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Research Article

Some Properties of Kantorovich-Stancu-Type Generalization of Szász Operators including Brenke-Type Polynomials via Power Series Summability Method

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In this paper, we study the Kantorovich-Stancu-type generalization of Szász-Mirakyan operators including Brenke-type polynomials and prove a Korovkin-type theorem via the T-statistical convergence and power series summability method. Moreover, we determine the rate of the convergence. Furthermore, we establish the Voronovskaya- and Grüss-Voronovskaya-type theorems for T-statistical convergence.

1. Introduction and Preliminaries

Let $\mathscr{K} \subseteq \mathbb{N}$ (set of natural numbers) and $\mathscr{K}_m = \{i \le m : i \in \mathscr{K}\}$. Then, the natural density or we can say asymptotic density of $\mathscr{K}$ is defined by $\sigma(\mathscr{K}) = \lim_m (1/m)|\mathscr{K}_m|$ whenever the limit exists, where $|\mathscr{K}_m|$ denotes the cardinality of the set $\mathscr{K}_m$. A sequence $\eta = (\eta_i)$ is statistically convergent to $\mathscr{L}$ if for every $\varepsilon > 0$

$$\lim_m \frac{1}{m} |\{i \le m : |\eta_i - \mathscr{L}| \ge \varepsilon\}| = 0, \qquad (1)$$

and we write $st - \lim_m \eta_m = \mathscr{L}$.

Let $T = (t_{nj})$ be a matrix and $\eta = (\eta_j)$ be a sequence. The $T-$ transform of the sequence $\eta = (\eta_n)$ is defined by $T\eta = (T_n(\eta))$, $(T\eta)_n = \sum_j t_{nj}\eta_j$ if the series converges for every $n \in \mathbb{N}$. We say that η is $T-$ summable to the number $\mathscr{L}$ if $(T\eta)_n$ converges to $\mathscr{L}$. The summability matrix T is regular whenever $\lim_j \eta_j = \mathscr{L} = \lim_n (T\eta)_n$.

Let $T = (t_{nj})$ be a regular matrix. A sequence $\eta = (\eta_j)$ is said to be T -statistically convergent (see [1]) to real number $\mathscr{L}$ if for any >0, $\lim_n \sum_{j:|\eta_j - \mathscr{L}| \ge \varepsilon} t_{nj} = 0$, and we write $st_T - \lim \eta = \mathscr{L}$. If T is Cesàro matrix of order 1, then T -statistical convergence is reduced to the statistical convergence.

In this paper, we also use the power series summability method which includes several known summability methods such as Abel and Borel (see [2–9]). Note that the power method is more effective than the ordinary convergence (see [10]).

Let (p_j) be a sequence of real numbers such that $p_0 > 0$, $p_1, p_2, \cdots \ge 0$, and the corresponding power series $p(u) = \sum_{j=0}^{\infty} p_j u^j$ has radius of convergence R with $0 < R \le \infty$. If $\lim_{u \to R^-} (1/p(u)) \sum_{j=0}^{\infty} \eta_j p_j u^j = L$ for all $t \in (0, R)$, then we say that $\eta = (\eta_j)$ is convergent in the sense of power series method (see [11, 12]). Define $A(u)B(\eta u)=\sum_{k\geq 0}p_k(\eta)u^k$, where $A(u)=\sum_{j\geq 0}a_j u^j$ and $B(u)=\sum_{j\geq 0}b_j u^j$ are analytical functions such that $a_0\neq 0$ and $b_j\neq 0$ for all $j\geq 0$ (see [13]). Clearly, $p_k(\eta)=\sum_{j=0}^{k}a_{k-j}b_j\eta^j$. Moreover, the power series method is regular if and only if $\lim_{u\to R^-}(p_j u^j/p(u))=0$ holds for each $j\in\{0,1,\cdots\}$ (see [14]).

We study a Korovkin-type theorem for the Kantorovich-Stancu-type Szász-Mirakyan operators via power series method. We determine the rate of convergence for these operators. Furthermore, we give a Voronovskaya-type theorem for $T-$ statistical convergence. Such type of operators is widely studied by several authors (see [15–19]).

We start by recalling the class of Kantorovich-Stancu-type generalization of Szász-Mirakyan operators, including Brenke-type polynomials. For every $h\in C_B[0,\infty)=C[0,\infty)\cap E$,

$$K_n^{\alpha,\beta}(h,\eta)=\frac{n+\beta}{A(1)B(nx)}\sum_{k=0}^{\infty}p_k(n\eta)\int_{(k+\alpha)/(n+\beta)}^{(k+1+\alpha)/(n+\beta)}h(u)du, \quad (2)$$

for $n\in\mathbb{N}$, $x\in 0,\infty)$ and $0\leq\alpha\leq\beta$, where $E=\{h: x\in[0,\infty), \lim_{x\to\infty}(h(x)/(1+x^2))<\infty\}$. In what follows, we calculate the moments and central moments for Kantorovich-Stancu of Szász-Mirakyan operators. Let us mention some properties of the functions $A(u)$ and $B(u)$ (see [13, 20]).

(1) $A(1)\neq 0$, $(a_{k-r}b_r/A(1))\geq 0$, for all $0\leq r\leq k$ and $k=0,1,2,\cdots$

(2) $B:[0,\infty)\longrightarrow(0,\infty)$

(3) Series $A(u)=\sum_{r=0}^{\infty}a_r u^r$, for $a_0\neq 0$, and $B(u)=\sum_{r=0}^{\infty}b_r u^r$, for $b_r\neq 0$ $(r\geq 0)$, are convergent for $|u|<R$ $(R>1)$ and $A(u), B(u)$ are analytic functions

The next lemma is followed immediately from the fact that $A(u)B(\eta u)=\sum_{k\geq 0}p_k(\eta)u^k$.

Lemma 1. *Let D be the operator $u(d/du)$. For all $m\geq 0$,*

$$\sum_{k\geq 0}k^m p_k(\eta)t^k=D^m(A(u)B(\eta u)). \quad (3)$$

For example, Lemma 1 for $m=0,1,2,3,4$ gives

$$\sum_{k\geq 0}p_k(\eta)=A(1)B(\eta),$$

$$\sum_{k\geq 0}kp_k(\eta)=uA^{(1)}(u)B(\eta u)+\eta uA(t)B^{(1)}(\eta u),$$

$$\begin{aligned}\sum_{k\geq 0}k^2p_k(\eta)&=u^2A^{(2)}(u)B(\eta u)+A^{(1)}(u)\left(uB(\eta u)+2\eta u^2B^{(1)}(\eta u)\right)\\&\quad+A(u)\left(\eta uB^{(1)}(\eta u)+\eta^2u^2B^{(2)}(\eta u)\right),\end{aligned}$$

$$\begin{aligned}\sum_{k\geq 0}k^3p_k(\eta)&=u^3A^{(3)}(u)B(\eta u)+A^{(2)}(u)\left(3u^2B(\eta u)+3\eta u^3B^{(1)}(\eta u)\right)\\&\quad+A^{(1)}(u)\left(uB(\eta u)+6\eta u^2B^{(1)}(\eta u)+3\eta^2u^3B^{(2)}(\eta u)\right)\\&\quad+A(u)\left(\eta uB^{(1)}(\eta u)+3\eta^2u^2B^{(2)}(\eta u)+\eta^3u^3B^{(3)}(\eta u)\right),\end{aligned}$$

$$\begin{aligned}\sum_{k\geq 0}k^4p_k(\eta)&=u^4A^{(4)}(u)B(\eta u)+A^{(3)}(u)\left(6u^3B(\eta u)+4\eta u^4B^{(1)}(\eta u)\right)\\&\quad+A^{(2)}(u)\left(7u^2B(\eta u)+18\eta u^3B^{(1)}(\eta t)+6\eta^2u^4B^{(2)}(\eta u)\right)\\&\quad+A^{(1)}(u)\left(uB(\eta u)+14\eta u^2B^{(1)}(\eta u)+18\eta^2u^3B^{(2)}(\eta u)\right.\\&\quad\left.+4\eta^3u^4B^{(3)}(\eta u)\right)+A(u)\left(\eta uB^{(1)}(\eta u)+7\eta^2u^2B^{(2)}(\eta u)\right.\\&\quad\left.+6\eta^3u^3B^{(3)}(\eta u)+\eta^4u^4B^{(4)}(\eta u)\right).\end{aligned} \quad (4)$$

Theorem 2. *Let $e_i=e_i(t)=t^i$ for all $i\geq 0$ and let D be the operator $t(d/dt)$. Then,*

$$\begin{aligned}K_n^{\alpha,\beta}(e_i,x)&=\frac{(n+\beta)^{-i}}{(i+1)A(1)B(nx)}\sum_{j=0}^{i+1}\binom{i+1}{j}\left((\alpha+1)^{i+1-j}\right.\\&\quad\left.-\alpha^{i+1-j}\right)D^jA(t)B(nxt)|_{t=1}.\end{aligned} \quad (5)$$

Proof. By the definition of the operators, we have

$$\begin{aligned}K_n^{\alpha,\beta}(e_i,x)&=\frac{n+\beta}{A(1)B(nx)}\sum_{k\geq 0}p_k(nx)\int_{(k+\alpha)/(n+\beta)}^{(k+1+\alpha)/(n+\beta)}t^i dt\\&=\frac{n+\beta}{A(1)B(nx)}\sum_{k\geq 0}p_k(nx)\\&\quad\cdot\left(\frac{(k+1+\alpha)^{i+1}}{(i+1)(n+\beta)^{i+1}}-\frac{(k+\alpha)^{i+1}}{(i+1)(n+\beta)^{i+1}}\right)\\&=\frac{(n+\beta)^{-i}}{(i+1)A(1)B(nx)}\sum_{k\geq 0}\sum_{j=0}^{i+1}\binom{i+1}{j}k^jp_k(nx)\\&\quad\cdot\left((\alpha+1)^{i+1-j}-\alpha^{i+1-j}\right)\\&=\frac{(n+\beta)^{-i}}{(i+1)A(1)B(nx)}\sum_{j=0}^{i+1}\binom{i+1}{j}\\&\quad\cdot\left((\alpha+1)^{i+1-j}-\alpha^{i+1-j}\right)\sum_{k\geq 0}k^jp_k(nx).\end{aligned} \quad (6)$$

Thus, by Lemma 1, we complete the proof.

Lemma 3 (for instance, see [21], equation (1.27)). *Let X, L be two operators on the set of functions defined by $X(f(u))=uf(u)$ and $L(f(u))=(d/du)f(u)$. Then,*

$$(XL)^m=\sum_{j=1}^{m}S(m,j)X^jL^j. \quad (7)$$

Moreover,

$$(XL)^m(f(u)g(u)) = \sum_{j=1}^{m}\sum_{i=0}^{j}\binom{j}{i}S(m,j)u^j\frac{d^i}{du^i}f(u) \cdot \frac{d^{j-i}}{du^{j-i}}g(u), \quad (8)$$

where $S(m,j)$ is the Stirling number of the second kind.

Define $a_j = (d^j/dt^j)A(t)|_{t=1}$ and $b_j = (d^j/dt^j)B(t)|_{t=nx}$, for all $j \geq 0$. Therefore, Theorem 2 with $D = XL$ and Lemma 3 imply the following theorem.

Theorem 4. *Let $e_i = e_i(t) = t^i$ for all $i \geq 0$. Then,*

$$\begin{aligned} K_n^{\alpha,\beta}(e_i,x) &= \frac{(n+\beta)^{-i}}{i+1}\left((\alpha+1)^{i+1} - \alpha^{i+1}\right) \\ &+ \frac{(n+\beta)^{-i}}{i+1}\sum_{j=1}^{i+1}\sum_{\ell=1}^{j}\sum_{s=0}^{\ell}\binom{i+1}{j}\binom{\ell}{s} \\ &\cdot S(j,\ell)\left((\alpha+1)^{i+1-j} - \alpha^{i+1-j}\right)\frac{n^{\ell-s}a_s b_{\ell-s}}{a_0 b_0}x^{2\ell-s}, \end{aligned} \quad (9)$$

where $S(m,\ell)$ is the Stirling number of the second kind.

Example 5. By applying Theorem 4 for $i = 0,1,2,3,4$ with using (2), we obtain

$$\begin{aligned} K_n^{\alpha,\beta}(e_0,x) &= 1, \\ K_n^{\alpha,\beta}(e_1,x) &= \frac{2\alpha+1}{2(n+\beta)} + \frac{a_1}{a_0(n+\beta)} + \frac{nb_1}{b_0(n+\beta)}x, \\ K_n^{\alpha,\beta}(e_2,x) &= \frac{3\alpha^2+3\alpha+1}{3(n+\beta)^2} + \frac{2a_1(\alpha+1)+a_2}{a_0(n+\beta)^2} + \frac{2nb_1(a_0(\alpha+1)+a_1)}{a_0b_0(n+\beta)^2}x + \frac{n^2b_2}{b_0(n+\beta)^2}x^2, \\ K_n^{\alpha,\beta}(e_3,x) &= \frac{4\alpha^3+6\alpha^2+4\alpha+1}{4(n+\beta)^3} + \frac{6a_1(\alpha+1)^2+6a_2(\alpha+1)+a_1+3a2+2a_3}{2a_0(n+\beta)^3} + \frac{nb_1\left(6a_0(\alpha+1)^2+12a_1(\alpha+1)+a_0+6a_1+6a_2\right)}{2a_0b_0(n+\beta)^3}x \\ &+ \frac{3n^2b_2(2a_0(\alpha+1)+a_0+2a_1)}{2a_0b_0(n+\beta)^3} + \frac{n^3b_3}{b_0(n+\beta)^3}x^3, \\ K_n^{\alpha,\beta}(e_4,x) &= \frac{5\alpha^4+10\alpha^3+10\alpha^2+5\alpha+1}{5(n+\beta)^4} + \frac{4a_1(\alpha+1)^3+6a_2(\alpha+1)^2+2(a_1+3a_2+2a_3)(\alpha+1)+3a_2+4a_3+a_4}{a_0(n+\beta)^4} \\ &+ \frac{2nb_1\left(2a_0(\alpha+1)^3+6a_1(\alpha+1)^2+(a_0+6a_1+6a_2)(\alpha+1)+3a_1+6a_2+2a_3\right)}{a_0b_0(n+\beta)^4}x \\ &+ \frac{3n^2b_2\left(2a_0(\alpha+1)^2+2(a_0+2a_1)(\alpha+1)+a_0+4a_1+2a_2\right)}{a_0b_0(n+\beta)^4}x^2 + \frac{4n^3b_3(a_0(\alpha+1)+a_0+a_1)}{a_0b_0(n+\beta)^4}x^3 \\ &+ \frac{n^4b_4}{b_0(n+\beta)^4}x^4. \end{aligned} \quad (10)$$

Theorem 6. *Let $y = \alpha - x(n+\beta)$, and let D be the operator $t(d/dt)$. Then,*

$$\begin{aligned} K_n^{\alpha,\beta}\left((t-x)^i,x\right) &= \frac{(n+\beta)^{-i}}{(i+1)A(1)B(nx)}\sum_{j=0}^{i+1}\binom{i+1}{j} \\ &\cdot \left((1+y)^{i+1-j} - y^{i+1-j}\right)D^jA(t)B(nxt)|_{t=1}. \end{aligned} \quad (11)$$

Proof. By the definitions, we have

$$\begin{aligned} K_n^{\alpha,\beta}\left((t-x)^i,x\right) &= \frac{n+\beta}{A(1)B(nx)}\sum_{k\geq 0}p_k(nx)\int_{(k+\alpha)/(n+\beta)}^{(k+1+\alpha)/(n+\beta)}(t-x)^i dt \\ &= \frac{n+\beta}{A(1)B(nx)}\sum_{k\geq 0}p_k(nx) \\ &\cdot \left(\frac{(k+1+y)^{i+1}}{(i+1)(n+\beta)^{i+1}} - \frac{(k+y)^{i+1}}{(i+1)(n+\beta)^{i+1}}\right) \end{aligned}$$

$$= \frac{(n+\beta)^{-i}}{(i+1)A(1)B(nx)} \sum_{k\geq 0} \sum_{j=0}^{i+1} \binom{i+1}{j} \cdot k^j p_k(nx) \left((1+y)^{i+1-j} - y^{i+1-j}\right) = \frac{(n+\beta)^{-i}}{(i+1)A(1)B(nx)} \sum_{j=0}^{i+1} \binom{i+1}{j} \cdot \left((1+y)^{i+1-j} - y^{i+1-j}\right) \sum_{k\geq 0} k^j p_k(nx). \quad (12)$$

Thus, Lemma 1 completes the proof.

By Theorem 6 and Lemma 3, we obtain the following result.

Theorem 7. *Let $y = \alpha - x(n+\beta)$. Then,*

$$K_n^{\alpha,\beta}\left((t-x)^i, x\right) = \frac{(n+\beta)^{-i}}{i+1}\left((y+1)^{i+1} - y^{i+1}\right) + \frac{(n+\beta)^{-i}}{i+1} \sum_{j=1}^{i+1} \sum_{\ell=1}^{j} \sum_{s=0}^{\ell} \binom{i+1}{j} \binom{\ell}{s} \cdot S(j,\ell)\left((y+1)^{i+1-j} - y^{i+1-j}\right) \frac{n^{\ell-s} a_s b_{\ell-s}}{a_0 b_0} x^{2\ell-s}, \quad (13)$$

where $S(m,\ell)$ is the Stirling number of the second kind.

Remark 8. By applying Theorem 7 for $i = 0, 1, 2, 3$, we obtain

$$\begin{aligned}
K_n^{\alpha,\beta}(1,x) &= 1,\\
K_n^{\alpha,\beta}(t-x,x) &= \frac{2a_0\alpha + a_0 + 2a_1}{2a_0(n+\beta)} - \frac{(b_0 n + b_0\beta - b_1 n)}{b_0(n+\beta)} x,\\
K_n^{\alpha,\beta}\left((t-x)^2,x\right) &= \frac{3a_0\alpha^2 + 3a_0\alpha + 6a_1\alpha + a_0 + 6a_1 + 3a_2}{3a_0(n+\beta)^2} - \frac{2a_0b_0\alpha(n+\beta) - 2a_0b_1n(\alpha+1) + a_0b_0(n+\beta) + 2a_1b_0(n+\beta) - 2a_1b_1n}{a_0b_0(n+\beta)^2} x\\
&\quad + \frac{b_0n^2 + 2b_0n\beta + b_0\beta^2 - 2b_1n^2 - 2b_1n\beta + b_2n^2}{b_0(n+\beta)^2} x^2,\\
K_n^{\alpha,\beta}\left((t-x)^3,x\right) &= \frac{4a_0\alpha^3 + 6a_0\alpha^2 + 12a_1\alpha^2 + 4a_0\alpha + 24a_1\alpha + 12a_2\alpha + a_0 + 14a_1 + 18a_2 + 4a_3}{4a_0(n+\beta)^3}\\
&\quad - \frac{6a_0b_0n\alpha^2 + 6a_0b_0\alpha^2\beta - 6a_0b_1n\alpha^2 + 6a_0b_0n\alpha + 6a_0b_0\alpha\beta}{2a_0b_0(n+\beta)^3} x\\
&\quad - \frac{-12a_0b_1n\alpha + 12a_1b_0n\alpha + 12a_1b_0\alpha\beta - 12a_1b_1n\alpha + 2a_0b_0n + 2a_0b_0\beta}{2a_0b_0(n+\beta)^3} x\\
&\quad - \frac{-7a_0b_1n + 12a_1b_0n + 12a_1b_0\beta - 18a_1b_1n + 6a_2b_0n + 6a_2b_0\beta - 6a_2b_1n}{2a_0b_0(n+\beta)^3} x\\
&\quad + \frac{3\left(2a_0b_0n^2\alpha + 4a_0b_0n\alpha\beta + 2a_0b_0\alpha\beta^2 - 4a_0b_1n^2\alpha - 4a_0b_1n\alpha\beta + 2a_0b_2n^2\alpha\right)}{2a_0b_0(n+\beta)^3} x^2\\
&\quad + \frac{3\left(a_0b_0n^2 + 2a_0b_0n\beta + a_0b_0\beta^2 - 4a_0b_1n^2 - 4a_0b_1n\beta + 3a_0b_2n^2 + 2a_1b_0n^2\right)}{2a_0b_0(n+\beta)^3} x^2\\
&\quad - \frac{b_0n^3 + 3b_0n^2\beta + 3b_0n\beta^2 + b_0\beta^3 - 3b_1n^3 - 6b_1n^2\beta - 3b_1n\beta^2 + 3b_2n^3 + 3b_2n^2\beta - b_3n^3}{b_0(n+\beta)^3} x^3.
\end{aligned} \quad (14)$$

Theorem 6 for $i = 0, 1, \cdots, 6$ (with the help of mathematical programming), we obtain the following result.

Proposition 9. *Let us consider that*

$$\lim_{n\to\infty} \frac{B^{(m)}(nx)}{B(nx)} = 1 \quad \text{for} \quad m = 0, 1, 2, \cdots, 6, \quad (15)$$

where $B^{(m)}(t)$ is the m-th derivative of $B(t)$. Then, we obtain

$$\lim_{n\to\infty} K_n^{\alpha,\beta}(1,x) = 1$$

$$\lim_{n\to\infty} n K_n^{\alpha,\beta}\left((t-x)^1, x\right) = a_1/a_0 + 1/2 + \alpha - \beta x$$

$$\lim_{n\to\infty} n K_n^{\alpha,\beta}\left((t-x)^2, x\right) = x$$

Table 1: The values of the functions $20K_{20}^{\alpha,\beta}(f;x)$ and $100K_{100}^{\alpha,\beta}(f;x)$ at $x = 0, 0.1, 0.2, \cdots, 1$.

x	0	0.1	0.2	0.3	0.4	0.5	0.6
$20K_{20}^{\alpha,\beta}(f;x)$	0.01511	0.10173	0.18926	0.27770	0.36704	0.45729	0.54845
$100K_{100}^{\alpha,\beta}(f;x)$	0.00326	0.10041	0.19775	0.29529	0.39303	0.49096	0.58909

$$\lim_{n\to\infty} n^2 K_n^{\alpha,\beta}\left((t-x)^3, x\right) = 3\alpha + 5/2 + 3a_1/a_0 - 3\beta x^2$$

$$\lim_{n\to\infty} n^2 K_n^{\alpha,\beta}\left((t-x)^4, x\right) = 3x^2$$

$$\lim_{n\to\infty} n^3 K_n^{\alpha,\beta}\left((t-x)^5, x\right) = (15\alpha + 35/2 + 15a_1/a_0)x^2 - 15\beta x^3$$

$$\lim_{n\to\infty} n^3 K_n^{\alpha,\beta}\left((t-x)^6, x\right) = 15x^3. \quad (16)$$

Example 10. Let $A(t) = 1$, $B(t) = e^t$, $\alpha = 0$, $\beta = 1$, and $f(t) = (t-x)^2$. By the fact that $A(t)B(xt) = \sum_{k\geq 0} p_k(x)t^k$, we have that $p_k(x) = x^k/k!$.

Table 1 presents the values of the functions $nK_n^{\alpha,\beta}(f;x)$ and x at $x = 0, 0.1, 0.2, \cdots, 0.6$ and $n = 20$, $n = 100$, where we approximated $K_n^{\alpha,\beta}(f;x)$ as

$$K_n^{\alpha,\beta}(f,x) = \frac{n+\beta}{A(1)B(nx)} \sum_{k=0}^{3000} p_k(nx) \int_{(k+\alpha)/(n+\beta)}^{(k+1+\alpha)/(n+\beta)} f(t)dt. \quad (17)$$

We note that the Korovkin-type theorems are very useful tools in approximation which were studied in several function spaces [3–8, 10, 22–29]. We say that sequence of operators $K_n^{\alpha,\beta}$ converges to L in the sense of power series if

$$\lim_{u\to R^-} \frac{1}{p(u)} \sum_{n=0}^{\infty} K_n^{\alpha,\beta}(f,x) p_n u^n = L, \quad (18)$$

for every $u \in (0, R)$.

2. Main Results

We study here $T-$ statistical convergence of the operators $K_n^{\alpha,\beta}$. Note that the Korovkin-type theorem for $T-$ statistical convergence was considered in [24] as follows:

Theorem 11. *Let (B_j) be a sequence of positive linear operators on $C[0,1]$ and let $T = (t_{nj})$ be a nonnegative regular summability matrix such that*

$$st_T - \lim_n \left\| B_j e_i - e_i \right\| = 0, i = 0, 1, 2. \quad (19)$$

Then, for any $f \in C[0,1]$

$$st_T - \lim_n \left\| B_j h - h \right\| = 0, \quad (20)$$

where $\|h\| = \max_{0\leq x\leq 1} |h(x)|$.

Based on the above theorem, we give the following result.

Theorem 12. *Let $T = (t_{nj})$ be a regular matrix and $(K_n^{\alpha,\beta})$ be as in (2) on $C[0,M] \cap E$ such that $\lim_{n\to\infty}(B^{(i)}(nx))/(B(nx)) = 1$, where $B^{(i)}(nx)$ denotes i^{th} derivative and*

$$st_A - \lim_n \left\| K_n^{\alpha,\beta} e_i - e_i \right\| = 0\ (i = 1, 2). \quad (21)$$

Then, for any $h \in C[0,M] \cap E$

$$st_A - \lim_n \left\| K_n^{\alpha,\beta} h - h \right\| = 0, \quad (22)$$

where $\|h\| = \max_{t\in[0,M]} |h(t)|$.

Proof. From Lemma 5, we have that $st_T - \lim_n \|K_n^{\alpha,\beta} e_0 - e_0\| = 0$. Now, we will estimate the following expressions:

$$\left\| K_n^{\alpha,\beta} e_1 - e_1 \right\| \leq \left\| x\left(\frac{n}{n+\beta} \cdot \frac{B'(nx)}{B(nx)} - 1\right) \right\| + \left\| \frac{A'(1)}{(n+\beta)A(1)} \right\| + \left\| \frac{2\alpha+1}{2(n+\beta)} \right\|. \quad (23)$$

Note that $\lim_{n\to\infty}(B'(nx))/(B(nx)) = 1$. So from the last two relations we have that $\|K_n^{\alpha,\beta} e_1 - e_1\| = 0$. Moreover,

$$\left\| K_n^{\alpha,\beta} e_2 - e_2 \right\| = \left\| \left(\frac{n}{n+\beta}\right)^2 \frac{B''(nx)}{B(nx)} \cdot x^2 + \frac{nB'(nx)\left[2A'(1) + (2\alpha+2)A(1)\right]}{(n+\beta)^2 A(1)B(nx)} x + \frac{1}{(n+\beta)^2 A(1)} \left\{ A''(1) + (2\alpha+2)A'(1) \cdot \left(\alpha^2 + \alpha + \frac{1}{3}\right) A(1) \right\} - x^2 \right\| \longrightarrow 0. \quad (24)$$

Now proof follows directly from Theorem 11.

This theorem is an extension of some known results for the Kantorovich-Stancu-type Szász-Mirakyan operators.

Example 13 (see [6]). Under conditions given in Theorem 12, we define the following operators

$$N_n(h,x) = (1+x_n)K_n^{\alpha,\beta}(h,x), \tag{25}$$

where the sequence (x_n) is given as follows:

$$(x_n) = \begin{cases} \frac{1}{n^3}; & m^2 - m \le n \le m^2 - 1 \\ \frac{1}{n^4}; & n = m^2;\, m \in \mathbb{N}\setminus\{1\} \\ 0; & \text{otherwise,} \end{cases} \tag{26}$$

then

$$\begin{aligned} N_n(e_0,x) &= (1+x_n), \\ N_n(e_1,x) &= (1+x_n)\left(\frac{2\alpha+1}{2(n+\beta)} + \frac{a_1}{a_0(n+\beta)} + \frac{nb_1}{b_0(n+\beta)}x\right), \\ N_n(e_2,x) &= (1+x_n)\left(\frac{3\alpha^2+3\alpha+1}{3(n+\beta)^2} + \frac{2a_1(\alpha+1)+a_2}{a_0(n+\beta)^2}\right. \\ &\quad \left. + \frac{2nb_1(a_0(\alpha+1)+a_1)}{a_0b_0(n+\beta)^2}x + \frac{n^2b_2}{b_0(n+\beta)^2}x^2\right). \end{aligned} \tag{27}$$

By Theorem 11 we obtain $st_T - \lim_n \|N_nh - h\| = 0$, but the operators $N_n(h,x)$ do not satisfy Theorem 12. Hence, the sequence (N_n) is not statistically convergent but it is $T-$ statistically convergent.

Remark 14. The sequence (x_n) is not statistically convergent and hence not convergent. As an example, consider the Cesáro matrix of order 2.

$$T = (t_{nk}) = \begin{cases} \frac{2(n+1-k)}{(n+1)(n+2)}; & 0 \le k \le n, \\ 0; & k > n, \end{cases} \tag{28}$$

where

$$\begin{aligned} 0 \le \lim_n \sum_{k:|x_k-\alpha|\ge\varepsilon} t_{nk} &= \lim_n \sum_{\substack{k=m^2-m,\cdots,m^2-1 \\ k=m^2;m\in\mathbb{N}\setminus\{1\}}} t_{nk} \\ &= \lim_n \frac{2}{(n+1)(n+2)}[1+\cdots+n] \\ &\le \lim_n \frac{2}{(n+1)(n+2)} \cdot \frac{n(n+1)}{2} = 1. \end{aligned} \tag{29}$$

This proves that $x = (x_n)$ is $T-$ statistically convergent. We have

$$\begin{aligned} N_n(e_0,x) &= (1+x_n), \\ N_n(e_1,x) &= (1+x_n)\left(\frac{2\alpha+1}{2(n+\beta)} + \frac{a_1}{a_0(n+\beta)} + \frac{nb_1}{b_0(n+\beta)}x\right), \\ N_n(e_2,x) &= (1+x_n)\left(\frac{3\alpha^2+3\alpha+1}{3(n+\beta)^2} + \frac{2a_1(\alpha+1)+a_2}{a_0(n+\beta)^2}\right. \\ &\quad \left. + \frac{2nb_1(a_0(\alpha+1)+a_1)}{a_0b_0(n+\beta)^2}x + \frac{n^2b_2}{b_0(n+\beta)^2}x^2\right). \end{aligned} \tag{30}$$

By Example 13, this shows that $N_n(h,x)$ does not satisfy Theorem 12.

In [27, 29], Korovkin-type theorems are proved by Abel summability method. Now, we discuss for power series method. Let $B[0,\infty)$ ($C[0,\infty)$) be the space of all bounded (continuous) functions on the interval $[0,\infty)$.

Theorem 15. *Let $(K_n^{\alpha,\beta})$ be a sequence of positive linear operators from $C[0,M]\cap E$ into $B[0,M]\cap E$ such that*

$$\lim_{t\to R^-} \frac{1}{p(t)} \left\| \sum_{n=0}^{\infty} \left(K_n^{\alpha,\beta}e_i - e_i\right)p_nt^n \right\| = 0,\ i = 0,1,2. \tag{31}$$

Then, for $\mathfrak{h} \in C[0,M]\cap E$,

$$\lim_{t\to R^-} \frac{1}{p(t)} \left\| \sum_{n=0}^{\infty} \left(K_n^{\alpha,\beta}\mathfrak{h} - \mathfrak{h}\right)p_nt^n \right\| = 0. \tag{32}$$

Proof. Clearly, from (32) follows (31). Now, we show the converse that (31) implies (32). Let $\mathfrak{h} \in C[0,M]\cap E$, then there exists a constant $K > 0$ such that $|\mathfrak{h}(u)| \le K$ for all $u \in 0, M]$. Therefore,

$$|\mathfrak{h}(u) - \mathfrak{h}(x)|[\in 0, M]. \tag{33}$$

For every given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|\mathfrak{h}(u) - \mathfrak{h}(x)| \le \varepsilon, \tag{34}$$

whenever $|u-x| < \delta$ for all $u \in 0, M]$. Define $\psi \equiv \psi(u,x) = (u-x)^2$. If $|u-x| \ge \delta$, then

$$|\mathfrak{h}(u) - \mathfrak{h}(x)| \le \frac{2K}{\delta^2}\psi(u,x). \tag{35}$$

From (33)–(35), we have that $|\mathfrak{h}(u) - \mathfrak{h}(x)| < \varepsilon + (2K/\delta^2)\psi(u,x)$, namely,

$$-\varepsilon - \frac{2K}{\delta^2}\psi(u,x) < \mathfrak{h}(t) - \mathfrak{h}(x) < \frac{2K}{\delta^2}\psi(u,x) + \varepsilon. \tag{36}$$

By applying the operator $K_n^{\alpha,\beta}(1,x)$, $K_n^{\alpha,\beta}(1,x)$ is a monotone and linear operator, we obtain

$$\begin{aligned} K_n^{\alpha,\beta}(1,x)\left(-\varepsilon-\frac{2K}{\delta^2}\psi\right) &< K_n^{\alpha,\beta}(1,x)(\mathfrak{h}(u)-\mathfrak{h}(x)) \\ &< K_n^{\alpha,\beta}(1,x)\left(\frac{2K}{\delta^2}\psi+\varepsilon\right), \end{aligned} \tag{37}$$

which implies

$$\begin{aligned} &-\varepsilon K_n^{\alpha,\beta}(1,x)-\frac{2K}{\delta^2}K_n^{\alpha,\beta}(\psi(u),x) \\ &\quad< K_n^{\alpha,\beta}(\mathfrak{h},x)-\mathfrak{h}(x)K_n^{\alpha,\beta}(1,x) \\ &\quad< \frac{2K}{\delta^2}K_n^{\alpha,\beta}(\psi(u),x)+\varepsilon K_n^{\alpha,\beta}(1,x). \end{aligned} \tag{38}$$

On the other hand,

$$\begin{aligned} K_n^{\alpha,\beta}(\mathfrak{h},x)-\mathfrak{h}(x) &= K_n^{\alpha,\beta}(\mathfrak{h},x)-\mathfrak{h}(x)K_n^{\alpha,\beta}(1,x) \\ &\quad+\mathfrak{h}(x)\left[K_n^{\alpha,\beta}(1,x)-1\right]. \end{aligned} \tag{39}$$

From (38) and (39) we get

$$\begin{aligned} K_n^{\alpha,\beta}(\mathfrak{h},x)-\mathfrak{h}(x) &< \frac{2K}{\delta^2}K_n^{\alpha,\beta}(\psi(u),x)+\varepsilon K_n^{\alpha,\beta}(1,x) \\ &\quad+\mathfrak{h}(x)\left[K_n^{\alpha,\beta}(1,x)-1\right]. \end{aligned} \tag{40}$$

Now, we estimate the following expression:

$$\begin{aligned} K_n^{\alpha,\beta}(\psi(u),x) &= K_n^{\alpha,\beta}\left((x-u)^2,x\right)=K_n^{\alpha,\beta}\left(\left(x^2-2xu+u^2\right),x\right) \\ &= x^2K_n^{\alpha,\beta}(1,x)-2xK_n^{\alpha,\beta}(u,x)+K_n^{\alpha,\beta}\left(u^2,x\right). \end{aligned} \tag{41}$$

By (40), we obtain

$$\begin{aligned} &K_n^{\alpha,\beta}(\mathfrak{h},x)-\mathfrak{h}(x) \\ &\quad< \frac{2K}{\delta^2}\left\{x^2\left[K_n^{\alpha,\beta}(1,x)-1\right]-2x\left[K_n^{\alpha,\beta}(u,x)-x\right]\right. \\ &\qquad\left.+\left[K_n^{\alpha,\beta}\left(u^2,x\right)-x^2\right]\right\}+\varepsilon K_n^{\alpha,\beta}(1,x) \\ &\qquad+f(x)\left[K_n^{\alpha,\beta}(1,x)-1\right] \\ &\quad= \varepsilon+\varepsilon\left[K_n^{\alpha,\beta}(1,x)-1\right]+\mathfrak{h}(x)\left[K_n^{\alpha,\beta}(1,x)-1\right]++\frac{2K}{\delta^2} \\ &\qquad\cdot\left\{x^2\left[K_n^{\alpha,\beta}(1,x)-1\right]-2x\left[K_n^{\alpha,\beta}(u,x)-x\right]\right. \\ &\qquad\left.+\left[K_n^{\alpha,\beta}\left(u^2,x\right)-x^2\right]\right\}. \end{aligned} \tag{42}$$

Therefore,

$$\begin{aligned} \left|K_n^{\alpha,\beta}(\mathfrak{h},x)-\mathfrak{h}(x)\right| &\le \varepsilon+\left(\varepsilon+K+\frac{2KM^2}{\delta^2}\right)\left|K_n^{\alpha,\beta}(1,x)-1\right| \\ &\quad+\frac{4KM}{\delta^2}\left|K_n^{\alpha,\beta}(u,x)-x\right| \\ &\quad+\frac{2K}{\delta^2}\left|K_n^{\alpha,\beta}\left(u^2,x\right)-x^2\right|. \end{aligned} \tag{43}$$

From the above relations and the linearity of $K_n^{\alpha,\beta}$, we obtain

$$\begin{aligned} &\frac{1}{p(v)}\left\|\sum_{n=0}^{\infty}\left(U_{n,p}(\mathfrak{h};x)-\mathfrak{h}(x)\right)p_nv^n\right\| \\ &\quad\le \varepsilon+\left(\varepsilon+K+\frac{2KM^2}{\delta^2}\right)\frac{1}{p(t)}\left\|\sum_{n=0}^{\infty}\left(K_n^{\alpha,\beta}(1;x)-1\right)p_nt^n\right\| \\ &\qquad+\frac{4KM}{\delta^2}\frac{1}{p(v)}\left\|\sum_{n=0}^{\infty}\left(K_n^{\alpha,\beta}(u;x)-x\right)p_nv^n\right\| \\ &\qquad+\frac{2K}{\delta^2}\frac{1}{p(v)}\left\|\sum_{n=0}^{\infty}\left(K_n^{\alpha,\beta}\left(u^2;x\right)-x^2\right)p_nv^n\right\|. \end{aligned} \tag{44}$$

Hence, (32) follows from the last relation and (31).

3. Rate of Convergence

Modulus of continuity is defined by

$$\omega(\mathfrak{h},\delta)=\sup_{|h|<\delta}|\mathfrak{h}(x+h)-\mathfrak{h}(x)|,\ \mathfrak{h}(x)\in C[0,M]\cap E. \tag{45}$$

It is not hard to verify

$$|\mathfrak{h}(x)-\mathfrak{h}(y)|\le\omega(\mathfrak{h},\delta)\left(\frac{|x-y|}{\delta}+1\right). \tag{46}$$

So, we can state the following.

Theorem 16. *Let $T=(t_{ij})$ be a nonnegative regular summability matrix and $\mathfrak{h}\in C[0,M]\cap E$. If (α_n) is a sequence of positive real numbers such that $\omega(\mathfrak{h},\delta_n)=st_T-0(\alpha_n)$, then*

$$\left\|K_n^{\alpha,\beta}\mathfrak{h}-\mathfrak{h}\right\|=st_A-0(\alpha_n), \tag{47}$$

where

$$\delta_n = \left\{ M^2 \left\| \left(\frac{n^2 B''(nx)}{(n+\beta)^2 B(nx)} - \frac{2nB'(nx)}{(n+\beta)B(nx)} + 1 \right) \right\| + M \left\| \left(\frac{nB'(nx)\left[2'(1) + (2\alpha+2)A(1)\right]}{(n+\beta)^2 A(1) B(nx)} - \frac{2A'(1)}{(n+\beta)A(1)} + \frac{2(2\alpha+1)}{2(n+\beta)} \right) \right\| + \left\| \frac{1}{(n+\beta)^2 A(1)} \cdot \left\{ A''(1) + (2\alpha+2)A'(1)\left(\alpha^2+\alpha+\frac{1}{3}\right)A(1) \right\} \right\| \right\}^2, \quad (48)$$

for any positive integer n.

Proof. Let $\mathfrak{h} \in C[0,M] \cap E$. By positivity and linearity of $K_n^{\alpha,\beta}$ and (46), we see

$$\begin{aligned}\left|K_n^{\alpha,\beta}(\mathfrak{h};x) - \mathfrak{h}\right| &\le K_n^{\alpha,\beta}(|\mathfrak{h}(t)-\mathfrak{h}(x)|;x) \\ &\le \frac{n+\beta}{A(1)B(nx)} \sum_{k=0}^{\infty} p_k(nx) \int_{(k+\alpha)/(n+\beta)}^{(k+1+\alpha)/(n+\beta)} \omega(\mathfrak{h},\delta) \cdot \left(1 + \frac{|t-x|}{\delta}\right) dt \\ &\le \omega(\mathfrak{h},\delta)\left[1 + \frac{1}{\delta}\frac{n+\beta}{A(1)B(nx)}\sum_{k=0}^{\infty} p_k(nx) \cdot \int_{(k+\alpha)/(n+\beta)}^{(k+1+\alpha)/(n+\beta)} (t-x)dt\right] (\text{by Lemma 1}) = \omega(\mathfrak{h},\delta) \cdot \left[1 + \frac{1}{\delta}K_n^{\alpha,\beta}(|t-x|;x)\right].\end{aligned} \quad (49)$$

By applying the Cauchy-Schwartz inequality, we have

$$\left|K_n^{\alpha,\beta}(\mathfrak{h};x) - \mathfrak{h}\right| \le \omega(\mathfrak{h},\delta)\left[1 + \frac{1}{\delta}\left(K_n^{\alpha,\beta}\left(|t-x|^2;x\right)\right)^{1/2}\right]. \quad (50)$$

Based on Examples 5 and Remark 8, we obtain

$$\begin{aligned}K_n^{\alpha,\beta}\left((u-x)^2;x\right) &= K_n^{\alpha,\beta}(e_2;x) - 2xK_n^{\alpha,\beta}(e_1;x) + x^2K_n^{\alpha,\beta}(e_0;x) \\ &\le M^2 \left\| \left(\frac{n^2 B''(nx)}{(n+\beta)^2 B(nx)} - \frac{2nB'(nx)}{(n+\beta)B(nx)} + 1 \right) \right\| + M \left\| \left(\frac{nB'(nx)\left[2A'(1) + (2\alpha+2)A(1)\right]}{(n+\beta)^2 A(1) B(nx)} - \frac{2A'(1)}{(n+\beta)A(1)} + \frac{2(2\alpha+1)}{2(n+\beta)} \right) \right\| + \left\| \frac{1}{(n+\beta)^2 A(1)} \cdot \left\{ A''(1) + (2\alpha+2)A'(1)\left(\alpha^2+\alpha+\frac{1}{3}\right)A(1) \right\} \right\|.\end{aligned} \quad (51)$$

By taking

$$\delta_n = \left\{ M^2 \left\| \left(\frac{n^2 B''(nx)}{(n+\beta)^2 B(nx)} - \frac{2nB'(nx)}{(n+\beta)B(nx)} + 1 \right) \right\| + M \left\| \left(\frac{nB'(nx)\left[2A'(1) + (2\alpha+2)A(1)\right]}{(n+\beta)^2 A(1) B(nx)} - \frac{2A'(1)}{(n+\beta)A(1)} + \frac{2(2\alpha+1)}{2(n+\beta)} \right) \right\| + \left\| \frac{1}{(n+\beta)^2 A(1)} \cdot \left\{ A''(1) + (2\alpha+2)A'(1)\left(\alpha^2+\alpha+\frac{1}{3}\right)A(1) \right\} \right\| \right\}^2, \quad (52)$$

we get that $\|K_n^{\alpha,\beta}\mathfrak{h} - \mathfrak{h}\| \le 2 \cdot \omega(\mathfrak{h},\delta_n)$. Therefore, for every $\varepsilon > 0$, we have

$$\frac{1}{\alpha_n} \sum_{\left\|K_n^{\alpha,\beta}\mathfrak{h}-\mathfrak{h}\right\| \ge \varepsilon} t_{nj} \le \frac{1}{\alpha_n} \sum_{2\cdot\omega(f,\delta_n)\ge\varepsilon} t_{nj}. \quad (53)$$

From the conditions that are given in the theorem, we have that $\|K_n^{\alpha,\beta}\mathfrak{h} - \mathfrak{h}\| = st_T - 0(\alpha_i)$, as claimed.

In the next result, we present the rate of convergence for the power summability method.

Theorem 17. *Let $\mathfrak{h} \in C[0,M] \cap E$ and let ϕ be a positive real function defined on $(0,M) \cap E$. If $\omega(\mathfrak{h},\psi(u)) = O(\phi(u))$, as $\nu \longrightarrow R^-$, then we have*

$$\frac{1}{p(\nu)} \left\| \sum_{n=0}^{\infty} \left(K_n^{\alpha,\beta}e_i - e_i\right) p_n \nu^n \right\| = O(\phi(\nu)), \quad (54)$$

where the function $\psi : (0,M) \cap E \longrightarrow \mathbb{R}$ is defined by relation

$$\psi(u) = \left\{ \sup_{\substack{x\in(0,M) \\ n\in\mathbb{N}}} \left\{ K_n^{\alpha,\beta}\left((u-x)^2;x\right) \right\} \right\}^{1/2}. \quad (55)$$

Proof. Let $\mathfrak{h} \in C[0, M] \cap E$. For any $u \in (0, R)$, $x \in (0, M)$, and $\delta > 0$, we have

$$\begin{aligned}
&\left|\sum_{n=0}^{\infty}\left[K_n^{\alpha,\beta}(\mathfrak{h}; x) - \mathfrak{h}(x)\right]p_n v^n\right| \\
&\quad\le \sum_{n=0}^{\infty} K_n^{\alpha,\beta}(|\mathfrak{h}(u) - \mathfrak{h}(x)|; x)p_n v^n \\
&\quad\le \sum_{n=0}^{\infty} K_n^{\alpha,\beta}\left(\omega\left(\mathfrak{h}, \frac{|u-x|}{\delta}\delta\right); x\right)p_n v^n \\
&\quad\le \sum_{n=0}^{\infty} K_n^{\alpha,\beta}\left(\left(1 + \left[\left|\frac{u-x}{\delta}\right|\right]\right)\omega(\mathfrak{h}, \delta); x\right)p_n v^n \\
&\quad\le \omega(\mathfrak{h}, \delta)\sum_{n=0}^{\infty} K_n^{\alpha,\beta}\left(1 + \frac{(u-x)^2}{\delta^2}; x\right)p_n v^n \\
&\quad\le \omega(\mathfrak{h}, \delta)\sum_{n=0}^{\infty} K_n^{\alpha,\beta}(e_0(u); x)p_n v^n \\
&\qquad + \frac{\omega(\mathfrak{h}, \delta)}{\delta^2}\sum_{n=0}^{\infty} K_n^{\alpha,\beta}\left((u-x)^2; x\right)p_n v^n \\
&\quad= p(v)\omega(\mathfrak{h}, \delta) + \frac{\omega(\mathfrak{h}, \delta)}{\delta^2}\sup_{\substack{0\le x\le 1\\ n\in\mathbb{N}}} \\
&\qquad\cdot\left\{K_n^{\alpha,\beta}\left((u-x)^2; x\right)\right\}\sum_{n=0}^{\infty} p_n v^n,
\end{aligned} \tag{56}$$

which leads to

$$\begin{aligned}
&\left|\sum_{n=0}^{\infty}\left[K_n^{\alpha,\beta}(f; x) - f(x)\right]p_n v^n\right| \\
&\quad\le p(v)\omega(f, \delta) + \frac{\omega(f, \delta)}{\delta^2}\sup_{0\le x\le 1} \\
&\qquad\cdot\left\{K_n^{\alpha,\beta}\left((u-x)^2; x\right)\right\}p(v).
\end{aligned} \tag{57}$$

If we set $\delta = \psi(u)$, then from the last inequality we have

$$0 \le \frac{1}{p(v)}\left\|\sum_{n=0}^{\infty}\left(K_n^{\alpha,\beta}\mathfrak{h} - \mathfrak{h}\right)p_n v^n\right\| \le 2\omega(\mathfrak{h}, \delta), \tag{58}$$

as required.

4. Voronovskaya-Type Theorems

First, we prove a Voronovskaya-type theorem for the operators under consideration.

Theorem 18. *Let* $\mathfrak{h}, \mathfrak{h}', \mathfrak{h}'' \in C[0, M] \cap E$ *and* $\lim_{n\to\infty} B^{(i)}(nx)/B(nx) = 1$, *for* $i \in \{1, 2\}$. *Then,*

$$\begin{aligned}
&\lim_{n\to\infty}(n+\beta)\left[K_n^{\alpha,\beta}(\mathfrak{h}; x) - \mathfrak{h}(x)\right] \\
&\quad= \left(\frac{A'(1)}{A(1)} + \frac{1}{2} + \alpha - \beta x\right)\mathfrak{h}'(x) + \frac{x}{2}\mathfrak{h}''(x),
\end{aligned} \tag{59}$$

for every $x \in [0, M]$.

Proof. Assume that $\mathfrak{h}', \mathfrak{h}'' \in C[0, M] \cap E$ and $x \in [0, M]$. By Taylor's formula, we have

$$\mathfrak{h}(y) = \mathfrak{h}(x) + (y-x)\mathfrak{h}'(x) + \frac{1}{2}(y-x)^2\mathfrak{h}''(x) + (y-x)^2\psi(y-x), \tag{60}$$

where $\psi(y-x) \longrightarrow 0$ and $y - x \longrightarrow 0$. Applying in both sides of the above relation operators $K_n^{\alpha,\beta}$, we obtain

$$\begin{aligned}
K_n^{\alpha,\beta}(\mathfrak{h}, x) - \mathfrak{h}(x) &= \mathfrak{h}'(x)K_n^{\alpha,\beta}(y-x; x) \\
&\quad + \frac{\mathfrak{h}''(x)}{2}K_n^{\alpha,\beta}\left((y-x)^2; x\right) \\
&\quad + K_n^{\alpha,\beta}\left((y-x)^2\psi(y-x); x\right),
\end{aligned} \tag{61}$$

which implies

$$\begin{aligned}
(n+\beta)\left[K_n^{\alpha,\beta}(\mathfrak{h}, x) - \mathfrak{h}(x)\right] &= (n+\beta)\mathfrak{h}'(x)K_n^{\alpha,\beta}(y-x; x) \\
&\quad + (n+\beta)\frac{\mathfrak{h}''(x)}{2}K_n^{\alpha,\beta}\left((y-x)^2; x\right) \\
&\quad + (n+\beta)K_n^{\alpha,\beta}\left((y-x)^2\psi(y-x); x\right).
\end{aligned} \tag{62}$$

Now, we will estimate this expression:

$$\lim_{n\to\infty}(n+\beta)K_n^{\alpha,\beta}\left((y-x)^2\psi(y-x); x\right). \tag{63}$$

Let $\varepsilon > 0$ and $\delta > 0$ such that $|\psi(y-x)| < \varepsilon$, where $|y-x| < \delta$. We will split the above relation in two parts:

$$\begin{aligned}
U_1 &= (n+\beta)\frac{n+\beta}{A(1)B(nx)}\sum_{k=0}^{\infty} p_k(nx)\int_{|y-x|\le\delta}(y-x)^2\psi(y-x)dy \\
U_2 &= (n+\beta)\frac{n+\beta}{A(1)B(nx)}\sum_{k=0}^{\infty} p_k(nx)\int_{|y-x|\ge\delta}(y-x)^2\psi(y-x)dy.
\end{aligned} \tag{64}$$

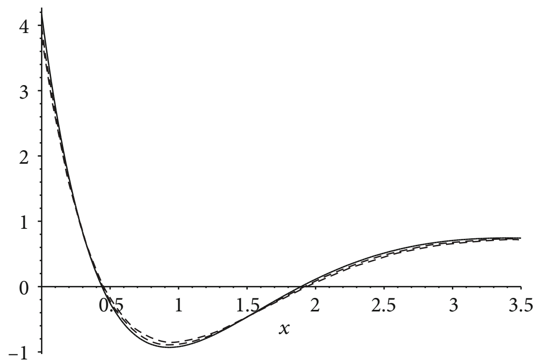

Figure 1: The graphs of $(10+\beta)[K_{10}^{\alpha,\beta}(f;x)-f(x)]$, $(20+\beta)[K_{20}^{\alpha,\beta}(f;x)-f(x)]$, and $((A'(1)/A(1))+(1/2)+\alpha-\beta x)f'(x)+(x/2)f''(x)$ when $x \in 0,3.5]$.

From the above conditions, we have

$$|U_1| \le \varepsilon(n+\beta)^2 \frac{1}{A(1)B(nx)} \sum_{k=0}^{\infty} p_k(nx) \int_{k+\alpha/n+\beta}^{k+\alpha+1/n+\beta} (y-x)^2 dy = \varepsilon(n+\beta) K_n^{\alpha,\beta}\left((y-x)^2;x\right). \tag{65}$$

On the other hand, from Proposition 9, condition (3), we get that $|U_1| \le \varepsilon_1$.

Let us denote by $N=\sup_x\{|\psi(y-x)|;|y-x|\ge\delta\}$. Then, we obtain

$$|U_2| \le \frac{N}{\delta^2}(n+\beta)\frac{n+\beta}{A(1)B(nx)}\sum_{k=0}^{\infty} p_k(nx)\int_{|y-x|\ge\delta}(y-x)^4 dy = \frac{N}{\delta^2}(n+\beta)K_n^{\alpha,\beta}\left((y-x)^4;x\right). \tag{66}$$

Condition (5) in Proposition 9 tells us that $\lim_{n\to\infty} |U_2| = 0$, which completes the proof.

Example 19. Let $A(t)=1$, $B(t)=e^t$, $\alpha=1/3$, $\beta=1/2$, and $\mathfrak{h}(x)=5xe^{-x}$. By $A(t)B(xt)=\sum_{k\ge 0} p_k(x)t^k$, we see that $p_k(x)=x^k/k!$. Figure 1 presents the graphs of the functions $(10+\beta)[K_{10}^{\alpha,\beta}(\mathfrak{h};x)-\mathfrak{h}(x)]$, $(20+\beta)[K_{20}^{\alpha,\beta}(\mathfrak{h};x)-\mathfrak{h}(x)]$, and $((A'(1)/A(1))+(1/2)+\alpha-\beta x)\mathfrak{h}'(x)+(x/2)\mathfrak{h}''(x)$.

We extend the Voronovskaya-type theorem for $T-$ statistical method for these operators. Consider operators N_n from Example 13. We start with the following lemma.

Lemma 20. *Let $\mathfrak{h} \in C[0,M]\cap E$ such that $\mathfrak{h}', \mathfrak{h}'' \in C[0,M]\cap E$, $x\in[0,M]$, and $\lim_{n\to\infty}(B^{(i)}(nx)/B(nx))=1$, for $i\in\{1,2,3,4\}$. Then, we obtain*

$$(n+\beta)^4 K_n\left((y-x)^4;x\right) \sim 0(st_A) \quad on \quad [0,M]. \tag{67}$$

Proof. The proposition follows directly from Proposition 9 (5).

Theorem 21. *Let $\mathfrak{h}\in C[0,M]\cap E$ such that $\mathfrak{h}',\mathfrak{h}''\in C[0,M]\cap E$, $x\in[0,M]$, for any finite M and let $\lim_{n\to\infty}(B^{(i)}(nx)/B(nx))=1$, for $i\in\{1,2,3,4\}$. Then, for $x\in[0,M]$,*

$$\begin{aligned}&\Bigg|(n+\beta)^2\Bigg[N_n(\mathfrak{h};x)-\mathfrak{h}(x)-\mathfrak{h}'(x)\left[\frac{2a_0\alpha+a_0+2a_1}{2a_0(n+\beta)}-\frac{(b_0n+b_0\beta-b_1n)}{b_0(n+\beta)}x\right]-\frac{\mathfrak{h}''(x)}{2}\\&\cdot\Bigg(-\frac{2a_0b_0\alpha(n+\beta)-2a_0b_1n(\alpha+1)+a_0b_0(n+\beta)+2a_1b_0(n+\beta)-2a_1b_1n}{a_0b_0(n+\beta)^2}x\\&+\frac{b_0n^2+2b_0n\beta+b_0\beta^2-2b_1n^2-2b_1n\beta+b_2n^2}{b_0(n+\beta)^2}x^2\Bigg)\Bigg]\Bigg|\\&\sim\frac{\mathfrak{h}''(x)}{2}\cdot\frac{3a_0\alpha^2+3a_0\alpha+6a_1\alpha+a_0+6a_1+3a_2}{3a_0(n+\beta)^2}.\end{aligned} \tag{68}$$

Proof. Taylor's formula gives

$$\mathfrak{h}(y)=\mathfrak{h}(x)+(y-x)\mathfrak{h}'(x)+\frac{1}{2}(y-x)^2\mathfrak{h}''(x)+(y-x)^2\psi(y-x), \tag{69}$$

where $\psi(y-x)\longrightarrow 0$, as $y-x\longrightarrow 0$. Taking into consideration Remark 8, after applying N_n in both sides of relation (69), we obtain

$$\begin{aligned}N_n(\mathfrak{h})&=(1+x_n)\mathfrak{h}(x)+(1+x_n)\mathfrak{h}'(x)\left(\frac{2a_0\alpha+a_0+2a_1}{2a_0(n+\beta)}-\frac{(b_0n+b_0\beta-b_1n)}{b_0(n+\beta)}x\right)+(1+x_n)\frac{\mathfrak{h}''(x)}{2}\\&\quad\cdot\left(\frac{3a_0\alpha^2+3a_0\alpha+6a_1\alpha+a_0+6a_1+3a_2}{3a_0(n+\beta)^2}-\frac{2a_0b_0\alpha(n+\beta)-2a_0b_1n(\alpha+1)+a_0b_0(n+\beta)+2a_1b_0(n+\beta)-2a_1b_1n}{a_0b_0(n+\beta)^2}x\right.\\&\quad\left.+\frac{b_0n^2+2b_0n\beta+b_0\beta^2-2b_1n^2-2b_1n\beta+b_2n^2}{b_0(n+\beta)^2}x^2\right)+(1+x_n)K_n^{\alpha,\beta}\left(\Phi^2\psi(y-x);x\right).\end{aligned} \tag{70}$$

This yields

$$\begin{aligned}(n+\beta)^2N_n(\mathfrak{h})&=(1+x_n)(n+\beta)^2\mathfrak{h}(x)+(1+x_n)\mathfrak{h}'(x)(n+\beta)^2\left(\frac{2a_0\alpha+a_0+2a_1}{2a_0(n+\beta)}-\frac{(b_0n+b_0\beta-b_1n)}{b_0(n+\beta)}x\right)+(1+x_n)\frac{\mathfrak{h}''(x)}{2}(n+\beta)^2\\&\quad\left(\frac{3a_0\alpha^2+3a_0\alpha+6a_1\alpha+a_0+6a_1+3a_2}{3a_0(n+\beta)^2}-\frac{2a_0b_0\alpha(n+\beta)-2a_0b_1n(\alpha+1)+a_0b_0(n+\beta)+2a_1b_0(n+\beta)-2a_1b_1n}{a_0b_0(n+\beta)^2}x\right.\\&\quad\left.+\frac{b_0n^2+2b_0n\beta+b_0\beta^2-2b_1n^2-2b_1n\beta+b_2n^2}{b_0(n+\beta)^2}x^2\right)+(1+x_n)(n+\beta)^2K_n^{\alpha,\beta}\left(\Phi^2\psi(y-x);x\right).\end{aligned} \tag{71}$$

Therefore,

$$\begin{aligned}&\left|(n+\beta)^2\left[N_n(\mathfrak{h};x)-\mathfrak{h}(x)-\mathfrak{h}'(x)\left[\frac{2a_0\alpha+a_0+2a_1}{2a_0(n+\beta)}-\frac{(b_0n+b_0\beta-b_1n)}{b_0(n+\beta)}x\right]\right.\right.\\&\quad-\frac{\mathfrak{h}''(x)}{2}\left(-\frac{2a_0b_0\alpha(n+\beta)-2a_0b_1n(\alpha+1)+a_0b_0(n+\beta)+2a_1b_0(n+\beta)-2a_1b_1n}{a_0b_0(n+\beta)^2}x\right.\\&\quad\left.\left.\left.+\frac{b_0n^2+2b_0n\beta+b_0\beta^2-2b_1n^2-2b_1n\beta+b_2n^2}{b_0(n+\beta)^2}x^2\right)\right]-\frac{\mathfrak{h}''(x)}{2}\cdot\frac{3a_0\alpha^2+3a_0\alpha+6a_1\alpha+a_0+6a_1+3a_2}{3a_0(n+\beta)^2}\right|\\&\le Mx_n(n+\beta)^2+M_1x_n\left[\frac{2a_0\alpha+a_0+2a_1}{2a_0(n+\beta)}-\frac{(b_0n+b_0\beta-b_1n)}{b_0(n+\beta)}x\right]+M_2\\&\quad\cdot x_n\left(\frac{3a_0\alpha^2+3a_0\alpha+6a_1\alpha+a_0+6a_1+3a_2}{3a_0(n+\beta)^2}-\frac{2a_0b_0\alpha(n+\beta)-2a_0b_1n(\alpha+1)+a_0b_0(n+\beta)+2a_1b_0(n+\beta)-2a_1b_1n}{a_0b_0(n+\beta)^2}x\right.\\&\quad\left.+\frac{b_0n^2+2b_0n\beta+b_0\beta^2-2b_1n^2-2b_1n\beta+b_2n^2}{b_0(n+\beta)^2}x^2\right)+(n+\beta)^2K_n^{\alpha,\beta}\left((y-x)^2\psi(y-x);x\right)\\&\quad+x_n\left((n+\beta)^2K_n^{\alpha,\beta}(y-x)^2\psi(y-x);x\right),\end{aligned} \tag{72}$$

where $M=\sup_{x\in[0,M]}|\mathfrak{h}(x)|$, $M_1=\sup_{x\in[0,M]}|\mathfrak{h}'(x)|$, and $M_2=\sup_{x\in[0,M]}|\mathfrak{h}''(x)|$.

Now, we have to prove that

$$\lim_{n\to\infty}(n+\beta)^2K_n^{\alpha,\beta}\left((y-x)^2\psi(y-x);x\right)=0. \quad (73)$$

By applying the Cauchy-Schwartz inequality, we obtain

$$(n+\beta)^2K_n^{\alpha,\beta}\left((y-x)^2\psi(y-x);x\right) \leq \left[(n+\beta)^4K_n^{\alpha,\beta}\left((y-x)^4;x\right)\right]^{1/2}\cdot\left[K_n^{\alpha,\beta}\left(\psi^2;x\right)\right]^{1/2}. \quad (74)$$

Also, by setting $\eta_x(y)=(\psi(y-x))^2$, we have that $\eta_x(x)=0$ and $\eta_x(\cdot)\in C[0,M]$. So

$$K_n^{\alpha,\beta}(\eta_x)\longrightarrow 0(st_A)\quad\text{on}\quad[0,M]. \quad (75)$$

Now, from the last relation, (74), (75), and Lemma 20, we obtain that

$$(n+\beta)^2K_n^{\alpha,\beta}\left((y-x)^2\psi(y-x);x\right)\longrightarrow 0(st_A)\quad\text{on}\quad[0,M]. \quad (76)$$

From the construction of (x_n), it follows that $(n+\beta)^2x_n\longrightarrow 0(st_A)$ on $[0,M]$.

For a given $\varepsilon>0$, we define the following sets:

$$\begin{aligned}
A&=\Bigg|\Bigg\{n:\Bigg|(n+\beta)^2\Bigg[N_n(\mathfrak{h};x)-\mathfrak{h}(x)-\mathfrak{h}'(x)\left[\frac{2a_0\alpha+a_0+2a_1}{2a_0(n+\beta)}-\frac{(b_0n+b_0\beta-b_1n)}{b_0(n+\beta)}x\right]\\
&\quad-\frac{\mathfrak{h}''(x)}{2}\Bigg(-\frac{2a_0b_0\alpha(n+\beta)-2a_0b_1n(\alpha+1)+a_0b_0(n+\beta)+2a_1b_0(n+\beta)-2a_1b_1n}{a_0b_0(n+\beta)^2}x\\
&\quad+\frac{b_0n^2+2b_0n\beta+b_0\beta^2-2b_1n^2-2b_1n\beta+b_2n^2}{b_0(n+\beta)^2}x^2\Bigg)\Bigg]-\frac{\mathfrak{h}''(x)}{2}\cdot\frac{3a_0\alpha^2+3a_0\alpha+6a_1\alpha+a_0+6a_1+3a_2}{3a_0(n+\beta)^2}\Bigg|\geq\varepsilon\Bigg\}\Bigg|,\\
A_1&=\left|\left\{n:\left|x_n(n+\beta)^2\right|\geq\frac{\varepsilon}{5M}\right\}\right|,\\
A_2&=\left|\left\{n:\left|x_n\left[\frac{2a_0\alpha+a_0+2a_1}{2a_0(n+\beta)}-\frac{(b_0n+b_0\beta-b_1n)}{b_0(n+\beta)}x\right]\right|\geq\frac{\varepsilon}{5M_1}\right\}\right|,\\
A_3&=\Bigg|\Bigg\{n:\Bigg|x_n\Bigg(\frac{3a_0\alpha^2+3a_0\alpha+6a_1\alpha+a_0+6a_1+3a_2}{3a_0(n+\beta)^2}-\frac{2a_0b_0\alpha(n+\beta)-2a_0b_1n(\alpha+1)+a_0b_0(n+\beta)+2a_1b_0(n+\beta)-2a_1b_1n}{a_0b_0(n+\beta)^2}x\\
&\quad+\frac{b_0n^2+2b_0n\beta+b_0\beta^2-2b_1n^2-2b_1n\beta+b_2n^2}{b_0(n+\beta)^2}x^2\Bigg)\Bigg|\geq\frac{\varepsilon}{5M_2}\Bigg\}\Bigg|,\\
A_4&=\left|\left\{n:\left|(n+\beta)^2K_n^{\alpha,\beta}\left((y-x)^2\psi(y-x);x\right)\right|\geq\frac{\varepsilon}{5}\right\}\right|,\\
A_5&=\left|\left\{n:\left|x_n(n+\beta)^2K_n^{\alpha,\beta}\left((y-x)^2\psi(y-x);x\right)\right|\geq\frac{\varepsilon}{5}\right\}\right|.
\end{aligned} \quad (77)$$

From last relations, we obtain that $A\leq A_1+A_2+A_3+A_4+A_5$. Hence, the result follows.

Remark 22. We see that the operators (N_n) (see Example 13) do not satisfy a Voronovskaya-type theorem in the usual sense.

5. Grüss-Voronovskaya-Type Theorems

This kind of result, for the first time, was shown in [30].

Theorem 23 (see [31]). *Let* $E=\{\mathfrak{h}:x\in[0,\infty),(\mathfrak{h}(x)/(1+x^2)),$ *is convergent as* $x\longrightarrow\infty\}$ *and*

$$\lim_{n\to\infty}\frac{B^{(m)}(nx)}{B(nx)}=1\quad\textit{for}\quad m=1,2. \quad (78)$$

If $\mathfrak{h}\in C[0,\infty)\cap E$, *then*

$$\lim_{n\to\infty}K_n^{\alpha,\beta}(f,x)=f(x), \quad (79)$$

and the operators $K_n^{\alpha,\beta}(\mathfrak{h},x)$ *converge uniformly in each compact subset of* $[0,\infty)$.

Now, we are ready to prove the following result.

Theorem 24. *For* $\mathfrak{h},\mathfrak{h}',\mathfrak{h}''\in C[0,\infty)$ *and any* $x\in[0,\infty)$, $\lim_{n\to\infty}B^{(i)}(nx)/B(nx)=1$, *for* $i\in\{1,2,3,4,5,6\}$, *where* $B^{(i)}$ *denotes the ith derivative of B. Then,*

$$\begin{aligned}&\left|(n+\beta)\left(K_n^{\alpha,\beta}(\mathfrak{h},x)-\mathfrak{h}(x)\right)-\mathfrak{h}'(x)\left(\frac{2a_0\alpha+a_0+2a_1}{2a_0(n+\beta)}-\frac{(b_0n+b_0\beta-b_1n)}{b_0(n+\beta)}x\right)-(n+\beta)\frac{\mathfrak{h}''(x)}{2}\right.\\&\cdot\left[\frac{3a_0\alpha^2+3a_0\alpha+6a_1\alpha+a_0+6a_1+3a_2}{3a_0(n+\beta)^2}-\frac{2a_0b_0\alpha(n+\beta)-2a_0b_1n(\alpha+1)+a_0b_0(n+\beta)+2a_1b_0(n+\beta)-2a_1b_1n}{a_0b_0(n+\beta)^2}x\right.\\&\left.\left.+\frac{b_0n^2+2b_0n\beta+b_0\beta^2-2b_1n^2-2b_1n\beta+b_2n^2}{b_0(n+\beta)^2}x^2\right]\right|=0(1)\omega\left(\mathfrak{h}'',n^{-1/2}\right),\end{aligned}\tag{80}$$

as $n\longrightarrow\infty$.

Proof. From Taylor's theorem, we have

$$\mathfrak{h}(u)=\mathfrak{h}(x)+\mathfrak{h}'(x)(u-x)+\frac{\mathfrak{h}''(x)}{2}(u-x)^2+R(u,x),\tag{81}$$

where $R(u,x)=((\mathfrak{h}''(\theta)-\mathfrak{h}''(x))/2)(u-x)^2$, for $\theta\in(u,x)$. Now, we obtain

$$\begin{aligned}&\left|K_n^{\alpha,\beta}(\mathfrak{h},x)-\mathfrak{h}(x)-\mathfrak{h}'(x)K_n^{\alpha,\beta}((u-x);x)\right.\\&\left.-\frac{\mathfrak{h}''(x)}{2}K_n^{\alpha,\beta}\left((u-x)^2;x\right)\right|\leq K_n^{\alpha,\beta}(|R(u,x)|,x).\end{aligned}\tag{82}$$

From which we get that

$$\begin{aligned}&\left|(n+\beta)\left(K_n^{\alpha,\beta}(\mathfrak{h},x)-\mathfrak{h}(x)\right)-\mathfrak{h}'(x)\left(\frac{2a_0\alpha+a_0+2a_1}{2a_0(n+\beta)}-\frac{(b_0n+b_0\beta-b_1n)}{b_0(n+\beta)}x\right)-(n+\beta)\frac{\mathfrak{h}''(x)}{2}\right.\\&\cdot\left[\frac{3a_0\alpha^2+3a_0\alpha+6a_1\alpha+a_0+6a_1+3a_2}{3a_0(n+\beta)^2}-\frac{2a_0b_0\alpha(n+\beta)-2a_0b_1n(\alpha+1)+a_0b_0(n+\beta)+2a_1b_0(n+\beta)-2a_1b_1n}{a_0b_0(n+\beta)^2}x\right.\\&\left.\left.+\frac{b_0n^2+2b_0n\beta+b_0\beta^2-2b_1n^2-2b_1n\beta+b_2n^2}{b_0(n+\beta)^2}x^2\right]\right|\leq(n+\beta)\cdot K_n^{\alpha,\beta}(|R(u,x)|,x).\end{aligned}\tag{83}$$

By the properties of modulus of continuity modulus, we have

$$\left|\frac{\mathfrak{h}''(\theta)-\mathfrak{h}''(x)}{2!}\right|\leq\frac{1}{2!}\left(1+\frac{|\theta-x|}{\delta}\right)\omega\left(\mathfrak{h}'',\delta\right).\tag{84}$$

On the other hand,

$$\left|\frac{\mathfrak{h}''(\theta)-\mathfrak{h}''(x)}{2!}\right|\leq\begin{cases}\omega\left(\mathfrak{h}'',\delta\right), & |u-x|\leq\delta,\\ \dfrac{(t-x)^4}{\delta^4}\omega\left(\mathfrak{h}'',\delta\right), & |u-x|\geq\delta.\end{cases}\tag{85}$$

For $0<\delta<1$, we obtain that

$$\left|\frac{\mathfrak{h}''(\theta)-\mathfrak{h}''(x)}{2!}\right|\leq\omega\left(\mathfrak{h}'',\delta\right)\left(1+\frac{(u-x)^4}{\delta^4}\right),\tag{86}$$

which gives

$$\begin{aligned}|R(u,x)|&\leq\omega\left(\mathfrak{h}'',\delta\right)\left(1+\frac{(u-x)^4}{\delta^4}\right)(u-x)^2\\&=\omega\left(\mathfrak{h}'',\delta\right)\left((u-x)^2+\frac{(u-x)^6}{\delta^4}\right).\end{aligned}\tag{87}$$

By the linearity of $K_n^{\alpha,\beta}$ and the above relation, we obtain

$$K_n^{\alpha,\beta}(|R(u,x)|,x) \leq \omega\left(\mathfrak{h}'',\delta\right)\left(K_n^{\alpha,\beta}\left((u-x)^2,x\right)+\frac{1}{\delta^4}K_n^{\alpha,\beta}\left((u-x)^6,x\right)\right). \tag{88}$$

Taking into consideration Proposition 9, we have

$$K_n^{\alpha,\beta}(|R(u,x)|,x) \leq \omega\left(\mathfrak{h}'',\delta\right)\left(0\left(\frac{1}{n}\right)+\frac{1}{\delta^4}0\left(\frac{1}{n^3}\right)\right) = 0\left(\frac{1}{n}\right)\omega\left(\mathfrak{h}'',\delta\right). \tag{89}$$

For $\delta = n^{-1/2}$, we complete the proof.

Theorem 25. *Let* $\mathfrak{h}'(x), g'(x), \mathfrak{h}''(x), g''(x), (\mathfrak{h}g)'(x), (\mathfrak{h}g)''(x) \in C[0,\infty)\cap E$, *and*

$$\lim_{n\to\infty}\frac{B^{(m)}(nx)}{B(nx)} = 1 \quad for \quad m=0,1,2,\cdots,6, \quad x\in[0,M], \tag{90}$$

where $B^{(m)}$ *is the mth derivative of B. Then,*

$$\lim_{n\to\infty}(n+\beta)\left[K_n^{\alpha,\beta}(\mathfrak{h}g,x)-K_n^{\alpha,\beta}(\mathfrak{h},x)\cdot K_n^{\alpha,\beta}(g,x)\right] = x\mathfrak{h}'(x)g'(x). \tag{91}$$

Proof. We know that

$$\begin{aligned}(n+\beta)&\left\{K_n^{\alpha,\beta}(\mathfrak{h}g,x)-K_n^{\alpha,\beta}(\mathfrak{h},x)K_n^{\alpha,\beta}(g,x)\right\}\\ &= (n+\beta)\left\{K_n^{\alpha,\beta}(\mathfrak{h}g,x)-(\mathfrak{h}g)(x)\right.\\ &\quad -(\mathfrak{h}g)'(x)K_n^{\alpha,\beta}((u-x),x)-\frac{(\mathfrak{h}g)''(x)}{2}K_n^{\alpha,\beta}\left((u-x)^2,x\right)\\ &\quad -g(x)\left[K_n^{\alpha,\beta}(\mathfrak{h},x)-\mathfrak{h}(x)-\mathfrak{h}'(x)K_n^{\alpha,\beta}((u-x),x)\right.\\ &\quad \left.-\frac{\mathfrak{h}''(x)}{2}K_n^{\alpha,\beta}\left((u-x)^2,x\right)\right]\\ &\quad -K_n^{\alpha,\beta}(\mathfrak{h},x)\left[K_n^{\alpha,\beta}(g,x)-g(x)-g'(x)K_n^{\alpha,\beta}((u-x),x)\right.\\ &\quad \left.-\frac{g''(x)}{2}K_n^{\alpha,\beta}\left((u-x)^2,x\right)\right]\\ &\quad +\frac{g''(x)}{2}K_n^{\alpha,\beta}\left((u-x)^2,x\right)\left[\mathfrak{h}(x)-K_n^{\alpha,\beta}(\mathfrak{h},x)\right]\\ &\quad +\mathfrak{h}'(x)g'(x)K_n^{\alpha,\beta}\left((u-x)^2,x\right)-g'(x)K_n^{\alpha,\beta}((u-x),x)\\ &\quad \left.\cdot\left[K_n^{\alpha,\beta}(\mathfrak{h},x)-\mathfrak{h}(x)\right]\right\}.\end{aligned} \tag{92}$$

From Proposition 9 and Theorems 23 and 24, we obtain

$$\lim_{n\to\infty}(n+\beta)\left\{K_n^{\alpha,\beta}(\mathfrak{h}g,x)-K_n^{\alpha,\beta}(\mathfrak{h},x)K_n^{\alpha,\beta}(g,x)\right\} = x\mathfrak{h}'(x)g'(x). \tag{93}$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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