Research Article

Iterative Algorithm for Mappings Satisfying $(B_{\gamma,\mu})$ Condition

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In this article, we introduce a novel iterative algorithm to approximate fixed point of mappings with $(B_{\gamma,\mu})$ condition. We establish some strong and weak convergence results in a uniformly convex Banach space. Using a numerical example, we compare the speed of the proposed algorithm with some leading iterative algorithms.

1. Introduction and Preliminaries

Throughout the article, $\mathbb{P}$ stands for the set of all positive numbers and $\mathbb{R}$ for the set of all real numbers. Let $\mathcal{V}$ be a Banach space, $\mathcal{K} \neq \emptyset$ a subset of $\mathcal{V}$ and $f : \mathcal{K} \rightarrow \mathcal{K}$. An element $q \in \mathcal{K}$ is called a fixed point of f if $q = fq$. We denote the set $\{q \in \mathcal{K} : q = fq\}$ by $\mathcal{F}(f)$. f is called quasi-nonexpansive (resp., nonexpansive) if $\mathcal{F}(f) \neq \emptyset$ and $\|fv - fq\| \leq \|v - q\|$ for all $v \in \mathcal{K}$ and $q \in \mathcal{F}(f)$ (resp., $\|fv - fw\| \leq \|v - w\|$, for all $v, w \in \mathcal{K}$). It is well known that if $\mathcal{V}$ is uniformly convex and $\mathcal{K}$ is convex closed bounded then $\mathcal{F}(f)$ is nonempty (see [1–3]). In [4], Suzuki generalized the concept of nonexpansive mappings as follows.

Definition 1. Let $\mathcal{K} \neq \emptyset$ be a subset of a Banach space $\mathcal{V}$. A mapping $f : \mathcal{K} \rightarrow \mathcal{K}$ is said to satisfy the (C) condition (or Suzuki mappings) if

$$\frac{1}{2} \|v - fv\| \leq \|v - w\| \Rightarrow \|fv - fw\| \leq \|v - w\|, \quad (1)$$

for each $v, w \in \mathcal{K}$.

Remark 2. It is easy to see that every nonexpansive mapping satisfies the (C) condition.

Using the following example, Suzuki [4] showed that (C) condition is more general than the nonexpansiveness.

Example 3 (see [5]). Define a mapping $f : [0, 3] \rightarrow \mathbb{R}$ by

$$fv = \begin{cases} 0 & \text{if } v \neq 3, \\ 1 & \text{if } v = 3. \end{cases} \quad (2)$$

Here, f satisfies (C) condition, but not nonexpansive.

Recently in 2018, Patir et al. [5] generalized the notion of (C) condition as follows.

Definition 4. Let $\mathcal{K} \neq \emptyset$ be a subset of a Banach space $\mathcal{V}$. A mapping $f : \mathcal{K} \rightarrow \mathcal{K}$ is said to satisfy the $(B_{\gamma,\mu})$ condition if there exists $\gamma \in [0, 1]$ and $\mu \in [0, 1/2]$ satisfying $2\mu \leq \gamma$ such that for each $v, w \in \mathcal{K}$,

$$\gamma \|v - fv\| \leq \|v - w\| + \mu \|w - fw\| \quad (3)$$

implies $\|fv - fw\| \leq (1 - \gamma) \|v - w\| + \mu (\|v - fw\| + \|w - fv\|)$.

Remark 5. It is easy to see that a mapping with (C) condition satisfies the $(B_{\gamma,\mu})$ condition.

Using the following example, Patir et al. [5] showed that $(B_{\gamma,\mu})$ condition is more general than the (C) condition.

Example 6 (see [5]). Define a mapping $f : [0, 2] \rightarrow \mathbb{R}$ by

$$fv = \begin{cases} 0 & \text{if } v \neq 2, \\ 1 & \text{if } v = 2. \end{cases} \quad (4)$$

Here, f satisfies $(B_{\gamma,\mu})$ condition, but not (C).

The famous theorem of Banach suggests a Picard iterative algorithm, that is, $v_{n+1} = fv_n$ to approximate the unique fixed point of f , where f is a contraction mapping. It is now well-known that for a nonexpansive mapping f the Picard iterative algorithm may fail to converge to the fixed point of f . Therefore, it is natural to investigate new algorithms to overcome such problem and to get better speed of convergence. Nowadays, the well-known algorithms are the Mann [6], Ishikawa [7], Agarwal et al. [8], Noor [9], Abbas and Nazir [10], Thakur et al. [11], and so on. For some recent literature on iterative algorithms, we refer the reader to [12–18].

Mann [6] iterative algorithm is defined as

$$\begin{cases} v_1 \in \mathcal{K}, \\ v_{n+1} = (1 - \alpha_n)v_n + \alpha_n f v_n, \quad n \in \mathbb{P}, \end{cases} \quad (5)$$

where $\alpha_n \in (0, 1)$.

Ishikawa [7] iterative algorithm reads as follows:

$$\begin{cases} v_1 \in \mathcal{K}, \\ w_n = (1 - \beta_n)v_n + \beta_n f v_n, \\ v_{n+1} = (1 - \alpha_n)v_n + \alpha_n f w_n, \quad n \in \mathbb{P}, \end{cases} \quad (6)$$

where $\alpha_n, \beta_n \in (0, 1)$.

Agarwal et al. [8] generalized iterative algorithm (6) is as follows:

$$\begin{cases} v_1 \in \mathcal{K}, \\ w_n = (1 - \beta_n)v_n + \beta_n f v_n, \\ v_{n+1} = (1 - \alpha_n)f v_n + \alpha_n f w_n, \quad n \in \mathbb{P}, \end{cases} \quad (7)$$

where $\alpha_n, \beta_n \in (0, 1)$.

In 2016, Thakur et al. [11] used a new iterative algorithm. With the help of a numerical example, they showed that this algorithm is better than the Picard, Mann, Ishikawa, Agarwal, Noor, and Abbas iterative algorithms for the class of Suzuki mappings. The algorithm reads as follows:

$$\begin{cases} v_1 \in \mathcal{K}, \\ x_n = (1 - \beta_n)v_n + \beta_n f v_n, \\ w_n = f((1 - \alpha_n)v_n + \alpha_n x_n), \\ v_{n+1} = f w_n, \quad n \in \mathbb{P}, \end{cases} \quad (8)$$

where $\alpha_n, \beta_n \in (0, 1)$.

Motivated by the above, in this article, we propose a new iterative algorithm and prove some weak and strong convergence results for mapping satisfying $(B_{\gamma,\mu})$ condition in the context of uniformly convex Banach spaces. We also present a new numerical example of a mapping satisfying $(B_{\gamma,\mu})$ condition and numerically compare speed of the proposed iterative algorithm with the speed of leading iterative algorithms. Our results are new and never investigated before.

Following are some basic definitions and results needed in the sequel.

Definition 7 (see [19]). Recall that a Banach space $\mathcal{V}$ is said to be uniformly convex if for each $\eta \in (0, 2]$ there is a $\lambda > 0$ such that for every two elements v, w in $\mathcal{V}$,

$$\left. \begin{aligned} \|v\| &\leq 1 \\ \|w\| &\leq 1 \\ \|v - w\| &> \eta \end{aligned} \right\} \text{implies } \frac{1}{2} \|v + w\| \leq (1 - \lambda). \quad (9)$$

Definition 8. Let $\mathcal{K} \neq \emptyset$ be a subset of a Banach space $\mathcal{V}$ and $\{v_n\}$ a bounded sequence in $\mathcal{V}$. For each $v \in \mathcal{V}$ define

- (i) asymptotic radius of $\{v_n\}$ at v by $r(v, \{v_n\}) := \limsup_{n \rightarrow \infty} \|v - v_n\|$
- (ii) asymptotic radius of $\{v_n\}$ relative to $\mathcal{K}$ by $r(\mathcal{K}, \{v_n\}) = \inf \{r(v, \{v_n\}) : v \in \mathcal{K}\}$
- (iii) asymptotic center of $\{v_n\}$ relative to $\mathcal{K}$ by $A(\mathcal{K}, \{v_n\}) = \{v \in \mathcal{K} : r(v, \{v_n\}) = r(\mathcal{K}, \{v_n\})\}$

The set $A(\mathcal{K}, \{v_n\})$ is singleton provided that $\mathcal{V}$ is uniformly convex. Also, $A(\mathcal{K}, \{v_n\})$ is nonempty convex provided that $\mathcal{K}$ is convex and weakly compact, see, e.g., [20, 21].

Definition 9 (see [22]). A Banach space $\mathcal{V}$ is said to have Opial's property if for each sequence $\{v_n\}$ in $\mathcal{V}$ which weakly converges to $v \in \mathcal{V}$ and for every $w \in \mathcal{V} - \{v\}$ it follows that

$$\limsup_{n \rightarrow \infty} \|v_n - w\| < \limsup_{n \rightarrow \infty} \|v_n - v\|. \quad (10)$$

Definition 10 (see [23]). Let $\mathcal{K} \neq \emptyset$ be a subset of a Banach space $\mathcal{V}$. A mapping $f : \mathcal{K} \rightarrow \mathcal{K}$ is said to satisfy condition (I) if there is a function $\zeta : [0, \infty) \rightarrow [0, \infty)$ satisfying $\zeta(0) = 0$ and $\zeta(r) > 0$ for all $r \in (0, \infty)$ such that $\|v - fv\| \geq \zeta(d(v, \mathcal{F}(f)))$ for all $v \in \mathcal{K}$, where $d(v, \mathcal{F}(f))$ represents distance of v from $\mathcal{F}(f)$.

The following lemma gives many examples of mappings with $(B_{\gamma,\mu})$ condition.

Lemma 11. Let $\mathcal{K} \neq \emptyset$ be a subset of a Banach space $\mathcal{V}$. If f satisfies (C) condition, then f satisfies the $(B_{\gamma,\mu})$ condition.

Lemma 12 (see [5]). Let $\mathcal{K} \neq \emptyset$ be a subset of a Banach space $\mathcal{V}$ and $f : \mathcal{K} \rightarrow \mathcal{K}$ satisfies $(B_{\gamma,\mu})$ condition. If q is a fixed point of f , then for each $v \in \mathcal{K}$

$$\|q - fv\| \leq \|q - v\|. \quad (11)$$

Theorem 13 (see [5]). Let $\mathcal{K} \neq \emptyset$ be a subset of a Banach space $\mathcal{V}$ having Opial property. Let $f : \mathcal{K} \rightarrow \mathcal{K}$ satisfy $(B_{\gamma,\mu})$ condition. If $\{v_n\} \subseteq \mathcal{K}$ be such that

- (i) $\{v_n\}$ converges weakly to z
- (ii) $\lim_{n \rightarrow \infty} \|fv_n - v_n\| = 0$

then $z = fz$.

Proposition 14 (see [5]). Let $\mathcal{K} \neq \emptyset$ be a subset of a Banach space $\mathcal{V}$. If $f : \mathcal{K} \rightarrow \mathcal{K}$ satisfies the $(B_{\gamma,\mu})$ condition on $\mathcal{K}$, then for all $v, w \in \mathcal{K}$ and $s \in [0, 1]$,

- (i) $fv - f^2v \leq v - fv$
- (ii) at least one of the following ((a) and (b)) holds:
 - (a) $(s/2)v - fv \leq v - w$
 - (b) $(s/2)fv - f^2v \leq fv - w$

Condition (a) implies $\|fv - fw\| \leq (1 - (s/2))\|v - w\| + \mu(\|v - fw\| + \|w - fv\|)$ and condition (b) implies $\|f^2v - fw\| \leq (1 - (s/2))\|fv - w\| + \mu(\|fv - fw\| + \|w - f^2v\|)$.

- (iii) $\|v - fw\| \leq (3 - s)\|v - fv\| + (1 - (s/2))\|v - w\| + \mu(2\|v - fv\| + \|v - fw\| + \|w - fv\| + 2\|fv - f^2v\|)$

We need the following useful lemma from [24].

Lemma 15. Let $\mathcal{V}$ be a uniformly convex Banach space and $0 < q \leq \delta_n \leq p < 1$ for every $n \in \mathbb{N}$. If $\{s_n\}$ and $\{w_n\}$ are two sequences in $\mathcal{V}$ such that $\limsup_{n \rightarrow \infty} \|s_n\| \leq l$, $\limsup_{n \rightarrow \infty} \|w_n\| \leq l$, and $\lim_{n \rightarrow \infty} \|\delta_n s_n + (1 - \delta_n)w_n\| = l$ for some $l \geq 0$, then $\lim_{n \rightarrow \infty} \|s_n - w_n\| = 0$.

2. Main Results

In this section, we first introduce a new iterative algorithm, namely, JA iteration, to approximate fixed point of mappings satisfying condition $(B_{\gamma,\mu})$ as follows:

$$\begin{cases} v_1 \in \mathcal{K}, \\ x_n = (1 - \beta_n)v_n + \beta_n fv_n, \\ w_n = fx_n, \\ v_{n+1} = f((1 - \alpha_n)fv_n + \alpha_n fw_n), \quad n \in \mathbb{N}, \end{cases} \quad (12)$$

where $\alpha_n, \beta_n \in (0, 1)$.

Now we construct a new example of mapping which satisfies $(B_{\gamma,\mu})$ condition but not (C). We then use this example to compare the rate of convergence of JA iterative algorithm (12) with the leading Thakur (8) and leading two-step Agarwal iterative algorithm (7). This example also shows that the converse of Lemma 11 does not hold in general.

Example 16. Let $\mathcal{K} = [1, 3]$ be endowed with absolute valued norm. Define a mapping $f : \mathcal{K} \rightarrow \mathcal{K}$ by

$$fv = \begin{cases} \frac{1+v}{2} & \text{if } v \neq 3, \\ 1 & \text{if } v = 3. \end{cases} \quad (13)$$

Choose $v = 25/10$ and $w = 3$. We see that $1/2|v - fv| = 3/8 < 1/2 = |v - w|$ but $|fv - fw| = 3/4 > 1/2 = |v - w|$. Thus, f is not Suzuki mapping. Choose $\gamma = 1$ and $\mu = 1/2$, we prove that f satisfies the $(B_{1,1/2})$ condition.

Case 1. For $v, w \in [1, 3]$, we have

$$\begin{aligned} & (1 - \gamma)|v - w| + \mu(|v - fw| + |w - fv|) \\ &= \frac{1}{2}(|v - fw| + |w - fv|) \\ &= \frac{1}{2} \left(\left| v - \left(\frac{1+w}{2} \right) \right| + \left| w - \left(\frac{1+v}{2} \right) \right| \right) \\ &\geq \frac{1}{2} \left| \frac{3v}{2} - \frac{3w}{2} \right| = \frac{3}{4}|v - w| \geq \frac{1}{2}|v - w| = |fv - fw|. \end{aligned} \quad (14)$$

Case 2. For $v \in [1, 3]$ and $w = 3$, we have

$$\begin{aligned} & (1 - \gamma)|v - w| + \mu(|v - fw| + |w - fv|) \\ &= \frac{1}{2}(|v - fw| + |w - fv|) = \frac{1}{2} \left(|v - 1| + \left| w - \left(\frac{1+v}{2} \right) \right| \right) \\ &= \frac{1}{2}|v - 1| + \frac{1}{2} \left| w - \left(\frac{1+v}{2} \right) \right| \geq \frac{1}{2}|v - 1| = |fv - fw|. \end{aligned} \quad (15)$$

Case 3. For $v = w = 3$, we have

$$(1 - \gamma)|v - w| + \mu(|v - fw| + |w - fv|) \geq 0 = |fv - fw|. \quad (16)$$

Hence, f satisfies the $(B_{1,1/2})$ condition. Now we show that algorithm (12) converges fast to $q = 1$ as compared to the leading Thakur (8) and Agarwal (7). Choose $v_1 = 1.5$,

$\alpha_n = \beta_n = 1/2$. Table 1 and Figure 1 show the efficiency of the proposed iterative algorithm.

Now using Example 16, we obtained the influence of initial points and parameters for the Agarwal (7), Thakur (8), and JA (12) iterations. In Table 2, items in bold show that the JA iteration (12) converges faster than others.

Now we establish some convergence theorems for mappings with $(B_{\gamma,\mu})$ condition using algorithm (12). Similar results for the other algorithms can be established on the same line of proofs.

Lemma 17. *Let $\mathcal{K} \neq \emptyset$ be a closed convex subset of a Banach space $\mathcal{V}$ and $f : \mathcal{K} \rightarrow \mathcal{K}$ satisfies the $(B_{\gamma,\mu})$ condition with $\mathcal{F}(f) \neq \emptyset$. If $\{v_n\}$ is a sequence generated by (12), then $\lim_{n \rightarrow \infty} \|v_n - q\|$ exists for each $q \in \mathcal{F}(f)$.*

Proof. Let $q \in \mathcal{F}(f)$. By Lemma 12, we have

$$\begin{aligned} \|x_n - q\| &= \|(1 - \beta_n)v_n + \beta_n f v_n - q\| \\ &\leq (1 - \beta_n)\|v_n - q\| + \beta_n \|f v_n - q\| \\ &\leq (1 - \beta_n)\|v_n - q\| + \beta_n \|v_n - q\| \leq \|v_n - q\|, \\ \|w_n - q\| &= \|f x_n - q\| \leq \|x_n - q\|. \end{aligned} \quad (17)$$

They imply that

$$\begin{aligned} \|v_{n+1} - q\| &= \|f((1 - \alpha_n)v_n + \alpha_n f w_n) - q\| \\ &\leq \|(1 - \alpha_n)f v_n + \alpha_n f w_n - q\| \\ &\leq (1 - \alpha_n)\|f v_n - q\| + \alpha_n \|f w_n - q\| \\ &\leq (1 - \alpha_n)\|v_n - q\| + \alpha_n \|w_n - q\| \\ &\leq (1 - \alpha_n)\|v_n - q\| + \alpha_n \|x_n - q\| \\ &\leq (1 - \alpha_n)\|v_n - q\| + \alpha_n \|v_n - q\| \\ &= \|v_n - q\|. \end{aligned} \quad (18)$$

Thus, $\{\|v_n - q\|\}$ is bounded and nonincreasing, and hence, $\lim_{n \rightarrow \infty} \|v_n - q\|$ exists for each $q \in \mathcal{F}(f)$.

Theorem 18. *Let $\mathcal{K} \neq \emptyset$ be a closed convex subset of a uniformly convex Banach space $\mathcal{V}$ and $f : \mathcal{K} \rightarrow \mathcal{K}$ a mapping satisfying the $(B_{\gamma,\mu})$ condition. If $\{v_n\}$ is a sequence generated by (12), then $\mathcal{F}(f) \neq \emptyset$ if and only if $\{v_n\}$ is bounded and $\lim_{n \rightarrow \infty} \|f v_n - v_n\| = 0$.*

Proof. Suppose $\mathcal{F}(f) \neq \emptyset$ and $q \in \mathcal{F}(f)$. Then, by Lemma 17, $\lim_{n \rightarrow \infty} \|v_n - q\|$ exists and $\{v_n\}$ is bounded. Put

$$\lim_{n \rightarrow \infty} \|v_n - q\| = l. \quad (19)$$

By the proof of Lemma 17 together with (19), we have

$$\limsup_{n \rightarrow \infty} \|x_n - q\| \leq \limsup_{n \rightarrow \infty} \|v_n - q\| = l. \quad (20)$$

TABLE 1: Sequences generated by JA (12), Thakur et al. (8), and Agarwal (7) iterative algorithms.

	JA (12)	Thakur et al. (8)	Agarwal (7)
v_1	1.500000000	1.500000000	1.500000000
v_2	1.085937500	1.109375000	1.218750000
v_3	1.014770507	1.023925781	1.0957031250
v_4	1.002538681	1.005233764	1.0418701171
v_5	1.000436335	1.001144886	1.0183181762
v_6	1.000074995	1.000250443	1.0080142021
v_7	1.000012889	1.000054784	1.0035062134
v_8	1.000002215	1.000011984	1.0015339683
v_9	1.000000380	1.000002621	1.0006711111
v_{10}	1.000000065	1.000000573	1.0002936111
v_{11}	1.000000011	1.000000125	1.0001284548
v_{12}	1.000000001	1.000000027	1.0000561990
v_{13}	1	1.000000006	1.0000245870
v_{14}	1	1.000000001	1.0000107568
v_{15}	1	1	1.0000047061

By Lemma 12, we have

$$\limsup_{n \rightarrow \infty} \|f v_n - q\| \leq \limsup_{n \rightarrow \infty} \|v_n - q\| = l. \quad (21)$$

Again by the proof of Lemma 17, we have

$$\|v_{n+1} - q\| \leq (1 - \alpha_n)\|v_n - q\| + \alpha_n \|x_n - q\|. \quad (22)$$

It follows that

$$\begin{aligned} \|v_{n+1} - q\| - \|v_n - q\| &\leq \frac{\|v_{n+1} - q\| - \|v_n - q\|}{\alpha_n} \\ &\leq \|x_n - q\| - \|v_n - q\|. \end{aligned} \quad (23)$$

So, we can get $\|v_{n+1} - q\| \leq \|x_n - q\|$. Therefore,

$$l \leq \liminf_{n \rightarrow \infty} \|x_n - q\|. \quad (24)$$

From (20) and (24), we obtain

$$l = \lim_{n \rightarrow \infty} \|x_n - q\|. \quad (25)$$

From (25), we have

$$\begin{aligned} l &= \lim_{n \rightarrow \infty} \|x_n - q\| = \lim_{n \rightarrow \infty} \|(1 - \beta_n)v_n + \beta_n f v_n - q\| \\ &= \lim_{n \rightarrow \infty} \|(1 - \beta_n)(v_n - q) + \beta_n(f v_n - q)\|. \end{aligned} \quad (26)$$

Hence,

$$l = \lim_{n \rightarrow \infty} \|(1 - \beta_n)(v_n - q) + \beta_n(f v_n - q)\|. \quad (27)$$

FIGURE 1: Convergence behavior of JA, Thakur et al., and Agarwal iterative algorithms for mapping f defined in Example 16.

Now from (19), (21), and (27) together with Lemma 15, we obtain

$$\lim_{n \rightarrow \infty} \|f v_n - v_n\| = 0. \quad (28)$$

Conversely, let $q \in A(\mathcal{K}, \{v_n\})$. By Proposition 14 (iii), for $\gamma = s/2$, $s \in [0, 1]$, $\|v_n - f q\| \leq (3 - s)\|v_n - f v_n\| + (1 - s/2)\|v_n - q\| + \mu(2\|v_n - f v_n\| + \|v_n - f q\| + \|q - f v_n\| + 2\|f v_n - f^2 v_n\|) \leq (3 - s)\|v_n - f v_n\| + (1 - s/2)\|v_n - q\| + \mu(2\|v_n - f v_n\| + \|v_n - f q\| + \|v_n - q\| + \|v_n - f v_n\| + 2\|v_n - f v_n\|)$ (by Proposition 14 (i))

$$\begin{aligned} \Rightarrow (1 - \mu) \limsup_{n \rightarrow \infty} \|v_n - f q\| &\leq \left(1 - \frac{s}{2} + \mu\right) \limsup_{n \rightarrow \infty} \|v_n - q\| \\ \Rightarrow \limsup_{n \rightarrow \infty} \|v_n - f q\| &\leq \left(\frac{1 - (s/2) + \mu}{1 - \mu}\right) \limsup_{n \rightarrow \infty} \|v_n - q\| \\ &\leq \limsup_{n \rightarrow \infty} \|v_n - q\| \left(\text{as } \frac{1 - (s/2) + \mu}{1 - \mu} \leq 1, \text{ for } 2\mu \leq \gamma = \frac{s}{2}\right) \\ \Rightarrow r(fq, \{v_n\}) &\leq r(q, \{v_n\}). \end{aligned} \quad (29)$$

So $f q \in A(\mathcal{K}, \{v_n\})$. Since $\mathcal{V}$ is uniformly convex Banach space, so $f q = q$.

Theorem 19. Let $\mathcal{K} \neq \emptyset$ be a closed convex subset of a uniformly convex Banach space $\mathcal{V}$ having Opial property. If $f : \mathcal{K} \rightarrow \mathcal{K}$ satisfies the $(B_{\gamma\mu})$ condition with $\mathcal{F}(f) \neq \emptyset$, then $\{v_n\}$ generated by (12) converges weakly to an element of $\mathcal{F}(f)$.

Proof. By Theorem 18, the sequence $\{v_n\}$ is bounded and $\lim_{n \rightarrow \infty} \|f v_n - v_n\| = 0$. Since $\mathcal{V}$ is uniformly convex, we have $\mathcal{V}$ that is reflexive. Thus, a subsequence $\{v_{n_i}\}$ of $\{v_n\}$ exists such that $\{v_{n_i}\}$ converges weakly to some $q_1 \in \mathcal{K}$. By Theorem 13, $q_1 \in \mathcal{F}(f)$. It is suffice to show that $\{v_n\}$ converges weakly to q_1 . Indeed, if $\{v_n\}$ does not converge weakly to q_1 , then there exists a subsequence $\{v_{n_j}\}$ of $\{v_n\}$ and $q_2 \in$

TABLE 2: Influence of parameters: comparison of various iteration processes.

Iterations	Initial points					
	1.2	1.5	1.9	2.2	2.5	2.9
For $\alpha_n = \frac{n}{(n+2)^{10/9}}, \beta_n = \frac{1}{(n+4)^{2/3}}$						
Agarwal	27	28	29	30	30	30
Thakur	14	15	15	15	16	16
JA	12	13	13	13	13	13
For $\alpha_n = \frac{n^{1/2}}{(n+5)^{1/5}}, \beta_n = \frac{1}{(7n+9)^{1/3}}$						
Agarwal	22	23	23	24	24	24
Thakur	13	13	14	14	14	14
JA	8	9	9	9	9	9
For $\alpha_n = \frac{1}{(3n+5)^{1/5}}, \beta_n = \frac{n}{7n+9}$						
Agarwal	28	29	30	30	31	31
Thakur	14	15	16	16	16	16
JA	12	13	13	13	13	13
For $\alpha_n = \left(\frac{n+1}{5n+1}\right)^{1/7}, \beta_n = \frac{1}{(2n+3)^{1/2}}$						
Agarwal	26	27	28	28	28	29
Thakur	14	14	15	15	15	15
JA	10	11	12	12	12	12

$\mathcal{K}$ such that $\{v_{n_j}\}$ converges weakly to q_2 and $q_2 \neq q_1$. Again by Theorem 13, $q_2 \in \mathcal{F}(f)$. By Lemma 17 together with Opial property, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \|v_n - q_1\| &= \lim_{i \rightarrow \infty} \|v_{n_i} - q_1\| < \lim_{i \rightarrow \infty} \|v_{n_i} - q_2\| \\ &= \lim_{n \rightarrow \infty} \|v_n - q_2\| = \lim_{j \rightarrow \infty} \|v_{n_j} - q_2\| \\ &< \lim_{j \rightarrow \infty} \|v_{n_j} - q_1\| = \lim_{n \rightarrow \infty} \|v_n - q_1\|. \end{aligned} \quad (30)$$

This is a contradiction. So, $q_1 = q_2$. Thus, $\{v_n\}$ converges weakly to $q_1 \in \mathcal{F}(f)$.

Theorem 20. Let $\mathcal{K} \neq \emptyset$ be a closed and convex subset of a uniformly convex Banach space $\mathcal{V}$ and $f : \mathcal{K} \rightarrow \mathcal{K}$ satisfies the $(B_{\gamma,\mu})$ condition with $\mathcal{F}(f) \neq \emptyset$. If $\{v_n\}$ is a sequence generated by (12), then $\{v_n\}$ converges to an element of $\mathcal{F}(f)$ if and only if $\liminf_{n \rightarrow \infty} d(v_n, \mathcal{F}(f)) = 0$ or $\limsup_{n \rightarrow \infty} d(v_n, \mathcal{F}(f)) = 0$.

Proof. The necessity is obvious.

Conversely, suppose that $\liminf_{n \rightarrow \infty} d(v_n, \mathcal{F}(f)) = 0$ and $q \in \mathcal{F}(f)$. From Lemma 17, $\lim_{n \rightarrow \infty} \|v_n - q\|$ exists for each $q \in \mathcal{F}(f)$. Therefore, $\lim_{n \rightarrow \infty} d(v_n, \mathcal{F}(f)) = 0$, by assumption. We prove that $\{v_n\}$ is a Cauchy sequence in $\mathcal{K}$. As $\lim_{n \rightarrow \infty} d(v_n, \mathcal{F}(f)) = 0$, for a given $\varepsilon > 0$, there exists $k_0 \in \mathbb{P}$ such that for each $n \geq k_0$,

$$\begin{aligned} d(v_n, \mathcal{F}(f)) &< \frac{\varepsilon}{2} \\ \Rightarrow \inf \{ \|v_n - q\| : q \in \mathcal{F}(f) \} &< \frac{\varepsilon}{2}. \end{aligned} \quad (31)$$

In particular, $\inf \{ \|v_{k_0} - q\| : q \in \mathcal{F}(f) \} < \varepsilon/2$. Therefore, there exists $q \in \mathcal{F}(f)$ such that $\|v_{k_0} - q\| < \varepsilon/2$.

Now for $k, n \geq k_0$,

$$\begin{aligned} \|v_{n+k} - v_n\| &\leq \|v_{n+k} - q\| + \|v_n - q\| \\ &\leq \|v_{k_0} - q\| + \|v_{k_0} - q\| = 2\|v_{k_0} - q\| < \varepsilon. \end{aligned} \quad (32)$$

This shows that $\{v_n\}$ is a Cauchy sequence in $\mathcal{K}$. As $\mathcal{K}$ is a closed subset of a Banach space $\mathcal{V}$, so there exists a point $p \in \mathcal{K}$ such that $\lim_{n \rightarrow \infty} v_n = p$. Now $\lim_{n \rightarrow \infty} d(v_n, \mathcal{F}(f)) = 0$ gives that $d(p, \mathcal{F}(f)) = 0$. The closeness of $\mathcal{F}(f)$ follows from Lemma 12. Hence, $p \in \mathcal{F}(f)$.

We now prove the following theorem using condition (I).

Theorem 21. Let $\mathcal{K} \neq \emptyset$ be a closed and convex subset of a uniformly convex Banach space $\mathcal{V}$ and $f : \mathcal{K} \rightarrow \mathcal{K}$ satisfies the $(B_{\gamma,\mu})$ condition with $\mathcal{F}(f) \neq \emptyset$. If $\{v_n\}$ is a sequence generated by (12), then $\{v_n\}$ converges strongly to an element of $\mathcal{F}(f)$ provided that f satisfies the condition (I).

Proof. From Theorem 18, we have

$$\lim_{n \rightarrow \infty} \|f v_n - v_n\| = 0. \quad (33)$$

Thus, by condition (I), we obtain

$$\lim_{n \rightarrow \infty} d(v_n, \mathcal{F}(f)) = 0. \quad (34)$$

Now all the requirements of Theorem 20 are satisfied; therefore, by its conclusion $\{v_n\}$ converges strongly to a fixed point of f .

Remark 22. In the view of Lemma 11, our results contain the results for mappings with (C) condition.

3. Conclusions

We have introduced a new faster iterative algorithm, namely, JA iteration, for the approximation of fixed points of the class of mapping with $(B_{\gamma,\mu})$ condition. Moreover, the class of mappings with $(B_{\gamma,\mu})$ condition is essentially more general than the class of Suzuki mappings. Thus, our results are better in both the senses (faster iteration and general mapping). Hence, our results are the good extension of the corresponding results of Abbas and Nazir [10], Thakur et al. [11], and references therein.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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