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*Research Article*

# A Characterization of Central BMO Space via the Commutator of Fractional Hardy Operator

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This paper is devoted in characterizing the central BMO $(\mathbb{R}^n)$ space via the commutator of the fractional Hardy operator with rough kernel. Precisely, by a more explicit decomposition of the operator and the kernel function, we will show that if the symbol function belongs to the central BMO $(\mathbb{R}^n)$ space, then the commutator are bounded on Lebesgue space. Conversely, the boundedness of the commutator implies that the symbol function belongs to the central BMO $(\mathbb{R}^n)$ space by exploiting the center symmetry of the Hardy operator deeply.

## 1. Introduction

In this paper, we focus on the need for characterizing the central BMO $(\mathbb{R}^n)$ space via the boundedness of the commutators of the following fractional Hardy operators

$$\begin{aligned} H_{\Omega,\alpha}f(x) &= \frac{1}{|x|^{n-\alpha}}\int_{|y|<|x|}\Omega(x-y)f(y)dy, \\ H^*_{\Omega,\alpha}f(x) &= \int_{|y|\geq|x|}\frac{\Omega(x-y)f(y)}{|y|^{n-\alpha}}dy, \\ &0<\alpha<n. \end{aligned} \tag{1}$$

$H^*_{\Omega,\alpha}$ is the dual operator of $H_{\Omega,\alpha}$. Here, $\Omega$ satisfies

$$\Omega(tx)=\Omega(x), \forall t>0, x\in\mathbb{R}^n; \tag{2}$$

$$\int_{\mathbb{S}^{n-1}}\Omega\left(x'\right)d\sigma\left(x'\right)=0; \tag{3}$$

$$\Omega\in L^q\left(\mathbb{S}^{n-1}\right)\forall q>1; \tag{4}$$

and $\mathbb{S}^{n-1}=\{x\in\mathbb{R}^n : |x|=1\}$ denotes the unit sphere in $\mathbb{R}^n$.

For a function $b$, the commutators of $H_{\Omega,\alpha}$ and $H^*_{\Omega,\alpha}$ can be written as

$$\begin{aligned} [b,H_{\Omega,\alpha}]f &:= b(H_{\Omega,\alpha}f)-H_{\Omega,\alpha}(bf), \\ [b,H^*_{\Omega,\alpha}]f &:= b\left(H^*_{\Omega,\alpha}f\right)-H^*_{\Omega,\alpha}(bf). \end{aligned} \tag{5}$$

In [1], Fu et al. considered the boundedness of $H_{\Omega,\alpha}$ and $[b,H_{\Omega,\alpha}]$ on homogeneous Herz spaces and Lebesgue spaces under the assumption that $\Omega$ satisfies (2) and (4). We recall the results from ([1], Proposition 3.1 and Theorem 3.1) as

Suppose that

$$\begin{cases} 1<p_1,p_2<\infty \text{ with } \dfrac{1}{p_1}=\dfrac{1}{p_2}+\dfrac{\alpha}{n}; \\ \dfrac{1}{p_1}+\dfrac{1}{s}+\dfrac{1}{q}=1 \text{ for some } s>p_1'=\dfrac{p_1}{p_1-1}; \\ \Omega \text{ satisfies } (1.1) \text{ and } (1.3); \\ b\in \mathrm{CBMO}^{\max\{p_2,s\}}(\mathbb{R}^n). \end{cases} \tag{6}$$

Then,

$$\begin{cases} H_{\Omega,\alpha} \text{ is bounded from } L^{p_1}(\mathbb{R}^n) \text{ to } L^{p_2}(\mathbb{R}^n); \\ [b; H_{\Omega,\alpha}] \text{ is bounded from } L^{p_1}(\mathbb{R}^n) \text{ to } L^{p_2}(\mathbb{R}^n). \end{cases} \tag{7}$$

For the boundedness of the classical fractional Hardy operator, see [2]. For a ball $B_r := B(0, r) \subset \mathbb{R}^n$ (i.e., a ball centered at 0 with radius $r > 0$) and $p \geq 1$, $\mathrm{CBMO}^p(\mathbb{R}^n)$ is the central BMO $(\mathbb{R}^n)$ function space, which was introduced by Lu and Yang [3] via the norm

$$\|b\|_{\mathrm{CBMO}^p(\mathbb{R}^n)} = \sup_{r>0} \left( \frac{1}{|B_r|} \int_{B_r} |b(x) - b_{B_r}|^p dx \right)^{1/p} \quad \text{with } b_{B_r} = \frac{1}{|B_r|} \int_{B_r} b(x) dx. \tag{8}$$

It is easy to see that CBMO $(\mathbb{R}^n)$ can be understood as a local version of the classical BMO $(\mathbb{R}^n)$ space at the origin [4] and

$$\begin{aligned} &\mathrm{BMO}(\mathbb{R}^n) \subset \mathrm{CBMO}(\mathbb{R}^n), \\ &\mathrm{CBMO}^{p>1}(\mathbb{R}^n) \subset \mathrm{CBMO}(\mathbb{R}^n). \end{aligned} \tag{9}$$

Hence, the famous John-Nirenberg inequality for BMO $(\mathbb{R}^n)$ space is not true for CBMO $(\mathbb{R}^n)$ space, which reveals that they have quite different properties.

In the study of harmonic analysis, the characterization of function spaces via the boundedness of the commutators plays an important role in the field of PDEs, see, for example, [5–11] and the references therein. However, there are less attention paid for the commutators with rough kernels since the characterization depends heavily on the smoothness of the kernel function $\Omega$. Under the premise that $\Omega$ is smooth enough, i.e., $\Omega \in C^\infty(\mathbb{S}^{n-1})$ or $\Omega \in \mathrm{Lip}_1(\mathbb{S}^{n-1})$, see, for example, [12–15]. It is difficult to weaken the smoothness of $\Omega$, Chen and Ding [16] considered a characterization of BMO $(\mathbb{R}^n)$ space under the condition that $\Omega$ satisfies the following Hölder condition of log type

$$\left|\Omega(x') - \Omega(y')\right| \leq \frac{A}{\left(\log\left(2/|x' - y'|\right)\right)^\gamma} \quad \text{with } A > 0, \gamma > 1, x', y' \in \mathbb{S}^{n-1}. \tag{10}$$

It is easy to check that (10) is weaker than the Lipschitz condition and stronger than the condition (4). For $q \geq 1$ and $\Omega \in L^q(\mathbb{S}^{n-1})$, we call $\Omega$ satisfies the $L^q$–Dini condition if $\int_0^1 ((w_q(\delta))/(\delta)) < \infty$, where $w_q(\delta)$ is defined as

$$w_q(\delta) = \sup_{\|\tau\|\leq\delta} \left( \int_{\mathbb{S}^{n-1}} \left|\Omega(\tau x') - \Omega(x')\right|^q d\sigma(x') \right)^{1/q} \quad \text{with } \|\tau\| = \sup_{x' \in \mathbb{S}^{n-1}} |\tau x' - x'|. \tag{11}$$

As a useful supplement of [1], we give a characterization of the CBMO $(\mathbb{R}^n)$ space via the boundedness of $[b, H_{\Omega,\alpha}]$ and $[b, H^*_{\Omega,\alpha}]$ as follows.

**Theorem 1.** *Let $1 < p_1, p_2 < \infty$ with $(1/p_1) = (1/p_2) + (\alpha/n)$ and $(1/p_1) + (1/s)(1/q) + 1$ for some $s > p_1'$.*

*(a) If*

$$\begin{cases} \Omega \text{ satisfies } (1.1) \text{ and } (1.3); \\ b \in CBMO^{\max\{p_2,s\}}(\mathbb{R}^n), \end{cases} \tag{12}$$

*then*

$$\begin{cases} [b, H_{\Omega,\alpha}] \text{ is bounded from } L^{p_1}(\mathbb{R}^n) \text{ to } L^{p_2}(\mathbb{R}^n); \\ [b, H_{\Omega,\alpha}] \text{ is bounded from } L^{p_1}(\mathbb{R}^n) \text{ to } L^{p_2}(\mathbb{R}^n). \end{cases} \tag{13}$$

*(b) if*

$$\begin{cases} \Omega \text{ satisfies } (1.1), (1.2), \text{and } (1.4); \\ [b, H_{\Omega,\alpha}] \text{ is bounded from } L^{p_1}(\mathbb{R}^n) \text{ to } L^{p_2}(\mathbb{R}^n); \\ [b, H_{\Omega,\alpha}] \text{ is bounded from } L^{p_1}(\mathbb{R}^n) \text{ to } L^{p_2}(\mathbb{R}^n), \end{cases} \text{ then } b \in CBMO(\mathbb{R}^n). \tag{14}$$

A part of Theorem 1 has been proven in [1], we will show the rest of Theorem 1 in Section 2. In what follows, we will denote $C$ by a positive constant which may vary from line to line. The symbol $A \lesssim B$ means $A \leq CB$ and $\mathbb{Z}$ for the set of all integers. Last, but not least, $B_r := B(0, r)$, $B_k := B_{2^k}$, $C_k := B_k \setminus B_{k-1}$, and $\chi_k := \chi_{C_k}$ with $k \in \mathbb{Z}$.

## 2. Proof of Theorem 1.1

We prove Theorem 1 in this section. To do so, we need one lemma about the estimates of the kernel function $\Omega$, which plays a key role in the proof.

**Lemma 2.** *Let $\Omega$ satisfy (2) and (10). Then,*

(a) $|\Omega(x-y)-\Omega(x)| \le C/(\log(|x|/|y|))^{\gamma}$ *with $\gamma$ be given in (10) and $|x| \ge 4|y|$.*

(b) *If $\Omega$ furthermore satisfies the $L^{q\ge 1}$–Dini condition, then, there is a $C>0$ such that for $0<C<1/2$, $r>0$, $x\in\mathbb{R}^n$ with $|x|<Cr$; we have*

$$\begin{cases} \left(\int_{r<|y|<2r} |\Omega(y-x)-\Omega(y)|^q dy\right)^{1/q} \le Cr^{n/q}\int_{|x|/2r}^{|x|/r} \frac{w_q(\delta)}{\delta} d\delta\,; \\ \left(\int_{r<|y|<2r} \frac{|\Omega(y-x)-\Omega(y)|^q}{|y|^{(n-\alpha)q}} dy\right)^{1/q} \le Cr^{-(n/q')+\alpha}\int_{|x|/2r}^{|x|/r} \frac{w_q(\delta)}{\delta} d\delta \quad \text{with}\, 0<\alpha<\frac{n}{p_1}. \end{cases} \tag{15}$$

*Proof.* We give the proof by a slight modification from [17]. For $|x|\ge 4|y|$, we first show that

$$\left|(x-y)'-x'\right| \le \frac{3|y|}{|y|}, \tag{16}$$

$$|1/|x|-1/|x-y|| \le C|y|/|x|^2. \tag{17}$$

Indeed, the first inequality can be obtained immediately from ([18], Lemma 2). Since

$$|1/|x|-1/|x-y|| \le 4|y|/3|x|^2 \le C|y|/|x|^2, \tag{18}$$

(a) can be shown by (2) and (10) as

$$\begin{aligned} |\Omega(x-y)-\Omega(x)| &\le \frac{C}{\left(\log\left(2/\left|(x-y)'-x'\right|\right)\right)^{\gamma}} \\ &\le \frac{C}{(\log(|x|/|y|))^{\gamma}}. \end{aligned} \tag{19}$$

We are left to show (b). The first estimate can be obtained directly by the $L^q$–Dini condition as

$$\begin{aligned} &\left(\int_{r<|y|<2r} |\Omega(y-x)-\Omega(y)|^q dy\right)^{1/q} \\ &= \left(\int_r^{2r}\int_{\mathbb{S}^{n-1}} \left|\Omega\left(y'-t^{-1}x'\right)-\Omega\left(y'\right)\right|^q d\delta\left(y'\right) t^{n-1} dt\right)^{1/q} \\ &\le Cr^{n/q}\int_{|x|/2r}^{|x|/r} \frac{w_q(\delta)}{\delta} d\delta, \end{aligned} \tag{20}$$

and the estimate

$$\int_{\mathbb{S}^{n-1}} \left|\Omega\left(y'-t^{-1}x'\right)-\Omega\left(y'\right)\right|^q d\delta\left(y'\right) \le Cw_q^q\left(\frac{|x|}{t}\right). \tag{21}$$

follows from ([19], p.65-77).

Accordingly, the second can be deduced similarly. In fact,

$$\begin{aligned} &\left(\int_{r<|y|<2r} \frac{|\Omega(y-x)-\Omega(y)|^q}{|y|^{(n-\alpha)q}} dy\right)^{1/q} \\ &= Cr^{-(n/q')+\alpha}\left(\int_r^{2r}\int_{\mathbb{S}^{n-1}} \left|\Omega\left(y'-t^{-1}x'\right) - \Omega\left(y'\right)\right|^q d\delta\left(y'\right)\frac{dt}{t}\right)^{1/q} \\ &\le Cr^{-(n/q')+\alpha}\int_{|x|/2r}^{|x|/r} \frac{w_q(\delta)}{\delta} d\delta, \end{aligned} \tag{22}$$

which is the desired one.

The following is the boundedness of the fractional Hardy operators.

**Theorem 3.** *Let $1<p_1,p_2<\infty$ with $(1/p_1)=(1/p_2)+(\alpha/n)$ and $\Omega$ satisfy (2) and (4). Then, both $H_{\Omega,\alpha}$ and $H^*_{\Omega,\alpha}$ are bounded from $L^{p_1}(\mathbb{R}^n)$ to $L^{p_2}(\mathbb{R}^n)$.*

*Proof.* Since the $(L^{p_1}(\mathbb{R}^n), L^{p_2}(\mathbb{R}^n))$ boundedness for $H_{\Omega,\alpha}$ is contained in ([1], Proposition 3.1), it is enough to check the boundedness for $H^*_{\Omega,\alpha}$. Namely, the task is now left to show that there exist constants $C>0$ such that for any $f\in L^{p_1}(\mathbb{R}^n)$, one has

$$\|H^*_{\Omega}f\|_{L^{p_2}(\mathbb{R}^n)} \le C\|f\|_{L^{p_1}(\mathbb{R}^n)}. \tag{23}$$

To do so, we first recall a useful estimates from [1] as

$$\begin{aligned} |x-y| \le |x|+2^i \le 2^{k+1} \,\&\, &\int_{C_i} |\Omega(x-y)|^q dy \\ &\le \int_0^{2^{k+1}}\int_{\mathbb{S}^{n-1}} \left|\Omega\left(x'\right)\right|^q d\delta\left(x'\right) t^{n-1} dt \le C2^{kn} \end{aligned} \tag{24}$$

for $x \in C_k$, $y \in C_i$, and $i \le k$. Since

$$\begin{aligned}\int_{\mathbb{R}^n} |H_\Omega^* f(x)|^{p_2} dx &\le \sum_{k=-\infty}^{+\infty} \int_{C_k} \left| \int_{|x|\le|y|\le 2^{kn}} \frac{\Omega(x-y)f(y)}{|y|^{n-\alpha}} dy \right|^{p_2} dx \\ &+ \sum_{k=-\infty}^{+\infty} \int_{C_k} \left| \int_{|y|>2^{kn}} \frac{\Omega(x-y)f(y)}{|y|^{n-\alpha}} dy \right|^{p_2} dx \\ &:= I_1 + I_2. \end{aligned} \tag{25}$$

Applying Hölder's inequality to $q, p_1, s$ for $s > 1$ and (24), we have

$$\begin{aligned} I_1 &\le \sum_{k=-\infty}^{+\infty} 2^{-k(n-\alpha)p_2} \int_{C_k} \left| \sum_{i=-\infty}^{k} \left( \int_{C_i} |\Omega(x-y)|^q dy \right)^{1/q} \right. \\ &\quad \left. \cdot \left( \int_{C_i} |f(y)|^{p_1} dy \right)^{1/p_1} \left( \int_{C_i} dy \right)^{1/s} \right|^{p_2} dx \\ &= C \sum_{k=-\infty}^{+\infty} \left| \sum_{i=-\infty}^{k} 2^{(i-k)n/s} \left( \int_{C_i} |f(y)|^{p_1} dy \right)^{1/p_1} \right|^{p_2} \\ &\le C \sum_{k=-\infty}^{+\infty} \sum_{i=-\infty}^{k} 2^{(i-k)np_2/2s} \|f\|_{L^{p_1}(B_i)}^{p_2} \left( \sum_{i=-\infty}^{k} 2^{(i-k)np_2'/2s} \right)^{p_2/p_2'} \\ &\le C \sum_{k=-\infty}^{+\infty} \sum_{i=-\infty}^{k} 2^{(i-k)np_2/2s} \|f\|_{L^{p_1}(B_i)}^{p_2} \\ &\le C \sum_{i=-\infty}^{+\infty} \|f\|_{L^{p_1}(B_i)}^{p_2} \sum_{k=i}^{+\infty} 2^{(i-k)np_2/2s} \le C \|f\|_{L^{p_1}(\mathbb{R}^n)}^{p_2}, \\ I_2 &\le \sum_{k=-\infty}^{+\infty} \int_{B_k} \left| \sum_{i=k}^{\infty} \left( \int_{C_i} |\Omega(x-y)|^q dy \right)^{1/q} \right. \\ &\quad \left. \cdot \left( \int_{C_i} |f(y)|^{p_1} dy \right)^{1/p_1} \left( \int_{C_i} |y|^{-(n-\alpha)s} dy \right)^{1/s} \right|^{p_2} dx \\ &\le C \sum_{k=-\infty}^{+\infty} \left| \sum_{i=k}^{+\infty} 2^{(k-i)n/p_2} \|f\|_{L^{p_1}(B_i)} \right|^{p_2} \\ &\le C \sum_{k=-\infty}^{+\infty} \sum_{i=k}^{+\infty} 2^{(k-i)n/2p_2} \|f\|_{L^{p_1}(B_i)}^{p_2} \\ &\le C \sum_{i=-\infty}^{+\infty} \|f\|_{L^{p_1}(B_i)}^{p^2} \sum_{k=-\infty}^{i} 2^{(k-i)n/2p_2} \le C \|f\|_{L^{p_1}(\mathbb{R}^n)}^{p_2}, \end{aligned} \tag{26}$$

which is our desired result.

Now, we can prove Theorem 1. Without loss of generality, we can assume that $\|b\|_{CBMO^{\max\{p_2,s\}}(\mathbb{R}^n)} = 1$ in the proof of (a) since $b \in \mathrm{CBMO}^{\max\{p_2,s\}}(\mathbb{R}^n)$. We see at once that the $(L^{p_1}(\mathbb{R}^n), L^{p_2}(\mathbb{R}^n))$ boundedness of $[b, H_{\Omega,\alpha}]$ is just ([1], Theorem 3.1). To complete the proof of (a), what is left is to show is that

$$\| [b, H_{\Omega,\alpha}^*] f \|_{L^{p_2}(\mathbb{R}^n)} \le C \|f\|_{L^{p_1}(\mathbb{R}^n)}. \tag{27}$$

It is easy to check that

$$\begin{aligned} &\int_{\mathbb{R}^n} \left| [b, H_{\Omega,\alpha}^*] f(x) \right|^{p_2} dx \\ &= \sum_{k=-\infty}^{+\infty} \int_{C_k} \left| \int_{|y|\ge|x|} \frac{\Omega(x-y)(b(x)-b(y))f(y)}{|y|^{n-\alpha}} dy \right|^{p_2} dx \\ &= \sum_{k=-\infty}^{+\infty} \int_{C_k} \left| \int_{2^{kn}\ge|y|\ge|x|} \frac{\Omega(x-y)(b(x)-b(y))f(y)}{|y|^{n-\alpha}} dy \right. \\ &\quad \left. + \int_{|y|>2^{kn}} \frac{\Omega(x-y)(b(x)-b(y))f(y)}{|y|^{n-\alpha}} dy \right|^{p_2} dx \\ &\le C \sum_{k=-\infty}^{+\infty} \int_{C_k} \left| \int_{|x|\le|y|\le 2^{kn}} \frac{\Omega(x-y)(b(x)-b(y))f(y)}{|y|^{n-\alpha}} dy \right|^{p_2} dx \\ &\quad + C \sum_{k=-\infty}^{+\infty} \int_{C_k} \left| \int_{|y|>2^{kn}} \frac{\Omega(x-y)(b(x)-b(y))f(y)}{|y|^{n-\alpha}} dy \right|^{p_2} dx \\ &:= J_1 + J_2. \end{aligned} \tag{28}$$

Using Hölder's inequality, we have

$$\begin{aligned} J_1 &\le C \sum_{k=-\infty}^{+\infty} \int_{C_k} \left| \frac{1}{|x|^{n-\alpha}} \int_{|y|\le 2^{kn}} \Omega(x-y)(b(x)-b(y))f(y) dy \right|^{p_2} dx \\ &\le C \sum_{k=-\infty}^{+\infty} 2^{-k(n-\alpha)p_2} \int_{B_k} \left| \sum_{i=-\infty}^{k} \int_{B_i} \Omega(x-y)(b(x) \right. \\ &\quad \left. - b(y))f(y) dy \right|^{p_2} dx \le C \|f\|_{L^{p_1}(\mathbb{R}^n)}^{p_2}. \end{aligned} \tag{29}$$

For the term $J_2$, we see at once that

$$\begin{aligned} J_2 &\le C \sum_{k=-\infty}^{+\infty} \int_{C_k} \left| \sum_{i=k}^{+\infty} \int_{C_i} \frac{\Omega(x-y)(b(x)-b_{B_k})f(y)}{|y|^{n-\alpha}} dy \right|^{p_2} dx \\ &\quad + C \sum_{k=-\infty}^{+\infty} \int_{C_k} \left| \sum_{i=k}^{+\infty} \int_{C_i} \frac{\Omega(x-y)(b(y)-b_{B_k})f(y)}{|y|^{n-\alpha}} dy \right|^{p_2} dx \\ &:= J_{21} + J_{22}. \end{aligned} \tag{30}$$

The Hölder inequality, along with (24), implies

$$\begin{aligned} J_{21} &\leq C \sum_{k=-\infty}^{+\infty} \int_{C_k} \left|b(x) - b_{B_k}\right|^{p_2} \left|\sum_{i=k}^{+\infty} \int_{C_i} \frac{\Omega(x-y) f(y)}{|y|^{n-\alpha}} dy\right|^{p_2} dx \\ &\leq C \sum_{k=-\infty}^{+\infty} \int_{B_k} \left|b(x) - b_{B_k}\right|^{p_2} \left|\sum_{i=k}^{+\infty} \|f\|_{L^{p_1}(B_i)} \right. \\ &\quad \cdot \left. \left(\int_{C_i} |\Omega(x-y)|^q dy\right)^{1/q} \left(\int_{C_i} |y|^{-(n-\alpha)s} dy\right)^{1/s} \right|^{p_2} dx \\ &\leq C \sum_{k=-\infty}^{+\infty} \sum_{i=k}^{+\infty} 2^{(k-i)n/2} \|f\|_{L^{p_1}(B_i)}^{p_2} \left(\sum_{i=k}^{+\infty} 2^{(k-i)np_2'/2p_2}\right)^{p_2/p_2'} \\ &\leq C\|f\|_{L^p(\mathbb{R}^n)}^p. \end{aligned} \tag{31}$$

The term $J_{22}$ need a further decomposition as follows:

$$\begin{aligned} J_{22} &\leq C \sum_{k=-\infty}^{+\infty} \int_{B_k} \left|\sum_{i=k}^{+\infty} \int_{C_i} \frac{\Omega(x-y)\left(b(y) - b_{B_i}\right) f(y)}{|y|^{n-\alpha}} dy\right|^{p_2} dx \\ &\quad + C \sum_{k=-\infty}^{+\infty} \int_{B_k} \left|\sum_{i=k}^{+\infty} \int_{C_i} \frac{\Omega(x-y)\left(b_{B_i} - b_{B_k}\right) f(y)}{|y|^{n-\alpha}} dy\right|^{p_2} dx \\ &:= J_{221} + J_{222}. \end{aligned} \tag{32}$$

Applying the Hölder inequality, we deduce

$$\begin{aligned} J_{221} &\leq C \sum_{k=-\infty}^{+\infty} \int_{B_k} \left|\sum_{i=k}^{+\infty} \left(\int_{C_i} \frac{|f(y)|^{p_1}}{|y|^{(n-\alpha)p_1}} dy\right)^{1/p} \right. \\ &\quad \cdot \left. \left(\int_{C_i} |\Omega(x-y)|^q dy\right)^{1/q} \left(\int_{C_i} \left|\left(b(y) - b_{B_i}\right)\right|^s dy\right)^{1/s} \right|^{p_2} dx \\ &\leq C\|b\|_{\mathrm{CBMO}^s(\mathbb{R}^n)}^p \sum_{k=-\infty}^{+\infty} \left|\sum_{i=k}^{+\infty} 2^{(k-i)n/p} \|f\|_{L^p(B_i)}\right|^p \\ &\leq C \sum_{k=-\infty}^{+\infty} \sum_{i=k}^{+\infty} 2^{(k-i)n/2} \|f\|_{L^{p_1}(B_i)}^{p_2} \left(\sum_{i=k}^{+\infty} 2^{(k-i)np_2'/2p_2}\right)^{p_2/p_2'} \\ &\leq C\|f\|_{L^p(\mathbb{R}^n)}^p. \end{aligned} \tag{33}$$

From the fact that

$$\begin{aligned} \left|b_{B_k} - b_{B_i}\right| &\leq \sum_{j=k}^{i-1} \left|b_{B_j} - b_{B_{j+1}}\right| \\ &\leq \sum_{j=k}^{i-1} \frac{1}{|B_j|} \int_{B_{j+1}} \left|b(y) - b_{B_{j+1}}\right| dy \\ &\leq C(i-k)\|b\|_{\mathrm{CBMO}^{p_2}(B_i)}, \end{aligned} \tag{34}$$

the term $J_{222}$ can be estimated as follows:

$$\begin{aligned} J_{222} &\leq C\|b\|_{CBMO^{p_2}(\mathbb{R}^n)}^{p_2} \sum_{k=-\infty}^{+\infty} \int_{B_k} \\ &\quad \cdot \left|\sum_{i=k}^{+\infty} (i-k) \int_{C_i} \frac{\Omega(x-y) f(y)}{|y|^{n-\alpha}} dy\right|^{p_2} dx \\ &\leq C \sum_{k=-\infty}^{+\infty} \int_{B_k} \left|\sum_{i=k}^{+\infty} (i-k) \|f\|_{L^{p_1}(B_i)} \left(\int_{C_i} |\Omega(x-y)|^q dy\right)^{1/q} \right. \\ &\quad \cdot \left. \left(\int_{C_i} |y|^{-(n-\alpha)s} dy\right)^{1/s} \right|^{p_2} dx \\ &\leq C \sum_{k=-\infty}^{+\infty} \left|\sum_{i=k}^{+\infty} 2^{(k-i)n/p} \|f\|_{L^p(B_i)}\right|^p \\ &\leq C \sum_{i=-\infty}^{+\infty} \|f\|_{L^{p_1}(B_i)}^{p_2} \sum_{k=-\infty}^{i} (i-k) 2^{(k-i)n/2} \\ &\leq C\|f\|_{L^{p_1}(\mathbb{R}^n)}^{p_2}, \end{aligned} \tag{35}$$

which is the desired result and (a) is obtained.

Next, we verify (b) inspired by [18]. Namely, we need to show that there is a constant $C(\Omega, \alpha, p, n)$ such that

$$\Theta := \Theta(b, r) := \frac{1}{|B_r|} \int_{B_r} \left|b(y) - b_{B_r}\right| dy \leq C \text{ for any } r \in \mathbb{R}^+. \tag{36}$$

For abbreviation, we assume that $\|[b, H_{\Omega,\alpha}]\|_{L^{p_1} \to L^{p_2}} = 1$, $\|[b, H_{\Omega,\alpha}^*]\|_{L^{p_1} \to L^{p_2}} = 1$, and $b_{B_r} = 0$ since $[b - b_{B_r}, H_{\Omega,\alpha}] = [b, H_{\Omega,\alpha}]$. Let $f(y) = \operatorname{sgn}(b(y))\chi_{B_r}(y)$. It is easy to check that $(1/|B_r|)\int_{\mathbb{R}^n} f(y) b(y) dy = \Theta$.

Applying (3) and (10), we deduce that for $A$ and $\gamma$ appearing in (10), there is a constant $C_1 < 1$ such that $\sigma(D) > 0$ for $D = \{x' \in \mathbb{S}^{n-1} : \Omega(x') \geq ((2A)/((\log(2/C_1))^\gamma))\}$. For $x' \in D$ and $y' \in \mathbb{S}^{n-1}$ with $|x' - y'| \leq C_1$, we obtain from (10) that

$$\begin{aligned} \Omega\left(y'\right) &= \Omega\left(x'\right) - \left(\Omega\left(x'\right) - \Omega\left(y'\right)\right) \\ &\geq \left|\Omega\left(x'\right)\right| - \left|\Omega\left(x'\right) - \Omega\left(y'\right)\right| \\ &\geq \frac{A}{(\log(2/C_1))^\gamma}. \end{aligned} \tag{37}$$

Writing $E = \{x \in \mathbb{R}^n : |x| > C_2 r \text{ and } \& x' \in D\}\ with\ C_2 = 3C_1^{-1} + 1 > 4$, we see at once that for $x \in E$, $|[b, H_{\Omega,\alpha}]f(x)| \geq |H_{\Omega,\alpha}(bf)(x)| - |b(x)||H_{\Omega,\alpha}f(x)| := K_1(x) - K_2(x)$. We conclude from (16) that $|(x-y)' - x'| \leq 3|y|/|x| \leq C_1$ since $|y| < r$ and $|x| > C_2|y| > 4|y|$, and hence, $\Omega((x-y)') \geq A/(\log(2/C_1))^\gamma$, and finally, that

$$K_1(x) \geq \left| \frac{A}{(\log(2/C_1))^{\gamma}} \frac{1}{|x|^{n-\alpha}} \int_{|y|<|x|} b(y) f(y) dy \right| = \frac{A}{(\log(2/C_1))^{\gamma} |x|^{n-\alpha}} \int_{B_r} b(y) f(y) dy = \frac{C_3 r^n}{|x|^{n-\alpha}} \Theta. \quad (38)$$

Furthermore,

$$\begin{aligned} K_2(x) &\leq \frac{|b(x)|}{|x|^{n-\alpha}} \int_{B_r} |f(y)| \left| \Omega\left((x-y)'\right) - \Omega\left(x'\right) \right| dy \\ &\leq \frac{C|b(x)|}{|x|^{n-\alpha}} \int_{B_r} \frac{|f(y)|}{(\log(|x|/|y|))^{\gamma}} dy \\ &\leq \frac{C|b(x)|}{|x|^{n-\alpha}} \int_{B_r} \frac{|f(y)|}{(\log(|x|/r))^{\gamma}} dy \\ &\leq \frac{C_4 r^n |b(x)|}{|x|^{n-\alpha} (\log(|x|/r))^{\gamma}}. \end{aligned} \quad (39)$$

For abbreviation, we write

$$\begin{cases} F = \left\{ x \in E : |b(x)| > \dfrac{C_3 \Theta}{2C_4} \left( \log\left(\dfrac{|x|}{r}\right) \right)^{\gamma}, |x| < \Theta^{p_2/n} r \right\}; \\ F_1 = \{E \setminus F\} \cap \left\{ |x| < \Theta^{p_2/n} r \right\}; \\ F_2 = \{E\} \cap \left\{ C_5 (|F| + (C_2 r)^n)^{1/n} < |x| < \Theta^{p_2/n} r \right\}. \end{cases} \quad (40)$$

This, along with the estimates for $K_1$ and $K_2$, one has

$$\begin{aligned} &\|\chi_F\|_{L^{p_1}(\mathbb{R}^n)} \|f\|_{L^{p_2'}(\mathbb{R}^n)} \\ &\quad \geq \|\chi_F\|_{L^{p_1}(\mathbb{R}^n)} \left\| [b, H_{\Omega,\alpha}] f \right\|_{L^{p_1'}(\mathbb{R}^n)} \\ &\quad \geq \int_{F_1} \left| [b, H_{\Omega,\alpha}] f(x) \right| dx \\ &\quad \geq \int_{F_1} \left| \frac{C_3 r^n \Theta}{|x|^{n-\alpha}} - \frac{C_3 r^n \Theta}{2|x|^{n-\alpha}} \right|^p dx \\ &\quad \geq \left( \frac{C_3 r^n \Theta}{2} \right) \int_{C_3 (|F| + (C_2 r)^n)^{1/n}}^{\Theta^{p_2/n} r} \rho^{\alpha-1} d\rho \int_D d\sigma\left(x'\right) \\ &\quad \geq \sigma(D) \frac{(C_3 r^n \Theta)}{\alpha} \left[ \Theta^{\alpha p_2/n} r^{\alpha} - C_5^{\alpha} (|F| + (C_2 r)^n)^{\alpha/n} \right]. \end{aligned} \quad (41)$$

Consequently,

$$\begin{aligned} (|F| + (C_2 r)^n)_{\alpha/n} &\geq C_5^{-\alpha} \Theta^{\alpha p_2/n} r^{\alpha} \\ &\quad - \frac{C_5^{-\alpha} \alpha}{C_3 \sigma(D)} \frac{\|\chi_F\|_{L^{p_1}(\mathbb{R}^n)} \|f\|_{L^{p_2'}(\mathbb{R}^n)}}{\Theta r^n} \\ &\geq C_6 \Theta^{\alpha p_2/n} r^{\alpha}. \end{aligned} \quad (42)$$

This in turn implies that $C_6 \Theta^{p_2} r^n \leq |F| + (C_2 r)^n$. Thus, $(b)$ is proved if $\Theta \leq (2C_6^{-1} C_2^n)^{1/p_2}$. If $\Theta > (2C_6^{-1} C_2^n)^{1/p_2}$, we see immediately that

$$|F| \geq C_6 \Theta^{p_2} r^n - C_2^n r^n \geq \frac{C_6 \Theta^{p_2} r^n}{2}. \quad (43)$$

According to $F \subset E$, $C_2 > 4$, $|y| > C_2 |x|$, Lemma 2 and (37), we can obtain that for $x \in B_r$ and $y \in F$,

$$|x-y| \approx |y|, \Omega\left((x-y)'\right) \geq \frac{A}{(\log(2/C_1))^{\gamma}}. \quad (44)$$

We continue to choose $f^*(y) = (sgn(b(y)))\chi_F(y)$ for $x \in B_r$ and get

$$\begin{aligned} \left| [b, H^*_{\Omega,\alpha}] f^*(x) \right| &\geq \int_F \frac{|\Omega(x-y) b(y)|}{|y|^{n-\alpha}} dy \\ &\quad - |b(x)| \int_F \frac{|\Omega(x-y) f^*(y)|}{|y|^{n-\alpha}} dy \\ &:= L_1(x) + L_2(x). \end{aligned} \quad (45)$$

It is easy to check that

$$\begin{aligned} L_2(x) &\leq C_7 |b(x)| \int_{C_2 r}^{\Theta^{p_2/n} r} \rho^{\alpha-1} d\rho \int_{\mathbb{S}^{n-1}} d\sigma\left(y'\right) \\ &\leq C_8 |b(x)| \left( \Theta^{\alpha p_2/n} r^{\alpha} - C_2^{\alpha} r^{\alpha} \right). \end{aligned} \quad (46)$$

To deal with the term $L_1(x)$, we first obtain from (44) that

$$L_1(x) \geq \frac{A C_3 \Theta}{2C_4 (\log 1/C_1)^{\gamma}} \int_F \frac{(\log(|y|/r))^{\gamma}}{|y|^{n-\alpha}} dy. \quad (47)$$

Then, the estimate for $L_1(x)$ consists of two cases.

*Case 1.* $\gamma \geq n$. Since $4r < C_2 r < |y| < \Theta^{p_2/n} r$ for $y \in F$, we conclude from (47) and $|F| \geq (C_6 \Theta^{p_2} r^n)/2$ that

$$\begin{aligned} L_1(x) &\geq \frac{C_9 \Theta}{r^n} \int_F \left( \frac{\log(|y|/r)}{|y|/r} \right)^n \frac{(\log(|y|/r))^{\gamma-n}}{|y|^{\alpha}} dy \\ &\geq \frac{C_9 \Theta^{1+(\alpha p_2/n)}}{r^{n-\alpha}} \left( \frac{\log(p_2/n)}{\Theta^{p_2/n}} \right)^n (\log C_2)^{\gamma-n} |F| \\ &\geq C_{10} \Theta^{1+\alpha p_2/n} (\log \Theta)^n r^{\alpha}. \end{aligned} \quad (48)$$

*Case 2.* $1 < \gamma < n$. In this case,

$$\begin{aligned} L_1(x) &\geq \frac{C_9 \Theta^{1+(\alpha p_2/n)}}{r^{n-\alpha}} \int_F \frac{(\log(|y|r))^{\gamma}}{(|y|/r)^n} dy \\ &\geq C_{11} \Theta^{1+\alpha p_2/n} (\log \Theta)^{\gamma} r^{\alpha}. \end{aligned} \quad (49)$$

This in turn reveals that

$$\begin{aligned} \left| [b, H^*_{\Omega,\alpha}] f^*(x) \right| &\geq C_{11} \Theta^{1+\alpha p_2/n} (\log \Theta)^{\gamma} r^{\alpha} \\ &\quad - C_9 |b(x)| r^{\alpha} \Theta^{\alpha p_2/n}, \quad \forall x \in B_r. \end{aligned} \quad (50)$$

Having disposed of the above estimates, we can now obtain

$$
\begin{aligned}
C_{12} r^{n+\alpha}\Theta^{1+\alpha p_2/n} &\geq \left(\int_{|x|<\Theta^{p_2/n} r} dx\right)^{1/p_1} r^{n/p_2'} \geq r^{n/p_2'} \|f^*\|_{L^{p_1}(\mathbb{R}^n)} \\
&\geq \left\|\chi_{B_r}\right\|_{L^{p_2'}(\mathbb{R}^n)} \left\|\left[b, H^*_{\Omega,\alpha}\right] f^*\right\|_{L^{p_2}(\mathbb{R}^n)} \\
&\geq \int_{B_r} \left|\left[b, H^*_{\Omega,\alpha}\right] f^*\right| dx \\
&\geq C_{11}\Theta(\log\Theta)^{\gamma} r^{\alpha}\int_{B_r} dx \\
&\quad - C_9\Theta^{\alpha p_2/n} r^{\alpha}\int_{B_r} |b(x)| dx \\
&\geq C_{11}\Theta^{1+\alpha p_2/n} r^{n+\alpha}(\log\Theta)^{\gamma} - C_9\Theta^{1+\alpha p_2/n} r^{n+\alpha}.
\end{aligned} \tag{51}
$$

Namely, $C_{11}(log\Theta)^{\gamma} - C_9 \leq C_{12}$. This in turn shows that there is a constant $C \coloneqq C(\Omega, \alpha, n, p, \gamma)$ such that $\Theta \leq C$, whence reaching the desired fact.

## Data Availability

The data used to support the findings of this study are included within the article.

## Conflicts of Interest

The author declares that there are no conflicts of interest.

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