\cdot)u(\cdot))(t), t \in (a, t_1], \end{aligned} \quad (14)$$

where the second term in the right side is a continuous function on $(a, T]$. In particular,

$$\begin{aligned} {}^\rho D_{a^+}^\beta x(t_1^-) = & \frac{x_0 - g(x)}{\Gamma(\alpha - \beta)} \left(\frac{t_1^\rho - a^\rho}{\rho} \right)^{\alpha-\beta-1} \\ & + {}^\rho I_{a^+}^{\alpha-\beta} (f(\cdot, x(\cdot)) + B(\cdot)u(\cdot))(t_1). \end{aligned} \quad (15)$$

Obviously, the impulsive condition ${}^\rho D_{a^+}^\beta x(t_k^+) - {}^\rho D_{a^+}^\beta x(t_k^-) = J_k(x(t_k))$ is a discontinuity condition, which means we can get

$$\begin{aligned} x(t) = & C_1 \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\alpha-1} + \frac{x_0 - g(x)}{\Gamma(\alpha)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\alpha-1} \\ & + {}^\rho I_{a^+}^\alpha (f(\cdot, x(\cdot)) + B(\cdot)u(\cdot))(t), t \in (t_1, t_2]. \end{aligned} \quad (16)$$

Since this function satisfies problem (1) in view of (8), for $t \in (t_1, t_2]$, we obtain

$$\begin{aligned} {}^\rho D_{a^+}^\beta x(t) = & \frac{C_1 \Gamma(\alpha)}{\Gamma(\alpha - \beta)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\alpha-\beta-1} \\ & + \frac{x_0 - g(x)}{\Gamma(\alpha - \beta)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\alpha-\beta-1} \\ & + {}^\rho I_{a^+}^{\alpha-\beta} (f(\cdot, x(\cdot)) + B(\cdot)u(\cdot))(t), \end{aligned} \quad (17)$$

so that

$$\begin{aligned} {}^\rho D_{a^+}^\beta x(t_1^+) = & \frac{C_1 \Gamma(\alpha)}{\Gamma(\alpha - \beta)} \left(\frac{t_1^\rho - a^\rho}{\rho} \right)^{\alpha-\beta-1} \\ & + \frac{x_0 - g(x)}{\Gamma(\alpha - \beta)} \left(\frac{t_1^\rho - a^\rho}{\rho} \right)^{\alpha-\beta-1} \\ & + {}^\rho I_{a^+}^{\alpha-\beta} (f(\cdot, x(\cdot)) + B(\cdot)u(\cdot))(t_1). \end{aligned} \quad (18)$$

Combining (15) and (18) and applying problem (1) yield

$$\begin{aligned} C_1 = & \frac{\Gamma(\alpha - \beta)}{\Gamma(\alpha)} \left(\frac{t_1^\rho - a^\rho}{\rho} \right)^{1+\beta-\alpha} J_1(x(t_1)), \\ x(t) = & \frac{x_0 - g(x)}{\Gamma(\alpha)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\alpha-1} + \frac{\Gamma(\alpha - \beta)}{\Gamma(\alpha)} \\ & \cdot \left(\frac{t_1^\rho - a^\rho}{\rho} \right)^{1+\beta-\alpha} J_1(x(t_1)) \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\alpha-1} \\ & + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} (f(\tau, x(\tau)) \\ & + B(\tau)u(\tau)) d\tau, t \in (t_1, t_2]. \end{aligned} \quad (19)$$

Repeating the above process, the solution $x(t)$ for $(t_{k-1}, t_k]$ can be written as

$$\begin{aligned} x(t) = & \frac{x_0 - g(x)}{\Gamma(\alpha)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\alpha-1} + \frac{\Gamma(\alpha - \beta)}{\Gamma(\alpha)} \\ & \cdot \left(\sum_{a < t_k < t} \left(\frac{t_k^\rho - a^\rho}{\rho} \right)^{1+\beta-\alpha} J_k(x(t_k)) \right) \\ & \cdot \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\alpha-1} + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} \\ & \cdot (f(\tau, x(\tau)) + B(\tau)u(\tau)) d\tau. \end{aligned} \quad (20)$$

Conversely, by a simple calculation of (12), we can easily get (1). This completes the proof.

Let $B_r := \{x \in X : \|x\|_\delta \leq r\}$ and $W_r := \{y \in PC_\delta((a, T]; X) : y(t) \in B_r, \forall t \in (a, T]\}$, where r is a finite positive constant.

We introduce the following hypotheses:

(H1). $f \in C((a, T] \{t_1, t_2, \dots, t_m\} \times X)$ and $f(t_k \pm, x)$ exist, $\forall x \in X, k = 1, \dots, m$.

(H2). There exists a function φ such that

$$\|f(t, x)\| \leq \varphi_r(t), \quad (21)$$

where r is a finite positive constant and $r > 0$. For a.e., $t \in (a, T]$ and $x \in B_r$, ${}^\rho I_{a^+}^\alpha \varphi_r(t)$ exists and is bounded and continuous.

(H3). For each bounded set $Q \subset X$,

$$\sigma(f(t, Q)) \leq q\sigma(Q), \quad (22)$$

where q is a constant, $q > 0$, and σ is the Hausdorff measure of noncompactness.

(H4). $B : (a, T] \longrightarrow \mathcal{E}(N, X)$ is an essentially bounded and compact mapping, i.e., $B \in L^\infty((a, T], \mathcal{E}(N, X))$.

(H5). $g : PC_\delta((a, T]; X) \longrightarrow X$ is a compact and continuous mapping.

(H6). For $\forall x_1, x_2 \in PC_\delta((a, T]; X)$, $J_k : X \longrightarrow X$ satisfies Lipschitz condition:

$$\|J_k(x_1) - J_k(x_2)\| \leq h_k \|x_1 - x_2\|, k = 1, 2, \dots, m, \quad (23)$$

where h_k is the Lipschitz constant and $h_k > 0$.

Remark 9.

(i) From the definition of U_{ad} and hypothesis (H4), we can easily obtain that $Bu \in L^p((a, T]; X)$ with $p > 1$ for any $u \in U_{ad}$. Therefore, $Bu \in L^1((a, T]; X)$ and $\|Bu\|_{L^1} < +\infty$

(ii) Under hypothesis (H5), $\sup_{x \in W_r} \|g(x)\| < +\infty$

Theorem 10. Suppose that hypotheses (H1)–(H6) be satisfied and there is a constant $r > 0$ such that

$$\begin{aligned} & \left(\frac{T^\rho - a^\rho}{\rho} \right)^{\alpha+\delta-1} \left(\sup_{x \in w_r} \|g(x)\| + \Gamma(\alpha - \beta) \right. \\ & \cdot \left(\sum_{1 \leq k \leq m} \left(\frac{t_k^\rho - a^\rho}{\rho} \right)^{1+\beta-\alpha} (\|J_k(0)\| + h_k r) \right) \\ & + \left(\frac{T^\rho - a^\rho}{\rho} \right)^\delta \sup_{t \in (a, T]} \{ {}^\rho I_a^\alpha |\varphi_r(t)| \} \\ & + \left(\frac{T^\rho - a^\rho}{\rho} \right)^{\alpha+\delta} \frac{\|Bu\|_{L^1}}{\alpha} + \|x_0\| \leq r, \end{aligned} \quad (24)$$

then problem (1) has at least one solution.

Proof. Let $x_0 \in X$ be fixed. We consider the operator V on $PC_\delta((a, T]; X)$ such that

$$\begin{aligned} Vx(t) = & \left[(x_0 - g(x)) + \Gamma(\alpha - \beta) \right. \\ & \cdot \left(\sum_{a < t_k < t} \left(\frac{t_k^\rho - a^\rho}{\rho} \right)^{1+\beta-\alpha} J_k(x(t_k)) \right) \\ & \cdot \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\alpha-1} + \rho^{1-\alpha} \int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} \\ & \cdot (f(\tau, x(\tau)) + B(\tau)u(\tau)) d\tau, t \in (a, T]. \end{aligned} \quad (25)$$

The proof is given in following several steps.

Step 1. V maps W_r into W_r .

For any $x \in W_r$, by (H1)–(H6) and (12), we can obtain

$$\begin{aligned} & \left| \left(\frac{t^\rho - a^\rho}{\rho} \right)^\delta Vx(t) \right| \\ & \leq \left(\frac{t^\rho - a^\rho}{\rho} \right)^\delta \left[\|x_0 - g(x)\| + \Gamma(\alpha - \beta) \right. \\ & \quad \cdot \left(\sum_{a < t_k < t} \left\| \left(\frac{t_k^\rho - a^\rho}{\rho} \right)^{1+\beta-\alpha} J_k(x(t_k)) \right\| \right) \\ & \quad \times \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\alpha-1} + \left(\frac{t^\rho - a^\rho}{\rho} \right)^\delta \rho^{1-\alpha} \int_a^t \\ & \quad \cdot \left\| \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} (f(\tau, x(\tau)) + B(\tau)u(\tau)) \right\| d\tau \\ & \leq \left[\|x_0\| + \|g(x)\| + \Gamma(\alpha - \beta) \left(\sum_{1 \leq k \leq m} \left(\frac{t_k^\rho - a^\rho}{\rho} \right)^{1+\beta-\alpha} \right. \right. \\ & \quad \cdot (\|J_k(0)\| + h_k r) \left. \left. \right) \right] \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\alpha+\delta-1} + \left(\frac{t^\rho - a^\rho}{\rho} \right)^\delta \rho^{1-\alpha} \\ & \quad \cdot \int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} (\|f(\tau, x(\tau))\| + \|B(\tau)u(\tau)\|) d\tau \\ & \leq \left[\|x_0\| + \|g(x)\| + \Gamma(\alpha - \beta) \left(\sum_{1 \leq k \leq m} \left(\frac{t_k^\rho - a^\rho}{\rho} \right)^{1+\beta-\alpha} \right. \right. \\ & \quad \cdot (\|J_k(0)\| + h_k r) \left. \left. \right) \right] \left(\frac{T^\rho - a^\rho}{\rho} \right)^{\alpha+\delta-1} \\ & \quad + \left(\frac{t^\rho - a^\rho}{\rho} \right)^\delta \rho^{1-\alpha} \int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} |\varphi_r(\tau)| d\tau \\ & \quad + \left(\frac{t^\rho - a^\rho}{\rho} \right)^\delta \rho^{1-\alpha} \|Bu\|_{L^1} \int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} d\tau \\ & \leq \left[\|x_0\| + \|g(x)\| + \Gamma(\alpha - \beta) \left(\sum_{1 \leq k \leq m} \left(\frac{t_k^\rho - a^\rho}{\rho} \right)^{1+\beta-\alpha} \right. \right. \\ & \quad \cdot (\|J_k(0)\| + h_k r) \left. \left. \right) \right] \left(\frac{T^\rho - a^\rho}{\rho} \right)^{\alpha+\delta-1} \\ & \quad + \left(\frac{t^\rho - a^\rho}{\rho} \right)^\delta {}^\rho I_a^\alpha |\varphi_r| + \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\alpha+\delta} \frac{\|Bu\|_{L^1}}{\alpha} \\ & \leq \left(\frac{T^\rho - a^\rho}{\rho} \right)^{\alpha+\delta-1} \left(\sup_{x \in w_r} \|g(x)\| + \Gamma(\alpha - \beta) \right. \\ & \quad \cdot \left(\sum_{1 \leq k \leq m} \left(\frac{t_k^\rho - a^\rho}{\rho} \right)^{1+\beta-\alpha} (\|J_k(0)\| + h_k r) \right) \\ & \quad + \left(\frac{T^\rho - a^\rho}{\rho} \right)^\delta \sup_{t \in (a, T]} \{ {}^\rho I_a^\alpha |\varphi_r(t)| \} \\ & \quad + \left(\frac{T^\rho - a^\rho}{\rho} \right)^{\alpha+\delta} \frac{\|Bu\|_{L^1}}{\alpha} + \|x_0\| \leq r. \end{aligned} \quad (26)$$

Step 2. V is continuous in W_r .

Let $x_1, x_2 \in W_r$; we can get

$$\begin{aligned}
 & \left| \left(\frac{t^\rho - a^\rho}{\rho} \right)^\delta Vx_1(t) - \left(\frac{t^\rho - a^\rho}{\rho} \right)^\delta Vx_2(t) \right| \\
 & \leq \left(\frac{t^\rho - a^\rho}{\rho} \right)^\delta \left[\|g(x_1) - g(x_2)\| + \Gamma(\alpha - \beta) \right. \\
 & \quad \cdot \left(\sum_{a < t_k < t} \left(\frac{t_k^\rho - a^\rho}{\rho} \right)^{1+\beta-\alpha} \|J_k(x_1(t_k)) - J_k(x_2(t_k))\| \right) \Big] \\
 & \quad \times \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\alpha-1} + \left(\frac{t^\rho - a^\rho}{\rho} \right)^\delta \rho^{1-\alpha} \int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} \\
 & \quad \cdot \|f(\tau, x_1(\tau)) - f(\tau, x_2(\tau))\| d\tau \\
 & \leq \left[\|g(x_1) - g(x_2)\| + \Gamma(\alpha - \beta) \right. \\
 & \quad \cdot \left(\sum_{a < t_k < t} \left(\frac{t_k^\rho - a^\rho}{\rho} \right)^{1+\beta-\alpha} \|J_k(x_1(t_k)) - J_k(x_2(t_k))\| \right) \Big] \\
 & \quad \cdot \left(\frac{T^\rho - a^\rho}{\rho} \right)^{\alpha+\delta-1} + \left(\frac{T^\rho - a^\rho}{\rho} \right)^\delta \rho^{1-\alpha} \int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} \\
 & \quad \cdot \|f(\tau, x_1(\tau)) - f(\tau, x_2(\tau))\| d\tau.
 \end{aligned} \tag{27}$$

Because g , J_k , and f are continuous, we can obtain V is continuous.

Step 3. V has at least one fixed point.

Let

$$\begin{aligned}
 I_1 x(t) &= (x_0 - g(x)) \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\alpha-1}, \\
 I_2 x(t) &= \Gamma(\alpha - \beta) \left(\sum_{a < t_k < t} \left(\frac{t_k^\rho - a^\rho}{\rho} \right)^{1+\beta-\alpha} J_k(x(t_k)) \right) \\
 & \quad \cdot \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\alpha-1}, \\
 I_3 x(t) &= \rho^{1-\alpha} \int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} (f(\tau, x(\tau)) + B(\tau)u(\tau)) d\tau,
 \end{aligned} \tag{28}$$

for each $x \in W_r$. By condition (H5), we can obtain that the operator I_1 is compact in W_r , and $\sigma(I_1 W_r) = 0$. From (H1) and (H3), we can get that $f(t, x(t))$ is a compact mapping in W_r ; we omit the proof of this step as it is similar to that of Theorem 4.34 in [38]. Combining with (H4), we know the operator I_3 is compact in W_r , and $\sigma(I_3 W_r) = 0$.

Then, by condition (H6), we can know that $\{J_k\}_{k=1}^m$ are Lipschitz continuous. For $x_1, x_2 \in W_r$, we have

$$\begin{aligned}
 & \left| \left(\frac{t^\rho - a^\rho}{\rho} \right)^\delta I_2 x_1(t) - \left(\frac{t^\rho - a^\rho}{\rho} \right)^\delta I_2 x_2(t) \right| \\
 & \leq \left(\frac{t^\rho - a^\rho}{\rho} \right)^\delta \Gamma(\alpha - \beta) \left(\sum_{a < t_k < t} \left(\frac{t_k^\rho - a^\rho}{\rho} \right)^{1+\beta-\alpha} \right. \\
 & \quad \cdot \|J_k(x_1(t_k)) - J_k(x_2(t_k))\| \left. \right) \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\alpha-1} \\
 & \leq \Gamma(\alpha - \beta) \left(\sum_{a < t_k < t} \left(\frac{t_k^\rho - a^\rho}{\rho} \right)^{1+\beta-\alpha} \|J_k(x_1(t_k)) - J_k(x_2(t_k))\| \right) \left(\frac{T^\rho - a^\rho}{\rho} \right)^{\alpha+\delta-1} \\
 & \leq \left(\frac{T^\rho - a^\rho}{\rho} \right)^{\alpha+\delta-1} \Gamma(\alpha - \beta) \sum_{a < t_k < t} \left(\frac{t_k^\rho - a^\rho}{\rho} \right)^{1+\beta-\alpha} h_k \|x_1 - x_2\|.
 \end{aligned} \tag{29}$$

Therefore,

$$\begin{aligned}
 \sigma(I_2 W_r) &\leq \left(\frac{T^\rho - a^\rho}{\rho} \right)^{\alpha+\delta-1} \Gamma(\alpha - \beta) \sum_{a < t_k < t} \left(\frac{t_k^\rho - a^\rho}{\rho} \right)^{1+\beta-\alpha} h_k \sigma(W_r), \\
 \sigma(V W_r) &\leq \sigma(I_1 W_r) + \sigma(I_2 W_r) + \sigma(I_3 W_r) \\
 &\leq \left(\frac{T^\rho - a^\rho}{\rho} \right)^{\alpha+\delta-1} \Gamma(\alpha - \beta) \left(\sum_{a < t_k < t} \left(\frac{t_k^\rho - a^\rho}{\rho} \right)^{1+\beta-\alpha} h_k \right) \sigma(W_r).
 \end{aligned} \tag{30}$$

By (12), we can obtain

$$\left(\frac{T^\rho - a^\rho}{\rho} \right)^{\alpha+\delta-1} \Gamma(\alpha - \beta) \sum_{a < t_k < t} \left(\frac{t_k^\rho - a^\rho}{\rho} \right)^{1+\beta-\alpha} h_k < 1. \tag{31}$$

By Lemma 3, we can easily obtain that the operator V has a fixed point in $x \in W_r$; i.e., (1) has at least one solution. This completes the proof.

4. Optimal Controls for Problem

In this section, we establish the optimal controls for a Lagrange problem.

Let $x^u \in W_r$ denote the solution of problem (1) associated with the control $u \in U_{ad}$.

We research the following limited Lagrange problem (P):

$$J(x^u, u) = \int_a^T \psi(t, x^u(t), u(t)) dt. \tag{32}$$

For all $u \in U_{ad}$, there exist a $x^0 \in W_r \subseteq PC_\delta((a, T]; X)$ and $u^0 \in U_{ad}$ such that

$$J(x^0, u^0) \leq J(x^u, u), \tag{33}$$

where $x^0 \in W_r$ denotes the solution of problem (1) associated with the control $u^0 \in U_{ad}$.

We introduce the following hypotheses:

The function $\psi : (a, T] \times X \times N$ satisfies the following:

(Ψ1). The function $\psi : (a, T] \times X \times N \longrightarrow R \cup \infty$ is Borel measurable.

(Ψ2). $\psi \in X \times N$ is sequentially lower continuous for a.e. $t \in (a, T]$.

(Ψ3). For all $x \in X$, $\psi \in N$ is convex a.e. $t \in (a, T]$.

(Ψ4). There exist two constants $c \geq 0$, $d > 0$ such that

$$\psi(t, x^u(t), u(t)) \geq \phi(t) + c\|x\| + d\|u\|_N^p, \quad (34)$$

where $\phi \in L^1((a, T]; R)$.

Remark 11. If the pair $(x(\cdot), u(\cdot))$ satisfies system (1), $\forall x(\cdot) \in W_r$, then the pair $(x(\cdot), u(\cdot))$ is feasible. If the pair (x^u, u) is feasible, then $x^u \subset W_r$.

Theorem 12. Suppose that hypotheses (Ψ1)–(Ψ4) are satisfied, if the problem (P) satisfies the condition of Theorem 10., then it has at least one optimal feasible pair.

Proof. Step 1. For each $u \in U_{ad}$, let

$$J(u) = \inf_{x^u \in \text{Sol}(u)} J(x^u, u). \quad (35)$$

If $\text{Sol}(u)$ has infinite elements, we can easily prove in the case of

$$J(u) = \inf_{x^u \in \text{Sol}(u)} J(x^u, u) = +\infty. \quad (36)$$

If $\text{Sol}(u)$ has finite elements, $J(x^\wedge u, u) = \inf_{x^u \in \text{Sol}(u)} J(x^u, u) = J(u)$, if $x^\wedge u \in \text{Sol}(u)$.

We suppose that $J(u) = \inf_{x^u \in \text{Sol}(u)} J(x^u, u) < +\infty$. By conditions (Ψ1) – (Ψ1), we can get $J(u) > -\infty$.

By the definition of the infimum, we can obtain a sequence $\{x_n^u\}_{n=1}^\infty \subseteq \text{Sol}(u)$, which satisfies $J(x_n^u, u) \rightarrow J(u)$ for $n \rightarrow \infty$.

Since $\{(x_n^u, u)\}_{n=1}^\infty$ is a sequence of feasible pairs, we obtain

$$\begin{aligned} x_n^u(t) = & \left[(x_0 - g(x_n^u)) + \Gamma(\alpha - \beta) \left(\sum_{a < t_k < t} \left(\frac{t_k^\rho - a^\rho}{\rho} \right)^{1+\beta-\alpha} J_k(x_n^u(t_k)) \right) \right] \\ & \cdot \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\alpha-1} + \rho^{1-\alpha} \int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} (f(\tau, x_n^u(\tau)) \\ & + B(\tau)u(\tau)) d\tau, t \in (a, T]. \end{aligned} \quad (37)$$

Step 2. There exists $x^\wedge u \in \text{Sol}(u)$ such $J(x^\wedge u, u) =$

$$\inf_{x^u \in \text{Sol}(u)} J(x^u, u) = J(u).$$

Claim 1. For $\forall u \in U_{ad}$, $\{x_n^u\}_{n=1}^\infty$ is precompact in $PC_\delta((a, T]; X)$.

Let

$$x_n^u = I_1 x_n^u + I_2 x_n^u + I_3 x_n^u, \quad (38)$$

we obtain that $\{I_1 x_n^u\}_{n=1}^\infty \subset PC_\delta((a, T]; X)$ and $\{I_3 x_n^u\}_{n=1}^\infty \subset PC_\delta((a, T]; X)$ are precompact (we omit the proof of this step as it is similar to that of Theorem 4.34 in [36]). Furthermore, by assumption (H6), the operator I_2 is Lipschitz continuous in $PC_\delta((a, T]; X)$, where Lipschitz constant $(T+1)^{\alpha+\delta-1} \Gamma(\alpha - \beta) \sum_{a < t_k < t} t_k^{1+\beta-\alpha} h_k$. Hence, from the given condition and the Hausdorff measure of noncompactness, we can get

$$\begin{aligned} \sigma(\{x_n^u\}_{n=1}^\infty) & \leq \sigma(\{I_1 x_n^u\}_{n=1}^\infty) + \sigma(\{I_2 x_n^u\}_{n=1}^\infty) + \sigma(\{I_3 x_n^u\}_{n=1}^\infty) \\ & \leq \left(\frac{T^\rho - a^\rho}{\rho} \right)^{\alpha+\delta-1} \Gamma(\alpha - \beta) \sum_{a < t_k < t} \\ & \quad \cdot \left(\frac{t_k^\rho - a^\rho}{\rho} \right)^{1+\beta-\alpha} h_k \sigma(\{x_n^u\}_{n=1}^\infty). \end{aligned} \quad (39)$$

Therefore, from the above inequality, we can obtain $\sigma(\{x_n^u\}_{n=1}^\infty) = 0$; then, the set $\{x_n^u\}_{n=1}^\infty$ is precompact in $PC_\delta((a, T]; X)$ for $u \in U_{ad}$.

Claim 2. $x^\wedge u \in \text{Sol}(u)$.

In general, for any $u \in U_{ad}$, we assume that when $n \rightarrow \infty$, $x_n^u \rightarrow x^\wedge u$ on $PC_\delta((a, T]; X)$. From the dominated convergence theorem and the continuity of g , J_i , and f , taking $n \rightarrow \infty$ on both sides for (37), we have

$$\begin{aligned} \hat{x}^u(t) = & \left[(x_0 - g(x^\wedge u)) + \Gamma(\alpha - \beta) \right. \\ & \cdot \left. \left(\sum_{a < t_k < t} \left(\frac{t_k^\rho - a^\rho}{\rho} \right)^{1+\beta-\alpha} J_k(\hat{x}^u(t_k)) \right) \right] \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\alpha-1} \\ & + \rho^{1-\alpha} \int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} (f(\tau, \hat{x}^u(\tau)) \\ & + B(\tau)u(\tau)) d\tau, a \leq t \leq T. \end{aligned} \quad (40)$$

Then, $x^\wedge u \in \text{Sol}(u)$.

Step 3. $J(x^\wedge u, u) = \inf_{x^u \in \text{Sol}(u)} J(x^u, u) = J(u)$, $\forall u \in U_{ad}$.

By using the definition of a feasible pair, from the Balder theorem and hypotheses (Ψ1)–(Ψ4), we have

$$\begin{aligned}
J(u) &= \lim_{n \rightarrow \infty} \int_a^T \psi(t, x_n^u(t), u(t)) dt \\
&\geq \int_a^T \psi(t, x \wedge^u(t), u(t)) dt \\
&= J(x \wedge^u, u) \geq J(u).
\end{aligned} \tag{41}$$

Hence, $J(x \wedge^u, u) \geq J(u)$. This implies that for $\forall u \in U_{\text{ad}}$, $J(u)$ at $x \wedge^u \in PC_\delta((a, T]; X)$ to the minimum.

Step 4. There exists $u_0 \in U_{\text{ad}}$ such that $J(u_0) \leq J(u)$ for all $u \in U_{\text{ad}}$.

If $\inf_{u \in U_{\text{ad}}} J(u) < +\infty$, we get that $\inf_{u \in U_{\text{ad}}} J(u) > -\infty$, and there is a sequence $\{u_n\}_{n=1}^\infty \subseteq U_{\text{ad}}$ satisfying $J(u_n) \rightarrow \inf_{u \in U_{\text{ad}}} J(u)$ as $n \rightarrow \infty$ (we omit the proof of this step as it is similar to that of Step 1). Since $L^p((a, T]; N)$ is a reflexive Banach space and $\{u_n\}_{n=1}^\infty$ is bounded in $L^p((a, T]; N)$, for some $u_0 \in L^p((a, T]; N)$, we can assume that $\{u_n\}_{n=1}^\infty$ converges weakly, as $n \rightarrow \infty$.

Since U_{ad} is convex and closed, by Mazur's lemma, we can deduce that $u_0 \in U_{\text{ad}}$. For $n \geq 1$, $x \wedge^{u_n}$ is the solution for (1) associated with u_n , where $J(u_n)$ attains its minimum. Hence, the pair $(x \wedge^{u_n}, u_n)$ is feasible, such that

$$\begin{aligned}
\hat{x}^{u_n}(t) &= \left[(x_0 - g(\hat{x}^{u_n})) + \Gamma(\alpha - \beta) \right. \\
&\quad \cdot \left(\sum_{a < t_k < t} \left(\frac{t_k^\rho - a^\rho}{\rho} \right)^{1+\beta-\alpha} J_k(\hat{x}^{u_n}(t_k)) \right) \Big] \\
&\quad \cdot \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\alpha-1} + \rho^{1-\alpha} \int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} \\
&\quad \cdot (f(\tau, \hat{x}^{u_n}(\tau)) + B(\tau)u(\tau)) d\tau.
\end{aligned} \tag{42}$$

where $a \leq t \leq T$.

Let

$$\begin{aligned}
I'_3 x \wedge^{u_n}(t) &= \rho^{1-\alpha} \int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} f(\tau, x \wedge^{u_n}(\tau)) d\tau, \\
Zu_n(t) &= \rho^{1-\alpha} \int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} B(\tau)u(\tau) d\tau.
\end{aligned} \tag{43}$$

We have

$$\begin{aligned}
x \wedge^{u_n}(t) &= I_1 x \wedge^{u_n}(t) + I_2 x \wedge^{u_n}(t) + I'_3 x \wedge^{u_n}(t) \\
&\quad + Zu_n(t), \quad a \leq t \leq T.
\end{aligned} \tag{44}$$

Since $\{I_1 x \wedge^{u_n}\}_{n=1}^\infty, Zu_n(t)$ are both precompact in $PC_\delta((a, T]; X)$ and I_2 satisfies the Lipschitz continuity in $PC_\delta((a,$

$T]; X)$ with the Lipschitz constant, then

$$\left(\frac{T^\rho - a^\rho}{\rho} \right)^{\alpha+\delta-1} \Gamma(\alpha - \beta) \sum_{a < t_k < t} \left(\frac{t_k^\rho - a^\rho}{\rho} \right)^{1+\beta-\alpha} h_k < 1. \tag{45}$$

Moreover, by given assumptions, we can obtain that $\{I'_3 x \wedge^{u_n}\}_{n=1}^\infty \subseteq PC_\delta((a, T]; X)$ is precompact.

We can obtain that $\rho(\{x \wedge^{u_n}\}_{n=1}^\infty) = 0$; i.e., $\{x \wedge^{u_n}\}_{n=1}^\infty \subseteq PC_\delta((a, T]; X)$ is precompact (we omit the proof of this step as it is similar to that of Theorem 4.34 in [36]). Thus, we can find a subsequence which is connected with $x \wedge^{u_0} \in PC_\delta((a, T]; X)$ and $\{x \wedge^{u_n}\}_{n=1}^\infty$ satisfying that $x \wedge^{u_n} \rightarrow x \wedge^{u_0}$, as $n \rightarrow \infty$ in $PC_\delta((a, T]; X)$.

Taking the limit $n \rightarrow \infty$ on both sides for (42), we obtain

$$\begin{aligned}
x \wedge^{u_0}(t) &= [(x_0 - g(x \wedge^{u_0})) + \Gamma(\alpha - \beta) \\
&\quad \cdot \left(\sum_{a < t_k < t} \left(\frac{t_k^\rho - a^\rho}{\rho} \right)^{1+\beta-\alpha} J_k(x \wedge^{u_0}(t_k)) \right) \Big] \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\alpha-1} \\
&\quad + \rho^{1-\alpha} \int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} (f(\tau, x \wedge^{u_0}(\tau)) \\
&\quad + B(\tau)u(\tau)) d\tau, \quad a \leq t \leq T.
\end{aligned} \tag{46}$$

Therefore, the pair $(x \wedge^{u_0}, u_0)$ is feasible.

By the Balder theorem and assumptions (Ψ1)–(Ψ4), we have

$$\begin{aligned}
\inf_{u \in U_{\text{ad}}} J(u) &= \lim_{n \rightarrow \infty} \int_a^T \psi(t, x \wedge^{u_n}(t), u_n(t)) dt \\
&\geq \int_a^T \psi(t, x \wedge^{u_0}(t), u_0(t)) dt = J(x \wedge^{u_0}, u_0) \\
&\geq \inf_{u \in U_{\text{ad}}} J(u).
\end{aligned} \tag{47}$$

Then,

$$J(x \wedge^{u_0}, u_0) = J(u_0) = \inf_{x^{u_0} \in \text{Sol}(u_0)} J(x^{u_0}, u_0), \tag{48}$$

which means J at $u_0 \in U_{\text{ad}}$ to the minimum.

5. An Example

Consider the following impulsive Katugampola fractional differential equations:

$$\begin{cases} {}^\rho D_{a^+}^\alpha x(t) = f(t, x(t)) + B(t)u(t), & t \in J' = (a, 1], a > 0, t \neq t_1, \\ {}^\rho I_{a^+}^{1-\alpha} x(a) = g(x), \\ {}^\rho D_{a^+}^\beta x(t_1^+) - {}^\rho D_{a^+}^\beta x(t_1^-) = J_1 x(t_1). \end{cases} \tag{49}$$

Taking

$f : C((a, 1]; X) \longrightarrow X$ and $f(t, x) = 2e^{-t}|x|/(1 + e^t)(1 + |x|)$,

$g : PC_\delta((a, 1]; X) \longrightarrow X$ and $g(x)$ is a continuous and compact map.

Supposing that

B is a bounded and compact map,

$J_1 : X \longrightarrow X$ and $J_1(x) = 3|x|/(3 + |x|)$.

Let $x_1, x_2 \in B_r$ and $t \in (a, 1]$; we have

$$\begin{aligned} |f(t, x_1) - f(t, x_2)| &= \frac{2e^{-t}}{(1 + e^t)} \left| \frac{|x_1|}{1 + |x_1|} - \frac{|x_2|}{1 + |x_2|} \right| \\ &= \frac{2e^{-t}||x_1| - |x_2||}{(1 + e^t)(1 + |x_1|)(1 + |x_2|)} \\ &\leq \frac{2e^{-t}}{1 + e^t} |x_1 - x_2| \leq e^{-t} |x_1 - x_2|. \end{aligned} \quad (50)$$

For each $x_1, x_2 \in B_r$ and $t \in (a, 1]$, we can easily get

$$\begin{aligned} |f(t, x)| &= \frac{2e^{-t}}{(1 + e^t)} \frac{|x|}{1 + |x|} \leq \frac{2e^{-t}}{(1 + e^t)} \leq e^{-t}, \\ |J_1(x_1) - J_1(x_2)| &\leq |x_1 - x_2|. \end{aligned} \quad (51)$$

So hypotheses (H1)–(H6) are satisfied.

We define the cost function $J(x^u, u) = \int_a^1 (\|x^u(t)\|_X^2 + \|u(t)\|_N^2) dt$, where (x^u, u) is a feasible pair. If the inequality

$$\begin{aligned} &\left(\frac{1^\rho - a^\rho}{\rho} \right)^{\alpha+\delta-1} \left(\sup_{x \in w_r} \|g(x)\| + r\Gamma(\alpha - \beta) t_1^{1+\beta-\alpha} \right) \\ &+ \left(\frac{1^\rho - a^\rho}{\rho} \right)^\delta \sup_{t \in (a, 1]} \{ {}^\rho I_a^\alpha |e^{-t}| \} + \left(\frac{1^\rho - a^\rho}{\rho} \right)^{\alpha+\delta} \frac{\|Bu\|_{L^1}}{\alpha} \\ &+ \|x_0\| \leq r \end{aligned} \quad (52)$$

holds, then it satisfied all the conditions in Theorem 10; our results can be applied to problem (49).

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interest.

Authors' Contributions

All authors read and approved the final manuscript.

Acknowledgments

This work is supported by the Natural Science Foundation of Xinjiang (2019D01A71), Scientific Research Programs of

Colleges in Xinjiang (XJEDU2018Y033), National Natural Science Foundation of China (11961069, 11861068), and Outstanding Young Science and technology personnel Training program of Xinjiang (2019Q022).

References

- [1] V. Lakshmikantham, S. Leela, and J. V. Devi, *Theory of Fractional Dynamic Systems*, Cambridge Scientific Publishers, 2009.
- [2] U. N. Katugampola, "New approach to a generalized fractional integral," *Applied Mathematics and Computation*, vol. 218, pp. 860–865, 2011.
- [3] U. N. Katugampola, "A new approach to generalized fractional derivatives," *Bulletin of Mathematical Analysis and Applications*, vol. 6, pp. 1–15, 2014.
- [4] U. N. Katugampola, "Existence and uniqueness results for a class of generalized fractional differential equations," 2014, <http://arxiv.org/abs/1411.5229>.
- [5] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer, New York, NY, USA, 2013.
- [6] S. Gaboury, R. Tremblay, and B.-J. Fugere, "Some relations involving a generalized fractional derivative operator," *Journal of Inequalities and Applications*, vol. 2013, no. 1, 2013.
- [7] R. Herrmann, *Fractional Calculus: An Introduction for Physicists*, World Scientific, River Edge, NJ, USA, 2 edition, 2014.
- [8] M. A. Noor, M. U. Awan, and K. I. Noor, "On some inequalities for relative semi-convex functions," *Journal of Inequalities and Applications*, vol. 2013, no. 1, 2013.
- [9] M. A. Noor, K. I. Noor, and M. U. Awan, "Generalized convexity and integral inequalities," *Applied Mathematics and Information Sciences*, vol. 9, no. 1, pp. 233–243, 2015.
- [10] M. A. Noor, K. I. Noor, M. U. Awan, and S. Khan, "Fractional Hermite-Hadamard inequalities for some new classes of Godunova-Levin functions," *Applied Mathematics and Information Sciences*, vol. 8, no. 6, pp. 2865–2872, 2014.
- [11] T. Odziejewicz, "Generalized fractional isoperimetric problem of several variables," *Discrete and Continuous Dynamical Systems-Series B*, vol. 19, no. 8, pp. 2617–2629, 2014.
- [12] T. Odziejewicz, A. B. Malinowska, and D. F. M. Torres, "Fractional calculus of variations in terms of a generalized fractional integral with applications to physics," *Abstract and Applied Analysis*, vol. 2012, Article ID 871912, 24 pages, 2012.
- [13] T. Odziejewicz, A. B. Malinowska, and D. F. M. Torres, "A generalized fractional calculus of variations," *Control and Cybernetics*, vol. 42, no. 2, pp. 443–458, 2013.
- [14] S. Pooseh, R. Almeida, and D. F. M. Torres, "Expansion formulas in terms of integer-order derivatives for the Hadamard fractional integral and derivative," *Numerical Functional Analysis and Optimization*, vol. 33, no. 3, pp. 301–319, 2012.
- [15] M. M. Rodrigues, "Generalized fractional integral transform with Whittaker's kernel," *AIP Conference Proceedings*, vol. 1561, pp. 194–200, 2013.
- [16] R. Ganesh, R. Sakthivel, and N. I. Mahmudov, "Approximate controllability of fractional functional equations with infinite delay," *Topological Methods in Nonlinear Analysis*, vol. 43, pp. 345–364, 2014.
- [17] N. I. Mahmudov, "Approximate controllability of evolution systems with nonlocal conditions," *Nonlinear Analysis*, vol. 68, no. 3, pp. 536–546, 2008.

- [18] J. Y. Park and S. N. Kang, "Approximate controllability of neutral functional differential system with unbounded delay," *International Journal of Mathematics and Mathematical Sciences*, vol. 26, no. 12, pp. 536–744, 2001.
- [19] R. Sakthivel, N. I. Mahmudov, and J. H. Kim, "Approximate controllability of nonlinear impulsive differential systems," *Reports on Mathematical Physics*, vol. 60, no. 1, pp. 85–96, 2007.
- [20] R. Sakthivel, Y. Ren, A. Debbouche, and N. I. Mahmudov, "Approximate controllability of fractional stochastic differential inclusions with nonlocal conditions," *Applicable Analysis*, vol. 95, no. 11, pp. 2361–2382, 2015.
- [21] R. P. Agarwal and F. Karakoc, "A survey on oscillation of impulsive delay differential equations," *Computers & Mathematics with Application*, vol. 60, no. 6, pp. 1648–1685, 2010.
- [22] M. Benchohra, J. Henderson, and S. K. Ntouyas, *Impulsive Differential Equations and Inclusions*, vol. 2, Hindawi Publishing Corporation, New York, NY, USA, 2006.
- [23] M. Akhmet, "Differential equations on time scales through impulsive differential equations," in *Almost Periodicity, Chaos, and Asymptotic Equivalence, Nonlinear Systems and Complexity*, Vol 27, pp. 201–222, Springer, Cham, 2020.
- [24] D. D. Bainov and P. S. Simeonov, "Systems with impulse effect: stability, theory and applications," in *Ellis Horwood Series: Mathematics and Its Applications*, Ellis Horwood, Chichester, 1989.
- [25] J. Chen, C. C. Tisdell, and R. Yuan, "On the solvability of periodic boundary value problems with impulse," *Journal of Mathematical Analysis and Applications*, vol. 331, no. 2, pp. 902–912, 2007.
- [26] E. R. Kaufmann, "Impulsive periodic boundary value problems for dynamic equations on time scale," *Advances in Difference Equations*, vol. 2009, Article ID 603271, 10 pages, 2009.
- [27] V. Lakshmikantham, D. D. Bainov, and P. S. Simeonov, *Theory of impulsive differential equations, Series in modern applied mathematics*, vol. 6, World Scientific, Hackensack, NJ, USA, 1994.
- [28] Z. Luo and Z. Jing, "Periodic boundary value problem for first-order impulsive functional differential equations," *Computers & Mathematics with Applications*, vol. 55, no. 9, pp. 2094–2107, 2008.
- [29] L. Qi and R. Yuan, "A generalization of Bochner's theorem and its applications in the study of impulsive differential equations," *Journal of Dynamics and Differential Equations*, vol. 31, no. 4, pp. 1955–1985, 2019.
- [30] S. Suresh and G. Thamizhendhi, "Some results on fractional semilinear impulsive integro-differential equations," *Malaya Journal of Matematik*, vol. 7, no. 2, pp. 259–263, 2019.
- [31] Y. Tian and W. Ge, "Variational methods to Sturm-Liouville boundary value problem for impulsive differential equations," *Nonlinear Analysis*, vol. 72, no. 1, pp. 277–287, 2010.
- [32] G. L. Zhang, "Asymptotical stability of numerical methods for semi-linear impulsive differential equations," *Computational and Applied Mathematics*, vol. 39, no. 1, p. 17, 2020.
- [33] J. R. Wang, M. Feckan, and A. Debbouche, "Time optimal control of a system governed by non-instantaneous impulsive differential equations," *Journal of Optimization Theory and Applications*, vol. 182, no. 2, pp. 573–587, 2019.
- [34] X. Wan and J. Sun, "Approximate controllability for abstract measure differential systems," *Systems and Control Letters*, vol. 61, no. 1, pp. 50–54, 2012.
- [35] Y. Sun, H. B. Gu, Y. H. Zhang, X. R. Chen, and X. Z. Wang, "Optimal controls for a class of impulsive fractional differential equations with nonlocal conditions," *Advances in Difference Equations*, vol. 2018, no. 125, pp. 50–54, 2018.
- [36] D. S. Oliveira and E. C. de Oliveira, "Hilfer-Katugampola fractional derivatives," *Computational and Applied Mathematics*, vol. 37, no. 3, pp. 3672–3690, 2018.
- [37] J. Banas and K. Goebel, *Measure of Noncompactness in Banach Space*, Dekker, New York, NY, USA, 1980.
- [38] Y. Zhou, *Basic Theory of Fractional Differential Equations*, World Scientific, Singapore, 2014.