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Research Article

A Remark on Isometries of Absolutely Continuous Spaces

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The purpose of this article is to study the isometries between vector-valued absolutely continuous function spaces, over compact subsets of the real line. Indeed, under certain conditions, it is shown that such isometries can be represented as a weighted composition operator.

1. Introduction

Suppose that $(V, \|\cdot\|)$ and $(W, \|\cdot\|)$ are Banach spaces (real or complex) and X and Y are compact subsets of the real line (with the induced distance and order from $\mathbf{R}$). A map $f\colon X \longrightarrow V$ is absolutely continuous, if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $\sum_{i=1}^{n} \|f(b_i) - f(a_i)\| < \varepsilon$, whenever $\{a_i, b_i\}_{1\le i\le n}$ is a (finite) sequence of mutually disjoint intervals in $\mathbf{R}$ whose end points are in X and $\sum_{i=1}^{n} |b_i - a_i| < \delta$. Also, the total variation of f (on X) is defined as

$$\mathcal{V}(f) \coloneqq \sup \sum_{i=1}^{n} \left\| f(x_{i+1}) - f(x_i) \right\|, \tag{1}$$

where the supremum is taken over all (finite) increasing sequences $\{x_i\}_{1\le i\le n}$ of points in X. The set of all absolutely continuous maps from X into V is denoted by $AC(X, V)$, and on this space, we consider the following norm:

$$\|f\|_{\mathcal{V}} \coloneqq \max\{\mathcal{V}(f), \|f\|_\infty\}, \tag{2}$$

where $\|f\|_\infty$ denotes the supremum norm of f (on X).

In this article, we study isometries on such function spaces. We attempt to characterize isometries on these function spaces, under certain conditions.

Suppose that $T\colon AC(X, V) \longrightarrow AC(Y, W)$ surjective linear isometry, where V and W are Banach spaces with trivial (one-dimensional) centralizers. We show that if T and T^{-1} have property $\mathcal{R}$, then T is a "weighted composition operator" of the form

$$Tf(y) = J(y)f(\phi(y)), \tag{3}$$

where $\phi\colon Y \longrightarrow X$ and its inverse are absolutely continuous, and J is an absolutely continuous map from Y into the space of surjective linear isometries from V into W. Compare with [1], Theorem 1.8 and [2], Theorem 5.3.

In particular, we present a direct proof for the main result of [3], which is somewhat shorter and more elementary than the one in [3]. Our proof is based on the principal ideas of [1] which convert the initial function space to the class of continuous maps (equipped with the supremum norm) and then appeal to the classical Banach–Stone theorem.

2. Preliminaries

We start this section with definitions of properties $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{R}$ for isometries between absolutely continuous function spaces. These properties are stated for isometries between Lipschitz spaces in [4], page 18, [5], Definition 2, and [2], Definition 2.1.

Suppose that $(V, \|\cdot\|)$ and $(W, \|\cdot\|)$ are Banach spaces and X and Y are complete metric spaces. Suppose that $T\colon AC(X, V) \longrightarrow AC(Y, W)$ is a linear map. It is said that T has property $\mathcal{P}$, if for every $y \in Y$, there exists $v \in V$ such that $T\widehat{v}(y) \ne 0$, where $\widehat{v} \in AC(X, V)$ denotes the constant map equal to v. Also, it is said that T has property $\mathcal{Q}$, if for every $y \in Y$ and $_s$, there exists $v \in V$ such that $T\widehat{v}(y) = w$. It is clear that if T has property $\mathcal{Q}$, then it has property $\mathcal{P}$ as well.

Definition 1. It is said that the linear map $T\colon AC(X,V) \longrightarrow AC(Y,W)$ has property $\mathscr{R}$, if for every $y \in Y$, $w \in W$ and $\varepsilon > 0$, there exists $v_{w,\varepsilon} \in V$ such that $\|v_{w,\varepsilon}\| \le \varepsilon$ and $\|w\| < \|w + T\widehat{v}_{w,\varepsilon}(y)\|$.

It is easy to see that property $\mathscr{R}$ is weaker than property $\mathscr{Q}$ and stronger than property $\mathscr{P}$. Also, if T has property $\mathscr{P}$ and W is strictly convex, then T has property $\mathscr{R}$.

Lemma 1. *Let X be a compact metric space, and let V be a Banach space. Let $f\colon X \longrightarrow V$ be a continuous map and $\varepsilon > 0$. Then, there exists a Lipschitz map $f_\varepsilon\colon X \longrightarrow V$ such that $\|f - f_\varepsilon\|_\infty \le \varepsilon$. In particular, when X is a compact subset of the real line, the set of all absolutely continuous maps $AC(X,V)$ is dense in the set of all continuous maps $C(X,V)$, equipped with the supremum norm.*

Proof 1. See [1], Lemma 1.6. Notice that every Lipschitz map is absolutely continuous, whenever X is a compact subset of the real line. □

Theorem 1. *Let X and Y be compact subsets of the real line, and let V and W be Banach spaces. Let $T\colon AC(X,V) \longrightarrow AC(Y,W)$ be a surjective linear isometry. In addition, suppose that both T and T^{-1} have property $\mathscr{R}$. Then, for every $f \in AC(X,V)$, we have $\|Tf\|_\infty = \|f\|_\infty$.*

Proof 2. It is similar to the proof of [2], Theorem 2.5, with minor modifications. □

3. Results

In this section, we characterize the surjective isometries between vector-valued absolutely continuous function spaces.

Theorem 2. *Let X and Y be compact subsets of the real line, and let V and W be Banach spaces with trivial (one-dimensional) centralizers (for definition see [5, 6]). Let $T\colon AC(X,V) \longrightarrow AC(Y,W)$ be a surjective linear isometry such that both T and T^{-1} have property $\mathscr{R}$. Then, T is a weighted composition operator of the form*

$$Tf(y) = J(y)f(\phi(y)), \tag{4}$$

for all $f \in AC(X,V)$ and $y \in Y$, where $\phi\colon Y \longrightarrow X$ and its inverse is absolutely continuous and J is an absolutely continuous map from Y into the space of surjective linear isometries from V into W.

Proof 3. Let $C(X,V)$ denote the set of all continuous maps from X into V, equipped with the supremum norm. By Theorem 1, we have

$$\|Tf\|_\infty = \|f\|_\infty, \tag{5}$$

for every $f \in AC(X,V)$. Then, by Lemma 1, we know that every continuous map can be uniformly approximated by absolutely continuous maps. So, we can easily extend T as a surjective linear isometry from $C(X,V)$ into $C(Y,W)$, and we denote this extension again by T.

By [6], Corollary 7.4.11, there exist a homeomorphism $\phi\colon Y \longrightarrow X$ and a map J from Y into the space of surjective linear isometries from V into W such that

$$Tf(y) = J(y)f(\phi(y)), \tag{6}$$

for all $f \in C(X,V)$ and $y \in Y$.

Now, consider the unit vector $v \in V$, and take f the constant function equal to v in the equation (6). Since $Tf \in AC(Y,W)$, we see that the map $y \longmapsto J(y)(v)$ is absolutely continuous and also, the total variation of this map is less than or equal to 1, uniformly with respect to v. This implies that the map $y \longmapsto J(y)$ is absolutely continuous.

On the other hand, it is clear that if $f \in AC(X,V)$, then $\|f\| \in AC(X,\mathbf{R})$. Define $f \in AC(X,V)$ as $f(x) := (x+\lambda)v$, where $v \in V$ is a unit vector and $\lambda > 0$ large enough such that $x + \lambda > 0$, for all $x \in X$. If we substitute such a function f in the equation (6), we see that ϕ is an absolutely continuous function (note that $\|J(y)(v)\| = 1$). Also, by considering T^{-1}, we can show that the inverse of ϕ is absolutely continuous as well. This completes the proof of Theorem 2.

It was noticed by the referee that Hosseini [[7], Theorem 2.1] has proven a similar result under a weaker assumption on T but under extra assumptions on Banach spaces V and W. Also, it is worth mentioning that during the proof of [7], Theorem 2.1, the author uses of [3], Theorem 4.1.

Finally, suppose that X and Y are compact subsets of the real line and V and W are strictly convex Banach spaces. Suppose that $T\colon AC(X,V) \longrightarrow AC(Y,W)$ is a surjective linear isometry such that T has property $\mathscr{P}$. Since every strictly convex Banach space has trivial centralizer (see [6], Theorem 7.4.14) and also by Lemma 3.6 and the proof of Lemma 3.7 in [3] (compare with [4], Remark 4.6), we see that T^{-1} has property $\mathscr{P}$ as well. Then, Theorem 2 implies that T is a "weighted composition operator" of the form (6). Notice that if T has property $\mathscr{P}$ and W is strictly convex, then T has property $\mathscr{R}$. Therefore, by [3], Lemmas 3.14 and 3.15, we are able to recover the main result of [3]. Our proof is somewhat shorter and more elementary than the one in [3]. □

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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