Research Article

Positive Solutions for Schrödinger-Poisson Systems with Sign-Changing Potential and Critical Growth

Pengfei He ^{1,2} and Hongmin Suo ³

¹School of Mathematics Science, Guizhou Normal University, Guiyang 550025, China

²School of Mathematics and Statistics, Guizhou University of Finance and Economics, Guiyang 550025, China

³School of Data Science and Information Engineering, Guizhou Minzu University, Guiyang 550025, China

Correspondence should be addressed to Pengfei He; gznchepengfei@163.com

Received 21 March 2020; Revised 16 May 2020; Accepted 10 June 2020; Published 2 July 2020

Academic Editor: Kasso Okoudjou

Copyright © 2020 Pengfei He and Hongmin Suo. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In this paper, we study the existence of positive solutions for Schrödinger-Poisson systems with sign-changing potential and critical growth. By using the analytic techniques and variational method, the existence and multiplicity of positive solutions are obtained.

1. Introduction and Main Result

In this paper, we consider the existence and multiplicity of positive solutions of the Schrödinger-Poisson systems with critical nonlinearity of the following form:

$$\begin{cases} -\Delta u - \phi u = \lambda f_{\lambda}(x)|u|^{q-2}u + |u|^4u, & \text{in } \Omega, \\ -\Delta \phi = u^2, & \text{in } \Omega, \\ u = \phi = 0, & \text{on } \partial\Omega, \end{cases} \quad (1)$$

where Ω is a smooth bounded domain in $\mathbb{R}^3$, $1 < q < 2$, $f_{\lambda} = \lambda f^+ + f^-$, $\lambda > 0$, and $f^{\pm} = \pm \max\{\pm f, 0\}$. When $N = 3$, $2^* = (2N/(N-2)) = 6$, the right hand side of the system of (1) has the critical term $|u|^4u$. It is pointed out that system (1) is related to the following Schrödinger-Poisson system:

$$\begin{cases} -\Delta u + V(x)u + k(x)\phi|u|^{q-2}u = f(x, u), & \text{in } \mathbb{R}^3, \\ -\Delta \phi = k(x)|u|^q, & \text{in } \mathbb{R}^3, \end{cases} \quad (2)$$

where the potentials $V(x)$, $k(x)$ satisfy some mild conditions. The existence, uniqueness, and multiplicity of positive solutions of systems like (2) have been extensively studied in

the last few decades, such as [1–14]. Furthermore, the Schrödinger-Poisson systems involving critical growth have been attracted many researchers, e.g., [15–24]. But there is little literature for liking the system (1) that considers the critical growth on bounded domains, such as [15–17].

In [24], the authors considered a Schrödinger-Poisson system with concave-convex nonlinearities of the form

$$\begin{cases} -\Delta u + \lambda \phi u = \lambda u^{q-1} + u^5, & \text{in } \Omega, \\ -\Delta \phi = |u|^2, & \text{in } \Omega, \\ u = \phi = 0, & \text{on } \partial\Omega, \end{cases} \quad (3)$$

where Ω is a smooth bounded domain in $\mathbb{R}^3$, $1 < q < 2$ and $\lambda > 0$. They obtained two positive solutions when λ is enough small by using the concentration-compactness principle.

Recently, in [25], the authors considered a Schrödinger-Newton system with critical nonlinearities of the form

$$\begin{cases} -\Delta u = f_{\lambda}(x)|u|^{q-2}u + \phi|u|^3u, & \text{in } \Omega, \\ -\Delta \phi = |u|^5, & \text{in } \Omega, \\ u = \phi = 0, & \text{on } \partial\Omega, \end{cases} \quad (4)$$

where Ω is a smooth bounded domain in $\mathbb{R}^3$, $1 < q < 2$, $f_\lambda = \lambda f^+ + f^-$, $\lambda > 0$, and $f^\pm = \pm \max \{\pm f, 0\}$. The authors studied the existence and multiplicity of positive solutions by the variational method and analytic techniques.

Motivated by results found in the above mentioned papers, in the present paper, we intend to study the existence of multiple positive solutions for system (1). Since system (1) contains a critical term and a nonlocal term, the critical value level becomes very difficult to estimate. The main objective of this paper is to look for new estimates and establish two positive solutions by analytic techniques.

The main result is as follows:

Theorem 1. Assume that $1 < q < 2$ and $f_\lambda \in L^\infty(\Omega)$ is a possibly sign-changing function. Then, there exists $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$, system (1) has at least two different positive solutions. Additionally, system (1) admits a positive ground state solution.

Throughout this paper, we use the following notations:

- (i) The space $H_0^1(\Omega)$ is equipped with the norm $\|u\|^2 = \int_\Omega |\nabla u|^2 dx$. The norm in $L^p(\Omega)$ is denoted by $|\cdot|_p$.
- (ii) $C, C_1, C_2, \dots$ denote various positive constants, which may vary from line to line.
- (iii) We denote by $B(0, r)$ (respectively, $\partial B(0, r)$) the closed ball (respectively, the sphere) of center zero and radius r , i.e., $B(0, r) = \{u \in H_0^1(\Omega) : \|u\| \leq r\}$, $\partial B(0, r) = \{u \in H_0^1(\Omega) : \|u\| = r\}$.
- (iv) Let S be the best Sobolev embedding constant for the embedding $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$, namely,

$$S = \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\left(\int_\Omega |u|^6 dx\right)^{1/3}} \quad (5)$$

2. Existence of a First Positive Solution of System (1)

For any given $u \in H_0^1(\Omega)$, by the Lax-Milgram theorem, the Dirichlet boundary problem $-\Delta \phi = u^2$ in Ω has a unique solution $\phi_u \in H_0^1(\Omega)$. Substitute ϕ_u in the first equation of system (1), system (1) transformed into the following variable equations:

$$\begin{cases} -\Delta u - \phi u = \lambda f_\lambda(x) |u|^{q-2} u + |u|^4 u, & \text{in } \Omega, \\ -\Delta \phi = u^2, & \text{on } \partial\Omega. \end{cases} \quad (6)$$

The formal energy functional corresponding to the system (6) is the following:

$$I_\lambda(u) = \frac{1}{2} \|u\|^2 - \frac{1}{4} \int_\Omega \phi_u |u|^2 dx - \frac{1}{q} \int_\Omega f_\lambda(x) |u|^q dx - \frac{1}{6} \int_\Omega |u|^6 dx. \quad (7)$$

If a function $u \in H_0^1(\Omega)$ satisfies

$$\begin{aligned} \int_\Omega (\nabla u, \nabla v) dx - \int_\Omega f_\lambda(x) u^{q-1} v dx - \int_\Omega \phi_u u v dx - \int_\Omega u^5 v dx \\ = 0, \quad \forall v \in H_0^1(\Omega), \end{aligned} \quad (8)$$

where $v \in H_0^1(\Omega)$, we can prove that $(u, \phi) \in H_0^1(\Omega) \times H_0^1(\Omega)$ is a solution of system (1) if and only if $\phi = \phi_u$, and $u \in H_0^1(\Omega)$ is called a weak solution of (6).

To prove our main theorem 1, some preliminary results are needed, let us first collect some properties.

Lemma 2 (see [17]). For each $u \in H_0^1(\Omega)$, there exists a unique solution $\phi_u \in H_0^1(\Omega)$ of

$$\begin{cases} -\Delta \phi = |u|^2, & \text{in } \Omega, \\ \phi = 0, & \text{on } \Omega. \end{cases} \quad (9)$$

Moreover,

- (1) $\phi_u \geq 0$ for $x \in \Omega$
- (2) For each $t \neq 0$, $\phi_{tu} = t^2 \phi_u$
- (3) $\int_\Omega \phi_u |u|^2 dx = \int_\Omega |\nabla \phi_u|^2 dx \leq S^{-3} \|u\|^4$
- (4) Suppose that $u_n \rightharpoonup u$ in $H_0^1(\Omega)$, then

$$\int_\Omega \phi_{u_n} |u_n|^2 dx - \int_\Omega \phi_{u_n - u} |u_n - u|^2 dx = \int_\Omega \phi_u |u|^2 dx + o(1) \quad (10)$$

Lemma 3 (see [18]). Let $r, s > 1$, $\psi \in L^s(\Omega)$, and $\psi^+ = \max(\psi, 0) \neq 0$, then there exists $\varphi_0 \in C_0^\infty(\Omega)$ such that $\int_\Omega \psi |\varphi_0|^r dx > 0$.

Lemma 4. There exist $\alpha, \rho, \Lambda_0 > 0$, such that every $\lambda \in (0, \Lambda_0)$, we have

$$\begin{aligned} I_\lambda(u) &> 0 \quad \text{for } u \in \partial B(0, \rho), \\ I_\lambda(u) &< 0 \quad \text{for } u \in B(0, \rho). \end{aligned} \quad (11)$$

Proof. According to the Sobolev and Hölder inequalities and Lemma 2, we have

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2} \|u\|^2 - \frac{1}{4} \int_\Omega \phi_u |u|^2 dx - \frac{1}{q} \int_\Omega f_\lambda(x) |u|^q dx - \frac{1}{6} \int_\Omega |u|^6 dx \\ &\geq \frac{1}{2} \|u\|^2 - \frac{1}{4} S^{-3} \|u\|^4 - \frac{\lambda}{q} |\Omega|^{(6-q)/6} S^{-q/2} |f^+|_\infty \|u\|^q \\ &\quad - \frac{1}{6} S^{-3} \|u\|^6 = \|u\|^q \left[\frac{1}{2} \|u\|^{2-q} - \frac{1}{4} S^{-3} \|u\|^{4-q} \right. \\ &\quad \left. - \frac{1}{6} S^{-3} \|u\|^{6-q} - \frac{\lambda}{q} |\Omega|^{(6-q)/6} S^{-q/2} |f^+|_\infty \right]. \end{aligned} \quad (12)$$

Setting $g(t) = (1/2)t^{2-q} - (1/4)S^{-3}t^{4-q} - (1/6)S^{-3}t^{6-q}$ for $t > 0$, there exists a positive constant

$$\rho = \left[\frac{(1/2)(4-q)S^{-3} - \sqrt{(1/4)(4-q)^2S^{-1} + (4/3)(6-q)(2-q)S^{-3}}}{(2/3)(6-q)S^{-3}} \right]^{1/2} > 0, \quad (13)$$

such that $\max_{t>0} g(t) = g(\rho) > 0$. Letting $\Lambda_0 = (qS^{q/2}/2|f^+|_\infty |\Omega|^{(6-q)/6})g(\rho)$. Consequently, it follows that $I_\lambda|_{\|u\|=\rho} \geq \alpha \triangleq (g(\rho)/6)\rho^q$ for any $\lambda \in (0, \Lambda_0)$.

Additionally, by Lemma 3, there exists $\varphi_0 \in C_0^\infty(\Omega) \subset H_0^1(\Omega)$ such that

$$\lim_{t \rightarrow 0^+} \frac{I_\lambda(tu)}{t^q} = \frac{1}{q} \int_\Omega f_\lambda(x) |\varphi_0|^q dx. \quad (14)$$

For u small enough, we have $I_\lambda(u) < 0$. The lemma is proved.

We recall that for $c \in \mathbb{R}$, a sequence $\{u_n\} \subset H_0^1(\Omega)$ is called a $(PS)_c$ sequence for I if $I(u_n) = c + o(1)$ and $I'(u_n) = o(1)$ strongly in the dual space of $H_0^1(\Omega)$ as $n \rightarrow \infty$. That is to say, I satisfies the $(PS)_c$ condition in $H_0^1(\Omega)$ if every $(PS)_c$ sequence in $H_0^1(\Omega)$ for I contains a convergent subsequence.

Lemma 5. *If $\{u_n\} \subset H_0^1(\Omega)$ is a $(PS)_c$ sequence for I_λ with $c < (1/3)S^{3/2} - D\lambda^{2/(2-q)}$ where $D = D(|\Omega|, S, q, |f^+|_\infty) > 0$. Then, there exists $u \in H_0^1(\Omega)$ such that $u_n \rightarrow u_*$ in $H_0^1(\Omega)$ and $\int_\Omega |u_n|^6 dx \rightarrow \int_\Omega |u|^6 dx$.*

Proof. Let $\{u\} \in H_0^1(\Omega)$ be such that

$$I_\lambda(u_n) \rightarrow c, I'_\lambda(u_n) \rightarrow 0. \quad (15)$$

We claim that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Indeed, by (15), we have that

$$\begin{aligned} 1 + c + o(1)\|u_n\| &\geq I_\lambda(u_n) - \frac{1}{4} \langle I'_\lambda(u_n), u_n \rangle \\ &\geq \frac{1}{4} \|u_n\|^2 - \left(\frac{1}{q} - \frac{1}{4} \right) \int_\Omega f_\lambda(x) |u_n|^q dx \\ &\geq \frac{1}{4} \|u_n\|^2 - \lambda \left(\frac{1}{q} - \frac{1}{4} \right) |\Omega|^{(6-q)/6} S^{-q/2} |f^+|_\infty \|u_n\|^q, \end{aligned} \quad (16)$$

which implies that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Thereby, there exists a subsequence, still denoted by $\{u_n\}$, and a function $u_* \in H_0^1(\Omega)$, such that

$$\begin{cases} u_n \rightharpoonup u, \text{ weakly in } H_0^1(\Omega), \\ u_n \rightarrow u, \text{ strongly in } L^s(\Omega) \ (1 \leq s < 6), \\ u_n(x) \rightarrow u(x), \text{ a.e. in } \Omega, \end{cases} \quad (17)$$

as $n \rightarrow \infty$. Set $w_n = u_n - u$, then $\|w_n\| \rightarrow 0$. Otherwise, there exists a subsequence (still denoted by w_n) such that

$$\lim_{n \rightarrow \infty} \|w_n\| = l > 0. \quad (18)$$

By (15), letting $n \rightarrow \infty$, for every $\varphi \in H_0^1(\Omega)$, it follows

$$\int_\Omega (\nabla u, \nabla \varphi) dx - \int_\Omega f_\lambda(x) |u|^{q-2} u \varphi dx - \int_\Omega \phi_u u \varphi dx - \int_\Omega |u|^4 u \varphi dx = 0. \quad (19)$$

In particular, taking the test function $\varphi = u$ in (19), we have

$$\|u\|^2 - \int_\Omega f_\lambda(x) |u|^q dx - \int_\Omega \phi_u |u|^2 dx - \int_\Omega |u|^6 dx = 0. \quad (20)$$

By Brézis–Lieb (see [26]) and Lemma 2, one has

$$\begin{cases} \|u_n\|^2 = \|w_n\|^2 + \|u\|^2 + o(1), \\ \int_\Omega |u_n|^6 dx = \int_\Omega |w_n|^6 dx + \int_\Omega |u|^6 dx + o(1). \end{cases} \quad (21)$$

Therefore, according to $\langle I'_\lambda(u_n), u_n \rangle \rightarrow 0$ as $n \rightarrow \infty$, we have

$$\begin{aligned} \|u\|^2 + \|w_n\|^2 + \int_\Omega \phi_u |u|^2 dx - \int_\Omega |u|^6 dx \\ - \int_\Omega |w_n|^6 dx - \int_\Omega f_\lambda(x) |u|^q dx = o(1). \end{aligned} \quad (22)$$

It follows from (20) and (22) implies that

$$\|w_n\|^2 - \int_\Omega |w_n|^6 dx = o(1). \quad (23)$$

Note that $\int_\Omega |w_n|^6 dx \leq S^{-3} \|w_n\|^6$, we deduce that

$$l^2 \geq S^{3/2}. \quad (24)$$

On the one hand, by using the Young inequality and (20), there holds

$$\begin{aligned} I_\lambda(u_*) &= \frac{1}{2} \|u\|^2 - \frac{1}{4} \int_\Omega \phi_u |u_*|^2 dx - \frac{1}{6} \int_\Omega |u|^6 dx \\ &\quad - \frac{1}{q} \int_\Omega f_\lambda(x) |u|^q dx \geq \frac{1}{4} \|u\|^2 - \left[\frac{1}{q} - \frac{1}{4} \right] \int_\Omega f_\lambda(x) |u|^q dx \\ &\geq \frac{1}{4} \|u\|^2 - \lambda \left[\frac{1}{q} - \frac{1}{4} \right] |\Omega|^{(6-q)/6} S^{-q/2} |f^+|_\infty \|u\|^q \\ &\geq -D\lambda^{2/(2-q)}, \end{aligned} \quad (25)$$

where $D = D(|\Omega|, S, q, |f^+|_\infty) > 0$ is a constant.

On the other hand, note that $c < (1/3)S^{3/2} - D\lambda^{2/(2-q)}$, by (23), one has

$$\begin{aligned} I_\lambda(u_*) &= I_\lambda(u_n) - \frac{1}{2}\|w_n\|^2 - \frac{1}{6}\int_\Omega |w_n|^6 dx + o(1) \\ &= I_\lambda(u_n) - \frac{1}{3}\|w_n\|^2 + o(1) \\ &\leq c - \frac{1}{3}l^2 < \frac{1}{3}S^{3/2} - D\lambda^{2/(2-q)} - \frac{1}{3}S^{3/2} \\ &= -D\lambda^{2/(2-q)}. \end{aligned} \quad (26)$$

From the above analysis, this is a contradiction. Therefore, $l = 0$, it implies that $u_n \rightarrow u$ in $H_0^1(\Omega)$, and

$$\begin{aligned} 0 &\leq \int_\Omega |u_n|^6 dx - \int_\Omega |u|^6 dx = \int_\Omega |w_n|^6 dx + o(1) \\ &\leq S^{-3}\|w_n\|^6 + o(1) \rightarrow 0. \end{aligned} \quad (27)$$

So $\int_\Omega |u_n|^6 dx \rightarrow \int_\Omega |u|^6 dx$ as $n \rightarrow \infty$. The lemma is proved.

Theorem 6. Assume $1 < q < 2$ and $f_\lambda \in L^\infty(\Omega)$. Then, system (1) has a positive solution $(u_*, \phi_{u_*}) \in H_0^1(\Omega) \times H_0^1(\Omega)$ which satisfies $I_\lambda(u_*) < 0$.

Proof. From Lemma 4 and applying Ekeland's variational principle in $\overline{B_\rho(0)}$, there exists a minimizing sequence $(u_n) \subset \overline{B_\rho(0)}$ such that

$$\begin{aligned} I_\lambda(u_n) &\leq \inf_{u_n \in \overline{B_\rho(0)}} I_\lambda(u) + \frac{1}{q} I_\lambda(v) \\ &\geq I_\lambda(u_n) - \frac{1}{n} \|v - u_n\|, \quad \forall u_n \in \overline{B_\rho(0)}. \end{aligned} \quad (28)$$

Therefore,

$$\begin{aligned} I'_\lambda(u_n) &\rightarrow 0, \\ I_\lambda(u_n) &< c. \end{aligned} \quad (29)$$

Since $\{u_n\}$ is bounded in $H_0^1(\Omega)$, there exist $u_* \in \overline{B_\rho(0)}$ and a subsequence still denoted by $\{u_n\}$ such that $u_n \rightharpoonup u_*$ in $H_0^1(\Omega)$ as $n \rightarrow \infty$.

Note that $I_\lambda(u_n) = I_\lambda(|u_n|)$ and Lemma 5, we can obtain $u_n \rightharpoonup u_*$ in $H_0^1(\Omega)$ and $u_* \geq 0$ and $u_* \neq 0$. Since $u_* \in H_0^1(\Omega)$, the embedding theorem means $u_* \in L^6(\Omega)$. Note that $f_\lambda \in L^\infty(\Omega)$, by the regularity of weak solutions, one has $u_* \in W^{2,6/q}(\Omega)$. Thereby, according to the classical bootstrap argument, we have $u_* \in W^{1,\alpha}$ for $r \in [1, +\infty)$. By the embedding theorem again, it holds that $u_* \in C^{1,\alpha}(\Omega)$ for $0 < \alpha < 1$. Since $\phi_{u_*} \geq 0$, we have

$$-\Delta u_* = u_*^5 + f_\lambda(x)u_*^{q-1} + \phi_{u_*}u_* \geq -f^-(x)|u_*^{q-1}| \geq -Cu_* \quad (30)$$

for some $C > 0$. Thereby, by the strong maximum principle, we obtain $u_* > 0$ in Ω , then (u_*, ϕ_{u_*}) is a positive solution of system (1) with $I_\lambda(u_*) < 0$ (by (11)). The theorem is proved.

For $u \in H_0^1(\Omega)$, we have

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2}\|u\|^2 - \frac{1}{4}\int_\Omega \phi_u |u|^2 dx - \frac{1}{6}\int_\Omega |u|^6 dx \\ &\quad - \frac{1}{q}\int_\Omega f_\lambda(x)|u|^q dx \leq \frac{1}{2}\|u\|^2 - \frac{1}{6}\int_\Omega |u|^6 dx \\ &\quad - \frac{1}{q}\int_\Omega f_\lambda(x)|u|^q dx. \end{aligned} \quad (31)$$

Now, if we consider the following problem:

$$\begin{cases} -\Delta u = f_\lambda(x)|u|^{q-2}u + |u|^4u, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \quad (32)$$

Then, we find the weak solutions of problem (32) correspond to the critical points of the functional J_λ , where

$$J_\lambda(u) = \frac{1}{2}\|u\|^2 - \frac{1}{q}\int_\Omega f_\lambda(x)|u|^q dx - \frac{1}{6}\int_\Omega |u|^6 dx. \quad (33)$$

Obviously, one has

$$I_\lambda(u) \leq J_\lambda(u), \quad \text{for } u \in H_0^1(\Omega). \quad (34)$$

Remark 7. Problem (32) has a positive local minimizer solution $v_0 \in B_\rho$ with $J_\lambda(v_0) < 0$ for each $\lambda \in (0, \Lambda_0)$ where ρ, Λ_0 are as in Lemma 4.

Indeed, by Lemma 4, we have

$$\frac{1}{2}\rho^{2-q} - \frac{1}{4}S^{-3}\rho^{4-q} - \frac{\lambda}{q}|\Omega|^{(6-q)/6}S^{-q/2}|f^+|_\infty > \frac{1}{6}S^{-3}\rho^{6-q}. \quad (35)$$

Therefore, when $\|u\| = \rho$, there holds

$$\begin{aligned} J_\lambda(u) &= \frac{1}{2}\|u\|^2 - \frac{1}{q}\int_\Omega f_\lambda(x)|u|^q dx - \frac{1}{6}\int_\Omega |u|^6 dx \\ &\geq \frac{1}{2}\|u\|^2 - \frac{S^{-3}}{6}\|u\|^6 - \frac{\lambda|\Omega|^{(6-q)/6}}{q}S^{-q/2}|f^+|_\infty\|u\|^q \\ &= \|u\|^q \left(\frac{1}{2}\|u\|^{2-q} - \frac{S^{-3}}{6}\|u\|^{6-q} - \frac{\lambda|\Omega|^{(6-q)/6}}{q}S^{-q/2}|f^+|_\infty \right) \\ &\geq \frac{1}{6}S^{-3}\rho^{6-q}\rho^q = \frac{1}{6}S^{-3}\rho^6. \end{aligned} \quad (36)$$

Similar to Lemma 4, we have

$$\lim_{t \rightarrow 0^+} \frac{J_\lambda(tu)}{t^q} = \frac{1}{q}\int_\Omega f_\lambda(x)|\varphi_0|^q dx. \quad (37)$$

So, we get $J_\lambda(u) < 0$. Similar to Theorem 6, problem (32) has a positive local minimizer solution $v_0 \in B_\rho$ with $J_\lambda(v_0) < 0$ for each $\lambda \in (0, \Lambda_0)$.

3. Existence of a Second Positive Solution of System (1)

As usual, we consider the following function:

$$v_\varepsilon = \frac{[3\varepsilon^2]^{1/4}}{(\varepsilon^2 + |x|^2)^{1/2}}, \quad (38)$$

where ε is a positive constant. Moreover, we know that v_ε is a positive solution of problem $-\Delta u = |u|^4 u$ in $\mathbb{R}^3$ and $\int_\Omega |\nabla v_\varepsilon|^2 dx = \int_\Omega |v_\varepsilon|^6 dx = S^{3/2}$. Let ζ be a smooth cut-off function $\zeta \in C_0^\infty(\Omega)$ such that $0 \leq \zeta(x) \leq 1$ in Ω , $\zeta(x) = 1$ near $x = 0$ and it is radially symmetric. Set $u_\varepsilon(x) = v_\varepsilon(x)\zeta(x)$ for $x \in \Omega$. Then, we have

Lemma 8. Assume $1 < q < 2$, there holds

$$\sup_{t \geq 0} I_\lambda(v_0 + tu_\varepsilon) < \frac{1}{3} S^{3/2} - D\lambda^{2/(2-q)}. \quad (39)$$

Proof. According to the definition and from [27], there hold

$$\begin{aligned} \|u_\varepsilon\|^2 &\leq S^{3/2} + O(\varepsilon), \\ \int_\Omega u_\varepsilon^6 dx &\geq S^{3/2} + O(\varepsilon). \end{aligned} \quad (40)$$

It is well known that the following inequality

$$(a+b)^6 \geq a^6 + b^6 + 6a^5b + 6ab^5 \quad (41)$$

holds for $a, b \geq 0$. Since $v_0 \in B_{\rho_1}$ is a positive solution of the problem (32), and $J_\lambda < 0$ for each $\lambda \in (0, \Lambda_1)$. Obviously, there exist two constants $n, M > 0$ such that $m \leq v_0(x) \leq M$, using (41), there holds

$$\begin{aligned} J_\lambda(v_0 + tu_\varepsilon) &= \frac{1}{2} (\|v_0\|^2 + t^2 \|u_\varepsilon\|^2) + \int_\Omega (\nabla v_0, \nabla tu_\varepsilon) dx \\ &\quad - \frac{1}{6} \int_\Omega |v_0 + tu_\varepsilon|^6 dx - \frac{1}{q} \int_\Omega f_\lambda(x) |v_0 + tu_\varepsilon|^q dx \\ &\leq J_\lambda(v_0) + \frac{t^2}{2} \|u_\varepsilon\|^2 - \frac{1}{6} \int_\Omega [|v_0 + tu_\varepsilon|^6 - v_0^6 - 6v_0^5 tu_\varepsilon] dx \\ &\quad + \frac{1}{q} \int_\Omega |f^-|_\infty \left\{ \int_0^{tu_\varepsilon} [(v_0 + \eta)^{q-1} - v_0^{q-1}] d\eta \right\} dx \\ &\leq \frac{t^2}{2} \|u_\varepsilon\|^2 - \frac{1}{6} \int_\Omega [|tu_\varepsilon|^6 + 6v_0(tu_\varepsilon)^5] dx \\ &\quad + \frac{1}{q} \int_\Omega \left(|f^-|_\infty \int_0^{tu_\varepsilon} \eta^{q-1} \right) dx \\ &\leq \frac{t^2}{2} \|u_\varepsilon\|^2 - \frac{t^2}{6} \int_\Omega u_\varepsilon^6 dx - C_1 t^5 \int_\Omega u_\varepsilon^5 dx \\ &\quad + C_2 t^q \int_\Omega u_\varepsilon^q dx, \end{aligned} \quad (42)$$

where $C_1, C_2 > 0$ (independent of ε, λ). Let

$$h(t) = \frac{t^2}{2} \|u_\varepsilon\|^2 - \frac{t^2}{6} \int_\Omega u_\varepsilon^6 dx - C_1 t^5 \int_\Omega u_\varepsilon^5 dx + C_2 t^q \int_\Omega u_\varepsilon^q dx. \quad (43)$$

Since $\lim_{t \rightarrow 0} h(t) = 0$, $\lim_{t \rightarrow \infty} h(t) = -\infty$, it is similar to [25] that we prove that there exist t_ε and positive constants t_1, t_2 independent of ε, λ , such that

$$\begin{aligned} 0 < t_0 \leq t_\varepsilon \leq t_2 < +\infty, \\ \sup_{t \geq 0} h(t) &\leq \frac{1}{3} S^{3/2} + C_3 \varepsilon - C_1 t_1^5 \int_\Omega u_\varepsilon^5 dx + C_2 t_2^q \int_\Omega u_\varepsilon^q dx. \end{aligned} \quad (44)$$

Note that $\int_\Omega u_\varepsilon^5 dx = C\varepsilon^{1/2} + O(\varepsilon^{1/2})$, $\int_\Omega u_\varepsilon^q dx = C\varepsilon^{q/2}$. Then,

$$\begin{aligned} \sup_{t \geq 0} h(t) &\leq \frac{1}{3} S^{3/2} + C_3 \varepsilon - C_4 \varepsilon^{1/2} + C_5 \varepsilon^{q/2} \\ &\leq \frac{1}{3} S^{3/2} + (C_3 + C_5) \varepsilon^{q/2} - C_4 \varepsilon^{1/2}. \end{aligned} \quad (45)$$

where $C_3, C_4, C_5 > 0$ (independent of ε, λ). Let $\varepsilon = \lambda^{4/(q(2-q))}$, $0 < \lambda < \Lambda_1 := (C_4/(C_3 + C_5 + D))^{2/(2(q-1))}$, there holds

$$\begin{aligned} (C_3 + C_5) \varepsilon^{q/2} - C_4 \varepsilon^{1/2} &= (C_3 + C_5) \lambda^{2/(2-q)} - C_4 \lambda^{2/(q(2-q))} \\ &= \lambda^{2/(2-q)} \left(C_3 + C_5 - C_4 \lambda^{2/(q(2-q))} \right) \\ &< -D\lambda^{2/(2-q)}. \end{aligned} \quad (46)$$

Recall that

$$I_\lambda(u) \leq J_\lambda(u) \quad (47)$$

for all $u \in H_0^1(\Omega)$. Therefore, from the above analysis, we conclude that (39) holds true when $\lambda < \Lambda_1$. The lemma is proved.

Theorem 9. Assume $1 < q < 2$ and $f_\lambda \in L^\infty(\Omega)$. Then, the system (1) admits a positive solution $(u_*, \phi_{u_*}) \in H_0^1(\Omega) \times H_0^1(\Omega)$ with $I_\lambda(u_*) > 0$.

Proof. Let $\lambda_* = \min \{\Lambda_0, \Lambda_1, 1\}$, then Lemma 4 and Theorem 6 and Lemma 8 hold for $0 < \lambda < \lambda_*$. By Remark 7, we can choose a large enough $T_0 > 0$ such that $I_\lambda(v_0 + T_0 u_\varepsilon) < 0$. Consequently, I_λ satisfies the mountain geometry structure. Applying the mountain-pass lemma [28], there exists a sequence $\{u_n\} \subset H_0^1(\Omega)$ such that

$$I'_\lambda(u_n) \rightarrow 0 \text{ and } I_\lambda(u_n) \rightarrow c > 0 \text{ as } n \rightarrow \infty, \quad (48)$$

where

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\lambda(\gamma(t)),$$

$$\Gamma = \{\gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = v_0, \gamma(1) = v_0 + T_0 u_\varepsilon\}. \quad (49)$$

From Lemma 5 and Lemma 8, $\{u_n\}$ has a convergent subsequence (still denoted by $\{u_n\}$) and there exists $u_{**} \in H_0^1(\Omega)$ such that $u_n \rightharpoonup u_{**}$ in $H_0^1(\Omega)$. Moreover, we can obtain u_{**} is a nonnegative weak solution of system (1) and

$$I_\lambda(u_{**}) = \lim_{n \rightarrow \infty} I_\lambda(u_n) = c > 0. \quad (50)$$

Therefore, we infer that $u_{**} \equiv 0$. Similar to Theorem 6, we obtain $u_{**} > 0$ in Ω , and so $(u_{**}, \phi_{u_{**}})$ is a positive solution of system (1). The theorem is proved.

4. Existence of a Ground State Solution of System (1)

Define

$$N = \inf \left\{ I_\lambda(u) : u \in H_0^1(\Omega), u \neq 0, I'_\lambda(u_n) = 0 \right\}. \quad (51)$$

By the definition of N , there exists $\{u\} \in H_0^1(\Omega)$ such that $u_n \neq 0$, and

$$I_\lambda(u_n) \rightarrow N, I'_\lambda(u_n) \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (52)$$

Obviously, we can easily deduce that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. There then exist a nonnegative subsequence of $\{u_n\}$ ($I_\lambda(u_n) = I_\lambda(|u_n|)$), still denoted by itself, and $u_\lambda \in H_0^1(\Omega)$ such that $u_n \rightharpoonup u_\lambda$ weakly in $H_0^1(\Omega)$. We claim that $u_\lambda \neq 0$. Arguing by contradiction, $u_n \rightharpoonup 0$ weakly in $H_0^1(\Omega)$ and $u_n \rightarrow 0$ strongly in $L^p(\Omega)$ ($1 \leq p < 6$). Then, we have

$$\int_\Omega f_\lambda(x) u_n^q dx = o(1). \quad (53)$$

Consequently, from (53), it holds that

$$\|u_n\|^2 - \int_\Omega |u_n|^6 dx = o(1). \quad (54)$$

Set $\lim_{n \rightarrow \infty} \|u_n\| = A$, note that $\int_\Omega |u_n|^6 dx \leq S^{-3} \|u_n\|^6$, it follows

$$A^2 \geq S^{3/2} \quad (55)$$

or

$$A = 0. \quad (56)$$

So it follows from (53) and (54) that

$$N = \lim_{n \rightarrow \infty} \left\{ \frac{1}{2} \|u_n\|^2 - \frac{1}{4} \int_\Omega \phi_{u_n} u_n^2 dx - \frac{1}{6} \int_\Omega |u_n|^6 dx - \frac{1}{q} \int_\Omega f_\lambda(x) u_n^q dx \right\} \quad (57)$$

$$= \lim_{n \rightarrow \infty} \frac{1}{4} \|u_n\|^2 = \frac{1}{4} A^2.$$

If $A = 0$, then $N = 0$, this contradicts to Theorem 6. Consequently, $N \geq (1/3)S^{3/2}$. According to Lemma 8, we have $(1/3)S^{3/2} \leq N < (1/3)S^{3/2} - D\lambda^{2/(2-q)}$, this is a contradiction. Consequently $u_n \rightharpoonup u_\lambda \neq 0$ in $H_0^1(\Omega)$. Moreover, by Lemma 5, we can prove that $u_n \rightharpoonup u_\lambda$ in $H_0^1(\Omega)$, and $(u_\lambda, \phi_{u_\lambda})$ is a positive solution of system (1) with $I_\lambda(u_n) \geq N$. Next, we prove that $I_\lambda(u_n) \leq N$. Indeed, since $I'_\lambda(u_\lambda) = 0$, it follows

$$I_\lambda(u_\lambda) = \frac{1}{2} \|u_\lambda\|^2 - \frac{1}{4} \int_\Omega \phi_{u_\lambda} u_\lambda^2 dx - \frac{1}{6} \int_\Omega |u_\lambda|^6 dx - \frac{1}{q} \int_\Omega f_\lambda(x) u_\lambda^q dx - \frac{1}{4} \left\{ \|u_\lambda\|^2 - \int_\Omega \phi_{u_\lambda} u_\lambda^2 dx - \int_\Omega |u_\lambda|^6 dx - \int_\Omega f_\lambda(x) u_\lambda^q dx \right\} \quad (58)$$

$$= \frac{1}{4} \|u_\lambda\|^2 - \left[\frac{1}{q} - \frac{1}{4} \right] \int_\Omega f_\lambda(x) u_\lambda^q dx.$$

In addition, it follows from (53) and the weak lower semi-continuity of the norm; it holds that

$$N = \lim_{n \rightarrow \infty} \left[\frac{1}{2} \|u_n\|^2 - \frac{1}{4} \int_\Omega \phi_{u_n} u_n^2 dx - \frac{1}{6} \int_\Omega |u_n|^6 dx - \frac{1}{q} \int_\Omega f_\lambda(x) u_n^q dx - \frac{1}{4} \left\{ \|u_n\|^2 - \int_\Omega \phi_{u_n} u_n^2 dx - \int_\Omega |u_n|^6 dx - \int_\Omega f_\lambda(x) u_n^q dx \right\} \right] \quad (59)$$

$$= \lim_{n \rightarrow \infty} \left[\frac{1}{2} \|u_\lambda\|^2 - \left(\frac{1}{q} - \frac{1}{4} \right) \int_\Omega f_\lambda(x) u_\lambda^q dx \right]$$

$$\geq \frac{1}{4} \|u_\lambda\|^2 - \left[\frac{1}{q} - \frac{1}{4} \right] \int_\Omega f_\lambda(x) u_\lambda^q dx.$$

Consequently, $u_\lambda \neq 0$ and $I_\lambda(u_\lambda) = N$, which implies that $(u_\lambda, \phi_{u_\lambda})$ is a positive ground state solution of system (1). Hence, we finish the proof of Theorem 1.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (No. 11661021).

References

- [1] H. Liu, "Positive solutions of an asymptotically periodic Schrödinger-Poisson system with critical exponent," *Nonlinear Analysis: Real World Applications*, vol. 32, pp. 198–212, 2016.
- [2] J. Sun and S. Ma, "Ground state solutions for some Schrödinger-Poisson systems with periodic potentials," *Journal of Differential Equations*, vol. 260, no. 3, pp. 2119–2149, 2016.
- [3] J. Zhang, "Ground state and multiple solutions for Schrödinger-Poisson equations with critical nonlinearity," *Journal of Mathematical Analysis and Applications*, vol. 440, no. 2, pp. 466–482, 2016.
- [4] Z. Liu, Z. Zhang, and S. Huang, "Existence and nonexistence of positive solutions for a static Schrödinger-Poisson-Slater equation," *Journal of Differential Equations*, vol. 266, no. 9, pp. 5912–5941, 2019.
- [5] S. Khoutir, "Infinitely many high energy radial solutions for a class of nonlinear Schrödinger-Poisson systems in $\mathbb{R}^3$," *Applied Mathematics Letters*, vol. 90, pp. 139–145, 2019.
- [6] E. H. Lieb, "Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation," *Studies in Applied Mathematics*, vol. 57, no. 2, pp. 93–105, 1977.
- [7] Z. Liu and S. Guo, "On ground state solutions for the Schrödinger-Poisson equations with critical growth," *Journal of Mathematical Analysis and Applications*, vol. 412, no. 1, pp. 435–448, 2014.
- [8] C. Y. Lei and J. F. Liao, "Multiple positive solutions for Schrödinger-Poisson system involving singularity and critical exponent," *Mathematical Methods in the Applied Sciences*, vol. 42, no. 7, pp. 2417–2430, 2019.
- [9] E. G. Murcia and G. Siciliano, "Least energy radial sign-changing solution for the Schrödinger-Poisson system in $\mathbb{R}^3$ under an asymptotically cubic nonlinearity," *Journal of Mathematical Analysis and Applications*, vol. 474, no. 1, pp. 544–571, 2019.
- [10] W. H. Xie, H. B. Chen, and H. X. Shi, "Multiplicity of positive solutions for Schrödinger-Poisson system with a critical nonlinearity in $\mathbb{R}^3$," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 42, pp. 2657–2680, 2019.
- [11] J. Wang, L. Tian, J. Xu, and F. Zhang, "Existence of multiple positive solutions for Schrödinger-Poisson systems with critical growth," *Zeitschrift für Angewandte Mathematik und Physik*, vol. 66, no. 5, pp. 2441–2471, 2015.
- [12] B. Almualemi, H. Chen, and S. Khoutir, "Existence of non-trivial solutions for Schrödinger-Poisson systems with critical exponent on bounded domains," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 42, no. 4, pp. 1675–1686, 2019.
- [13] C. O. Alves and G. M. Figueiredo, "Existence of positive solution for a planar Schrödinger-Poisson system with exponential growth," *Journal of Mathematical Physics*, vol. 60, no. 1, article 011503, 2019.
- [14] H. Fan, "Multiple positive solutions for a Schrödinger-Poisson involving concave-convex nonlinearities," *Electronic Journal of Differential Equations*, vol. 2019, no. 86, pp. 1–19, 2019.
- [15] A. Azzollini and P. d'Avenia, "On a system involving a critically growing nonlinearity," *Journal of Mathematical Analysis and Applications*, vol. 387, no. 1, pp. 433–438, 2012.
- [16] A. A. Pomponio, "Ground state solutions for the nonlinear Schrödinger-Maxwell equations," *Journal of Mathematical Analysis and Applications*, vol. 345, pp. 90–108, 2008.
- [17] A. Azzollini, P. d'Avenia, and G. Vaira, "Generalized Schrödinger-Newton system in dimension $N \geq 3$: Critical case," *Journal of Mathematical Analysis and Applications*, vol. 449, no. 1, pp. 531–552, 2017.
- [18] C. M. Chu and C. L. Tang, "Multiple results for critical quasi-linear elliptic systems involving concave-convex nonlinearities and sign-changing weight functions," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 36, pp. 789–805, 2013.
- [19] K. Teng, "Ground state solutions for the non-linear fractional Schrödinger-Poisson system," *Applicable Analysis*, vol. 98, no. 11, pp. 1959–1996, 2018.
- [20] S. Chen and X. Tang, "Ground state solutions of Schrödinger-Poisson systems with variable potential and convolution nonlinearity," *Journal of Mathematical Analysis and Applications*, vol. 473, no. 1, pp. 87–111, 2019.
- [21] Q. Zhang, "Existence of positive solution to Kirchhoff-Schrödinger-Poisson system with strong singular term," *Journal of Mathematical Physics*, vol. 60, no. 4, article 041504, 2019.
- [22] W. Gan and S. Liu, "Multiple positive solutions of a class of Schrödinger-Poisson equation involving indefinite nonlinearity in $\mathbb{R}^3$," *Applied Mathematics Letters*, vol. 93, pp. 111–116, 2019.
- [23] A. M. Batista and M. F. Furtado, "Positive and nodal solutions for a nonlinear Schrödinger-Poisson system with sign-changing potentials," *Nonlinear Analysis: Real World Applications*, vol. 39, pp. 142–156, 2018.
- [24] C.-Y. Lei and H. M. Suo, "Positive solutions for a Schrödinger-Poisson system involving concave-convex nonlinearities," *Computers & Mathematics with Applications*, vol. 74, no. 6, pp. 1516–1524, 2017.
- [25] C. Y. Lei and G. S. Liu, "Multiple Positive solutions for a Schrödinger-Newton system with sign-changing potential," *Computers & Mathematics with Applications*, vol. 77, no. 3, pp. 631–640, 2019.
- [26] H. Brézis and E. Lieb, "A relation between pointwise convergence of functions and convergence of functionals," *Proceedings of the American Mathematical Society*, vol. 88, no. 3, pp. 486–490, 1983.
- [27] H. Brézis and L. Nirenberg, "Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents," *Communications on Pure and Applied Mathematics*, vol. 36, no. 4, pp. 437–477, 1983.
- [28] A. Ambrosetti and P. H. Rabinowitz, "Dual variational methods in critical point theory and applications," *Journal of Functional Analysis*, vol. 14, no. 4, pp. 349–381, 1973.