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Research Article

Exact Combined Solutions for the $(2+1)$-Dimensional Dispersive Long Water-Wave Equations

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The homogeneous balance of undetermined coefficient (HBUC) method is presented to obtain not only the linear, bilinear, or homogeneous forms but also the exact traveling wave solutions of nonlinear partial differential equations. Linear equation is obtained by applying the proposed method to the $(2+1)$-dimensional dispersive long water-wave equations. Accordingly, the multiple soliton solutions, periodic solutions, singular solutions, rational solutions, and combined solutions of the $(2+1)$-dimensional dispersive long water-wave equations are obtained directly. The HBUC method, which can be used to handle some nonlinear partial differential equations, is a standard, computable, and powerful method.

1. Introduction

Nonlinear partial differential equations (NLPDEs) are used to describe a variety of phenomena not only in physics [1, 2], thermodynamics [3], fluid dynamics [4, 5], and practical engineering [6, 7] but also in several other fields [8]. How to obtain the traveling wave solutions for NLPDEs is very important in the nonlinear phenomena [1, 9, 10]. In recent decades, there are many excellent methods, such as the (G'/G)-expansion method [11, 12], the homotopy perturbation method [13, 14], the Riccati-Bernoulli sub-ODE method [15, 16], the three-wave method [17, 18], the inverse scattering method [19, 20], the first integral method [21, 22], Hirota's bilinear method [23, 24], the homogeneous balance method [25, 26], the iteration method [27, 28], the tanh-sech method [29, 30], and the extended homoclinic test method [31, 32], which are applied to obtain the exact traveling wave solutions of some NLPDEs.

The above traditional methods can be used to handle some well-known NLPDEs. However, there is no unified approach, which can be dealt with all NLPDEs. To obtain the traveling wave solutions of NLPDEs, Hirota's bilinear method, the three-wave method, and the (G'/G)-expansion method are employed to investigate the traveling wave solutions of many NLPDEs. Unfortunately, some exact solutions are omitted by using Hirota's bilinear method, the three-wave method, and the (G'/G)-expansion method if the NLPDEs can be linearized. To solve this problem, the HBUC method is proposed to derive the linear forms of NLPDEs.

In this paper, the $(2+1)$-dimensional dispersive long water-wave equations (DLWEs) [33, 34] are investigated as follows:

$$u_{yt}+u_{xxy}-2v_{xx}-\left(u^2\right)_{xy}=0, \tag{1}$$

$$v_t-v_{xx}-2(uv)_x=0, \tag{2}$$

where $u=u(x,y,t)$ represents the surface velocity of water along the x-direction and $v=v(x,y,t)$ gives the surface velocity of water along the y-direction.

The DLWEs can also be derived from the well-known Kadomtsev-Petviashvili equation using the symmetry constraint. The DLWEs were used to model nonlinear and dispersive long gravity waves traveling in two horizontal directions on shallow waters of uniform depth. The DLWEs also appear in many scientific applications such as nonlinear fiber optics, plasma physics, fluid dynamics, and coastal engineering. Moreover, the solutions of the DLWEs are very helpful for coastal and engineers to apply the nonlinear water model to coastal and harbor design [33].

The DLWEs were investigated where different approaches were exploited. Wu et al. reported that the DLWEs exist of many nonpropagating hydrodynamical solitons both in theory and in experiment, and the DLWEs have no Painleve property, though the system is Lax or inverse scattering transformation integrable [35]. Paquin and Winternitz investigated the DLWEs by the Lie group method [36]. The extended mapping approach [37], the extended projective approach [38], and the tanh-sech method [33] are among many other methods that were used to handle the DLWEs. Much effort has been focused on the existence of propagating solitons [36], multiple soliton solutions, and rational solutions [33].

In this paper, the linear equation of the DLWEs is derived by the HBUC method. Then, the multiple soliton solutions, periodic solutions, singular solutions, rational solutions, and combined solutions of the DLWEs are obtained directly.

The remainder of this paper is organized as follows: the HBUC method is presented in Section 2. In Section 3, the HBUC method is used to obtain N-multiple soliton solutions, periodic solutions, singular solutions, rational solutions, and combined solutions of Equations (1) and (2) directly. In Section 4, some important conclusions are given.

2. Description of the HBUC Method

In this section, the following general NLPDE in two variables is considered:

$$P(u, u_t, u_x, u_{xx}, u_{xt}, \cdots) = 0, \tag{3}$$

where P is a polynomial function of its arguments; the subscripts x and t denote the partial derivatives of u, respectively. The HBUC method consists of three steps as follows:

Step 1. Assume that the Equation (3) has a solution of the following form:

$$u = a_{mn}(\ln w)_{m,n} + \sum_{\substack{i,j=0 \\ i+j\neq 0,m+n}}^{i=m,j=n} a_{ij}(\ln w)_{i,j} + a_{00}, \tag{4}$$

where $u = u(x, t)$, $w = w(x, t)$, and $(\ln w)_{i,j} = (\partial^{i+j}(\ln w(x, t)))/\partial x^i \partial t^j$, and m, n (balance numbers) and a_{ij} $(i = 0, 1, \cdots, m\,;\ j = 0, 1, \cdots, n)$ (balance coefficients) $(a_{mn} \neq 0)$ are constants to be determined later.

The balance numbers can be determined by balancing the highest nonlinear terms and the highest order partial derivative terms. A set of algebraic equations for the balance coefficients is obtained by substituting Equation (4) into Equation (3) and balancing the terms with $(w_x/w)^i (w_t/w)^j$.

Step 2. If the NLPDEs can be linearized, the linear equation can be obtained by solving the set of algebraic equations and simplifying Equation (3) directly or after integrating some time (generally, integrating times equal to the orders of the lowest partial derivative of Equation (3)) with respect to x and t.

Step 3. Based on Step 1 and Step 2, by using traveling wave transformations

$$w(x, t) = w(\xi), \tag{5}$$

and

$$\xi = x - Vt, \tag{6}$$

Equation (3) can be reduced to a linear partial differential equation

$$w_t + \alpha_1 w_{xx} + \alpha_2 w_x + \alpha_3 w, \tag{7}$$

where α_1, α_2, and α_3 are constants. Then, solving the linear partial differential equation (7) yields the exact combined solutions of Equation (3). Next, Equations (1) and (2) are chosen to obtain the combined solutions by applying the HBUC method.

3. Application to the $(2 + 1)$-Dimensional DLWEs

Assume that the solutions of Equations (1) and (2) are of the forms

$$u = a_{mnl}(\ln w)_{m,n,l} + \sum_{\substack{i,j,k=0 \\ i+j+k\neq 0,m+n+l}}^{i=m,j=n,k=l} a_{ijk}(\ln w)_{i,j,k} + a_{000}, \tag{8}$$

$$v = b_{pqr}(\ln w)_{p,q,r} + \sum_{\substack{i,j,k=0 \\ i+j+k\neq 0,p+q+r}}^{i=p,j=q,k=r} b_{ijk}(\ln w)_{i,j,k} + b_{000}, \tag{9}$$

where $u = u(x, y, t)$, $w = w(x, y, t)$, and $(\ln w)_{i,j,k} = (\partial^{i+j+k}(\ln w(x, y, t)))/\partial x^i \partial y^j \partial t^k$, and m, n, l, p, q, r (balance numbers) and a_{ijk} $(i = 0, 1, \cdots, m\,;\ j = 0, 1, \cdots, n\,;\ k = 0, 1, \cdots, l)(a_{mnl} \neq 0)$ and b_{ijk} $(i = 0, 1, \cdots, p\,;\ j = 0, 1, \cdots, q\,;\ k = 0, 1, \cdots, r)(b_{pqr} \neq 0)$ (balance coefficients) are constants to be determined later.

Balancing u_{xxy}, v_{xx}, and $(u^2)_{xy}$ in Equation (1) and v_{xx} and $(uv)_x$ in Equation (2), it is required that

$$\begin{aligned} m+2&=2m+1=p+2,\\ n+1&=2n+1=q,\\ l&=2l=r,\\ p+2&=m+p+1,\\ q&=q+n,\\ l+r&=l. \end{aligned} \tag{10}$$

Solving the above algebraic equations, we get $m=1$, $n=0, l=0, p=1, q=1, r=0$. Then, Equations (8) and (9) can be written as

$$\begin{aligned} u&=a_1(\ln w)_x+a_0,\\ v&=b_3(\ln w)_{xy}+b_2(\ln w)_x+b_1(\ln w)_y+b_0, \end{aligned} \tag{11}$$

where $a_i (i=0,1)$ and $b_i (i=0,1,2,3)$ are constants to be determined later.

Substituting Equation (11) into Equations (1) and (2) and equating the coefficients of $w_x^3 w_y/w^4$ on the left-hand side of Equations (1) and (2) to zero yield a set of algebraic equations for a_1 and b_3 as follows:

$$\begin{aligned} -6\left(a_1^2+a_1-2b_3\right)&=0,\\ -6b_3(a_1-1)&=0. \end{aligned} \tag{12}$$

Solving the above algebraic equations and noticing $a_1 b_3 \neq 0$, we get $a_1=b_3=1$. Substituting a_1 and b_3 back into Equation (11), we get

$$\begin{aligned} u&=(\ln w)_x+a_0,\\ v&=(\ln w)_{xy}+b_2(\ln w)_x+b_1(\ln w)_y+b_0. \end{aligned} \tag{13}$$

Substituting Equation (13) back into Equations (1) and (2), Equation (1) minus Equation (2) is

$$\frac{A_1}{w}+\frac{A_2}{w^2}+\frac{A_3}{w^3}=0, \tag{14}$$

where

$$\begin{aligned} A_1&=b_2(2a_0w_{xx}-w_{xxx}-w_{xt})\\ &\quad+b_1\left(2a_0w_{xy}-w_{xxy}-w_{yt}\right)\\ &\quad+2b_0w_{xx},\\ A_3&=-6b_2w_x^3-6b_1w_x^2w_y,\\ A_2&=b_2\left(w_xw_t+7w_{xx}w_x-2a_0w_x^2\right)\\ &\quad+b_1\left(3w_{xx}w_y+w_yw_t+4w_{xy}w_x\right.\\ &\quad\left.-2a_0w_xw_y\right)-2b_0w_x^2. \end{aligned} \tag{15}$$

Obviously, setting $b_i=0\,(i=0,1,2)$, we find that Equation (1) coincides with Equation (2). According to the above analysis, suppose that the solutions of Equations (1) and (2) are of the forms

$$\begin{aligned} u(x,y,t)&=(\ln w)_x+a_0,\\ v(x,y,t)&=(\ln w)_{xy}, \end{aligned} \tag{16}$$

where a_0 is an arbitrary constant and $w=w(x,y,t)$ is a function of x, y, t that will be determined later.

Substituting Equation (16) into Equations (1) and (2) yields a single NLPDE

$$K_0+K_1+K_2+K_3=0, \tag{17}$$

where

$$\begin{aligned} K_0&=a_0\left(\frac{-2w_{xxy}}{w}+\frac{2w_{xx}w_y+4w_xw_{xy}}{w^2}-\frac{4w_x^2w_y}{w^3}\right),\\ K_1&=\frac{w_{xyt}-w_{xxxy}}{w},\\ K_2&=\frac{w_xw_{xxy}-w_{xy}w_t-w_{xt}w_y+w_{xx}w_{xy}+w_{xxx}w_y-w_xw_{yt}}{w^2},\\ K_3&=\frac{2w_xw_yw_t-2w_xw_{xx}w_y}{w^3}. \end{aligned} \tag{18}$$

Simplifying Equation (17) and integrating with respect to x once, we get

$$\frac{\partial}{\partial x}\left(\frac{w_{yt}w-w_yw_t}{w^2}-\frac{w_{xxy}w-w_{xx}w_y}{w^2}-\frac{2a_0\left(w_{xy}w-w_xw_y\right)}{w^2}\right)=0. \tag{19}$$

Equation (19) is identical to

$$\begin{aligned} &\left(w_{yt}w-w_yw_t\right)-\left(w_{xxy}w-w_{xx}w_y\right)\\ &\quad-2a_0\left(w_{xy}w-w_xw_y\right)-p(y,t)w^2=0. \end{aligned} \tag{20}$$

where $p(y,t)$ is an arbitrary function of y, t.

Particularly, taking $p(y,t)=0$ in Equation (20), the bilinear equation of Equations (1) and (2) is obtained as follows:

$$\left(w_{yt}w-w_yw_t\right)-\left(w_{xxy}w-w_{xx}w_y\right)-2a_0\left(w_{xy}w-w_xw_y\right)=0. \tag{21}$$

Equation (21) can be written concisely in terms of D-operator as

$$D_y\left(w_t-w_{xx}-2a_0w_x\right)\cdot w=0, \tag{22}$$

where

$$D_x^m D_t^n a \cdot b = \left(\partial_x - \partial_{x}'\right)^m \left(\partial_t - \partial_{t}'\right)^n a(x,t) b\left(x', t'\right)\Big|_{x'=x, t'=t}. \tag{23}$$

By using the property of D-operator, Equation (22) is identical to

$$w_t - w_{xx} - 2a_0 w_x - q(x,t) w = 0, \tag{24}$$

where $q(x,t)$ is an arbitrary function of x, t.

Particularly, taking $q(x,t) = \alpha = \text{constant}$ in Equation (24), we get a linear equation

$$w_t - w_{xx} - 2a_0 w_x - \alpha w = 0. \tag{25}$$

Remark 1. We note that Equation (25) does not depend on the variable y, instead it depends only on the variables x, t.

Using the transformation

$$w(x,y,t) = w(\xi) = w(x + l(y) - ct), \tag{26}$$

Equation (25) is reduced to

$$w'' + (2a_0 + c) w' + \alpha w = 0, \tag{27}$$

where $l(y)$ is an arbitrary function of y and the prime denotes the derivation with respect to ξ.

There are three types of traveling wave solutions of Equation (27) as follows.

When $\Delta = (2a_0 + c)^2 - 4\alpha > 0$,

$$w = C_1 e^{((-(2a_0+c)+\sqrt{\Delta})/2)\xi} + C_2 e^{((-(2a_0+c)-\sqrt{\Delta})/2)\xi}, \tag{28}$$

where $\xi = x + l(y) - ct$; C_1, C_2, a_0, α, and c are arbitrary constants; and $l(y)$ is an arbitrary function of y.

Generally, noticing the linear property of Equation (25), we can get the exact solution of Equation (25) as follows:

$$\begin{aligned} w_1 = \sum_{i=1}^{n_1} \Big(& C_{1,1i} e^{\left(\left(-(2a_0+c_{1i})+\sqrt{(2a_0+c_{1i})^2-4\alpha}\right)/2\right)\xi_{1i}} \\ & + C_{2,1i} e^{\left(\left(-(2a_0+c_{1i})-\sqrt{(2a_0+c_{1i})^2-4\alpha}\right)/2\right)\xi_{1i}} \Big), \end{aligned} \tag{29}$$

where $\Delta_{1i} = (2a_0 + c_{1i})^2 - 4\alpha > 0$ and $\xi_{1i} = x + l_{1i}(y) - c_{1i} t$; $C_{1,1i}, C_{2,1i}, a_0, \alpha$, and c_{1i} are arbitrary constants; and $l_{1i}(y)$ $(i = 1, 2, \cdots, n_1)$ are arbitrary functions of y.

When $\Delta = (2a_0 + c)^2 - 4\alpha < 0$,

$$w = e^{(-(2a_0+c)/2)\xi} \left(C_1 \cos\left(\frac{\sqrt{-\Delta}}{2}\xi\right) + C_2 \sin\left(\frac{\sqrt{-\Delta}}{2}\xi\right)\right), \tag{30}$$

where $\xi = x + l(y) - ct$; C_1, C_2, a_0, α, and c are arbitrary constants; and $l(y)$ is an arbitrary function of y.

Generally, noticing the linear property of Equation (25), we can get the exact solution as follows:

$$\begin{aligned} w_2 = \sum_{i=1}^{n_2} e^{(-(2a_0+c_{2i})/2)\xi_{2i}} \Bigg(& C_{1,2i} \cos\left(\frac{\sqrt{-\Delta_{2i}}}{2}\xi_{2i}\right) \\ & + C_{2,2i} \sin\left(\frac{\sqrt{-\Delta_{2i}}}{2}\xi_{2i}\right)\Bigg), \end{aligned} \tag{31}$$

where $\Delta_{2i} = (2a_0 + c_{1i})^2 - 4\alpha < 0$ and $\xi_{2i} = x + l_{2i}(y) - c_{2i} t$; $C_{1,2i}, C_{2,2i}, a_0, \alpha$, and c_{2i} are arbitrary constants; and $l_{2i}(y)$ $(i = 1, 2, \cdots, n_2)$ are arbitrary functions of y.

When $\Delta = (2a_0 + c)^2 - 4\alpha = 0$,

$$w = (C_1 + C_2 \xi) e^{(-(2a_0+c)/2)\xi}, \tag{32}$$

where $\xi = x + l(y) - ct$; C_1, C_2, a_0, α, and c are arbitrary constants; and $l(y)$ is an arbitrary function of y.

Generally, noticing the linear property of Equation (25), we can get the exact solution as follows:

$$w_3 = \sum_{i=1}^{2} (C_{1,3i} + C_{2,3i}\xi_{3i}) e^{(-(2a_0+c_{3i})/2)\xi_{3i}}, \tag{33}$$

where $\Delta_{3i} = (2a_0 + c_{3i})^2 - 4\alpha = 0$ and $\xi_{3i} = x + l_{3i}(y) - c_{3i} t$; $C_{1,3i}, C_{2,3i}, a_0, \alpha$, and c_{3i} are arbitrary constants; and $l_{3i}(y)$ $(i = 1, 2)$ are arbitrary functions of y.

Generally, we can get combined solutions of Equation (25) as follows:

$$w = w_1 + w_2 + w_3, \tag{34}$$

where $w_1, w_2,$ and w_3 are given by Equations (29), (31), and (33), respectively.

Accordingly, we can get combined solutions of Equations (1) and (2)

$$\begin{aligned} w &= w_1 + w_2 + w_3, \\ u(x,y,t) &= (\ln w)_x + a_0, \\ v(x,y,t) &= (\ln w)_{xy} = u_y. \end{aligned} \tag{35}$$

Choosing the appropriate parameters in Equation (35) can obtain all solutions of DLWEs in Ref. [15]. For example, setting $C_{1,2i} = C_{2,2i} = 0\ (i = 1, 2, \cdots, n_2)$, $C_{1,3i} = C_{2,3i} = 0\ (i = 1, 2)$, $n_1 = N$, $C_{1,1i} = 1/N$, $C_{2,1i} = 1$, $\alpha = a_0 = 0$, and $l_{1i}(y) = l_{1i} y$, we get the N-kink solutions and the N-soliton solutions

$$u(x,y,t)=\frac{-\sum_{i=1}^{N}c_{1i}e^{-c_{1i}(x+l_{1i}y-c_{1i}t)}}{1+\sum_{i=1}^{N}e^{-c_{1i}(x+l_{1i}y-c_{1i}t)}},$$
$$v(x,y,t)=-\frac{\left(\sum_{i=1}^{N}c_{1i}e^{-c_{1i}(x+l_{1i}y-c_{1i}t)}\right)\left(\sum_{i=1}^{N}c_{1i}l_{1i}e^{-c_{1i}(x+l_{1i}y-c_{1i}t)}\right)}{\left(1+\sum_{i=1}^{N}e^{-c_{1i}(x+l_{1i}y-c_{1i}t)}\right)^2}-\frac{\sum_{i=1}^{N}c_{1i}l_{1i}e^{-c_{1i}(x+l_{1i}y-c_{1i}t)}}{1+\sum_{i=1}^{N}e^{-c_{1i}(x+l_{1i}y-c_{1i}t)}}, \tag{36}$$

where c_{1i} and l_{1i} $(i=1,\cdots,N)$ are arbitrary constants.

Setting $C_{1,1i}=C_{2,1i}=0$ $(i=1,2,\cdots,n_1)$, $C_{1,3i}=C_{2,3i}=0$ $(i=1,2)$, $n_2=1$, $\xi_{20}=-\arctan(C_{2,21}/C_{1,21})$, and $l_{2i}(y)=l_{2i}y$, we get the singular solution (periodic solutions)

$$u(x,y,t)=-\frac{\sqrt{4\alpha-(2a_0+c_2)^2}}{2}\tan\cdot\left(\frac{\sqrt{4\alpha-(2a_0+c_2)^2}}{2}(x+l_2y-c_2t)+\xi_{20}\right)-\frac{c_2}{2},$$
$$v(x,y,t)=u_y. \tag{37}$$

where $C_{1,21}$, $C_{2,21}$, l_2, α, a_0, and c_2 are arbitrary constants.

Setting $C_{1,1i}=C_{2,1i}=0$ $(i=1,2,\cdots,n_1)$, $C_{1,2i}=C_{2,2i}=0$ $(i=1,2,\cdots,n_2)$, $C_{1,31}=C_1$, $C_{1,32}=C_2$, and $C_{1,32}=C_{2,32}=0$, we get the rational solutions

$$u(x,y,t)=\frac{C_2}{C_1+C_2(x+l_3y-c_3t)}-\frac{c_3}{2},\quad v(x,y,t)=u_y, \tag{38}$$

where $C_1, C_2, l_3, \alpha, a_0$, and $c_3((2a_0+c_3)^2-4\alpha=0)$ are arbitrary constants.

Remark 2. We can deal with Equation (25) by using some assumptions. For example, suppose that $w=-t\beta(y)+W(x)$, $\alpha=0$, and $a_0\neq 0$, we get

$$w=-t\beta(y)-\frac{C_1(y)e^{-2a_0x}}{2a_0}-\frac{\beta(y)x}{2a_0}+C_2(y),$$
$$u(x,y,t)=\frac{\beta(y)\left(2a_0^2t+a_0x+1\right)-a_0C_1(y)e^{-2a_0x}-2C_2(y)a_0^2}{\beta(y)(2a_0t+x)+C_1(y)e^{-2a_0x}-2a_0C_2(y)},$$
$$v(x,y,t)=u_y, \tag{39}$$

where $C_1(y)$, $C_2(y)$, and $\beta(y)$ are arbitrary functions of y, and $a_0\neq 0$ is an arbitrary constant.

Suppose that $w=-\beta(y)t+W(x)$ and $\alpha=a_0=0$, we get

$$w=-t\beta(y)-\frac{\beta(y)x^2}{2}+xC_1(y)+C_2(y),$$
$$u(x,y,t)=\frac{2x\beta(y)-2C_1(y)}{\beta(y)(x^2+2t)-2xC_1(y)-2C_2(y)}, \tag{40}$$
$$v(x,y,t)=u_y,$$

where $C_1(y)$, $C_2(y)$, and $\beta(y)$ are arbitrary functions of y.

Similarly, when $\alpha=a_0=0$, Equation (25) is reduced to $w_t-w_{xx}=0$. We can get the exact solution

$$w=C_1(y)-\frac{C_2(y)}{\sqrt{t}}e^{-(x^2/4t)},$$
$$u(x,y,t)=-\frac{xC_2(y)e^{-(x^2/4t)}}{2t\left(\sqrt{t}C_1(y)+C_2(y)e^{-(x^2/4t)}\right)}, \tag{41}$$
$$v(x,y,t)=u_y,$$

where $C_1(y)$ and $C_2(y)$ are arbitrary functions of y.

Similarly, we can assume that $w=\sum_{i=1}^{n}p_i(x)q_i(t)$; then, a new solution of Equation (25) can be obtained. Being similar to the above process, we omit it.

4. Conclusions

The $(2+1)$-dimensional dispersive long water-wave equations can be linearized by the HBUC method. Then, the N-multiple soliton solutions, periodic solutions, singular solutions, rational solutions, and combined solutions of Equations (1) and (2) can be obtained. Many well-known NLPDEs, such as the Whitham-Broer-Kaup equations, the Broer-Kaup equations, and the variant Boussinesq equations, can be handled by the HBUC method. The performance of the HBUC method is found to be simple and efficient. The HBUC method is also a standard, computable, and powerful method, which allows us to solve complicated and tedious algebraic calculations by the availability of computer systems like Maple.

Data Availability

The authors confirm that the data supporting the findings of this article are available within the article and are available on request from the corresponding author.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors completed the paper together. All authors read and approved the final manuscript.

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