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Research Article

New Estimates of q_1q_2-Ostrowski-Type Inequalities within a Class of n-Polynomial Prevexity of Functions

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In this article, we develop a novel framework to study for a new class of preinvex functions depending on arbitrary nonnegative function, which is called n-polynomial preinvex functions. We use the n-polynomial preinvex functions to develop q_1q_2 -analogues of the Ostrowski-type integral inequalities on coordinates. Different features and properties of excitement for quantum calculus have been examined through a systematic way. We are discussing about the suggestions and different results of the quantum inequalities of the Ostrowski-type by inferring a new identity for q_1q_2-differentiable function. However, the problem has been proven to utilize the obtained identity, we give q_1q_2-analogues of the Ostrowski-type integrals inequalities which are connected with the n-polynomial preinvex functions on coordinates. Our results are the generalizations of the results in earlier papers.

1. Introduction

Calculus is an imperative study of the derivatives and integrals. The classical derivative was convoluted with the strength regulation kind kernel, and eventually, this gave upward thrust to new calculus referred to as the quantum calculus. In mathematics, quantum calculus (named q-calculus) is the study of calculus without limits. The interest in this subject has exploded, and the q-calculus has in the last twenty years served as a bridge between mathematics and physics. The q -calculus has numerous applications in various fields of mathematics, for example, dynamical systems, number theory, combinatorics, special functions, fractals, and also for scientific problems in some applied areas such as computer science, quantum mechanics, and quantum physics. Jackson [1] defined the q-analogue of derivative and integral operator as well as provided some of their applications. It is imperative to mention that quantum integral inequalities are more practical and informative than their classical counterparts. It has been mainly due to the fact that quantum integral inequalities can describe the hereditary properties of the processes and phenomena under investigation. Historically, the subject of quantum calculus can be traced back to Euler and Jacobi, but in recent decades, it has experienced a rapid development. As a result, new generalizations of the classical concepts of quantum calculus have been initiated and reviewed in many literature. Tariboon and Ntouyas [2, 3] proposed the quantum calculus concepts on finite intervals and obtained several q-analogues of classical mathematical objects, which inspired many other researchers to study the subject in depth, and as a consequence, numerous novel results concerning quantum analogues of classical mathematical results have been launched. Noor et al. [4] obtained new q-analogues of inequality utilizing the first-order q-differentiable convex function.

Inequality plays an irreplaceable role in the development of mathematics. Very recently, many new inequalities such as the Hermite-Hadamard-type inequality [5–9], Petrović-type inequality [10], Pólya-Szegö and Ćebyšev-type inequalities [11], Ostrowski-type inequality [12], reverse Minkowski inequality [13], Jensen-type inequality [14–16], Bessel function inequality [17], trigonometric and hyperbolic functions inequalities [18], fractional integral inequality [19–22], complete and generalized elliptic integrals inequalities [23–28], generalized convex function inequality [29–31], and mean values inequality [32–34] have been discovered by many researchers. While the concept of classical convexity has been brought into a streamline by mathematical inequalities [35–50]. In fact, convex function and its connection with mathematical inequalities have wide applications in the estimation of some parameters in scientific observations and calculations [51–65]. In recent years, the classical concept of convexity has been extended and generalized in different directions, one of the important generalization of convexity is the invexity, which was studied by Hanson [66]; this work has greatly expanded the role of invexity in optimization. In [67, 68], the authors introduced a class of functions, which is called preinvexity as a generalization of convex functions.

Now, we recall the classical and well-known Hermite-Hadamard inequality [69], which can be stated as

$$\Phi\left(\frac{\xi_1+\xi_2}{2}\right) \le \frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\Phi(\gamma)d\gamma \le \frac{\Phi(\xi_1)+\phi(\xi_2)}{2}, \qquad (1)$$

for all $\xi_1, \xi_2 \in I$ if $\Phi : I \longrightarrow \mathbb{R}$ is a convex function.

Ostrowski [70] established an integral inequality for continuous and differentiable function as follows.

Theorem 1 (See [70]). *Let $\Phi : [\xi 1, \xi 2] \longrightarrow \mathbb{R}$ be continuous and differentiable on (ξ_1, ξ_2) such that $|\Phi'(\gamma)| \le M$ for all $\gamma \in (\xi_1, \xi_2)$. Then, one has*

$$\left|\Phi(\varrho) - \frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\Phi(\gamma)d\gamma\right| \le \left[\frac{1}{4} + \frac{(\varrho-\xi_1+\xi_2/2)^2}{(\xi_2-\xi_1)^2}\right](\xi_2-\xi_1)M, \qquad (2)$$

for all $\varrho \in [\xi_1, \xi_2]$ with the best possible constant 1/4.

The inequality (2) can be described in an identical kind as

$$\left|\Phi(\varrho) - \frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\Phi(\gamma)d\gamma\right| \le \frac{M}{\xi_2-\xi_1}\left[\frac{(\varrho-\xi_1)^2+(\varrho)^2}{2}\right]. \qquad (3)$$

Latif et al. [71] generalized the Ostrowski inequality (2) to the coordinated convex function by establishing an identity as follows.

Theorem 2 (See [71]). *Let $\xi_1 < \xi_2, \xi_3 < \xi_4$ and $\Phi : [\xi_1, \xi_2] \times [\xi_3, \xi_4] \longrightarrow \mathbb{R}$ be continuous and differentiable on $(\xi_1, \xi_2) \times (\xi_3, \xi_4)$ such that $\partial^2\Phi/\partial_z\partial_w \in L([\xi_1, \xi_2] \times [\xi_3, \xi_4])$.*

Then the identity

$$\begin{aligned}
\Phi(\varrho,\rho) &+ \frac{1}{(\xi_2-\xi_1)(\xi_4-\xi_3)}\int_{\xi_1}^{\xi_2}\int_{\xi_3}^{\xi_4}\Phi(u,v)dudv - y \\
&= \frac{(\varrho-\xi_1)^2(\rho-\xi_3)^2}{(\xi_2-\xi_1)(\xi_4-\xi_3)}\int_0^1\int_0^1 zw\frac{\partial^2}{\partial z\partial w}\Phi \\
&\quad\cdot(z\varrho+(1-z)\xi_1, w\rho+(1-w)\xi_3)dzdw \\
&\quad - \frac{(\varrho-\xi_1)^2(\xi_4-\rho)^2}{(\xi_2-\xi_1)(\xi_4-\xi_3)}\int_0^1\int_0^1 zw\frac{\partial^2}{\partial z\partial w}\Phi \\
&\quad\cdot(z\varrho+(1-z)\xi_1, w\rho+(1-w)\xi_4)dzdw \\
&\quad - \frac{(\xi_2-\varrho)^2(\rho-\xi_3)^2}{(\xi_2-\xi_1)(\xi_4-\xi_3)}\int_0^1\int_0^1 zw\frac{\partial^2}{\partial z\partial w}\Phi \\
&\quad\cdot(z\varrho+(1-z)\xi_2, w\rho+(1-w)\xi_3)dzdw \\
&\quad + \frac{(\xi_2-\varrho)^2(\xi_4-\rho)^2}{(\xi_2-\xi_1)(\xi_4-\xi_3)}\int_0^1\int_0^1 zw\frac{\partial^2}{\partial z\partial w}\Phi \\
&\quad\cdot(z\varrho+(1-z)\xi_2, w\rho+(1-w)\xi_4)dzdw,
\end{aligned} \qquad (4)$$

holds for all $(\varrho, \rho) \in [\xi_1, \xi_2] \times [\xi_3, \xi_4]$, where

$$y = \frac{1}{\xi_4-\xi_3}\int_{\xi_3}^{\xi_4}\Phi(\varrho,v)dv + \frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\Phi(u,\rho)du. \qquad (5)$$

Noor et al. [4] presented the Ostrowski-type inequality for quantum calculus.

Theorem 3 (See [4]). *Let $q \in (0,1), \xi_1 < \xi_2$ and $\Phi[\xi_1, \xi_2] \longrightarrow \mathbb{R}$ be continuous such that ${}_{\xi_1}D_q\Phi$ is integrable on (ξ_1, ξ_2). Then*

$$\begin{aligned}
\Phi(\varrho) &- \frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\Phi(u)\xi_1 d_q u = \frac{q(\varrho-\xi_1)^2}{\xi_2-\xi_1}\int_0^1 z_{\xi_1}D_q\Phi \\
&\cdot((1-z)\xi_1+z\varrho)_0 d_q z + \frac{q(\xi_2-\varrho)^2}{\xi_2-\xi_1}\int_0^1 z_{\xi 1}D_q\Phi \\
&\cdot((1-z)\xi_2+z\varrho)0 d_q z.
\end{aligned} \qquad (6)$$

The following quantum integral version of the Hermite-Hadamard-type inequality for the coordinated convex function was proved by Alp and Sarıkaya [72].

Theorem 4 (See [72]). *Let $q_1, q_2 \in (0,1)$, $\xi_1 < \xi_2$, $\xi_3 < \xi_4$ and $\Phi : \mathcal{N} = [\xi_1, \xi_2] \times [\xi_3, \xi_4] \longrightarrow \mathbb{R}$ be a coordinated convex* *function on $\mathcal{N}$. Then one has*

$$
\begin{aligned}
&\Phi\left(\frac{q_1\xi_1+\xi_2}{1+q_1}, \frac{q_2\xi_3+\xi_4}{1+q_2}\right) \\
&\quad \le \frac{1}{2}\left[\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\Phi\left(z, \frac{q_2\xi_3+\xi_4}{1+q_2}\right){}_{\xi_1}d_{q1}z \right.\\
&\qquad \left. + \frac{1}{\xi_3-\xi_4}\int_{\xi_3}^{\xi_4}\Phi\left(\frac{q_1\xi_1+\xi_2}{1+q_1}, w\right){}_{\xi_3}d_{q_2}w\right] \\
&\quad \le \frac{1}{(\xi_1-\xi_2)(\xi_3-\xi_4)}\int_{\xi_1}^{\xi_2}\int_{\xi_3}^{\xi_4}\Phi(z,w){}_{\xi_1}d_{q_1}z_{\xi_3}d_{q_2}w \\
&\quad \le \frac{1}{4}\left[\frac{1}{(1+q_1)(\xi_2-\xi_1)}\left(q_2\int_{\xi_1}^{\xi_2}\Phi(z,\xi_3){}_{\xi_1}d_{q_1}z + \int_{\xi_1}^{\xi_2}\Phi(z,\xi_4){}_{\xi_1}d_{q_1}z\right)\right.\\
&\qquad \left. + \frac{1}{(1+q_2)(\xi_3-\xi_4)}\left(q_1\int_{\xi_3}^{\xi_3}\Phi(\xi_1,w){}_{\xi_3}d_{q_2}w + \int_{\xi_3}^{\xi_4}\Phi(\xi_2,w){}_{\xi_3}d_{q_2}w\right)\right] \\
&\quad \le \frac{q_1q_2\Phi(\xi_1,\xi_3)+q_2\Phi(\xi_2,\xi_3)+q_1\Phi(\xi_1,\xi_4)+\Phi(\xi_2,\xi_4)}{(1+q_1)(1+q_2)}.
\end{aligned} \tag{7}
$$

Kalsoom et al. [73] found the quantum integral inequality for two parameters function on the finite rectangle.

Next, we present the definitions of q_1q_2-derivative and integral, and their two known results.

Definition 5. Let $q_1, q_2 \in (0,1)$, $\xi_1<\xi_2$, $\xi_3<\xi_4$ and $\Phi : [\xi_1, \xi_2] \times [\xi_3, \xi_4] \longrightarrow \mathbb{R}$ be a continuous function. Then, the partially q_1-derivative, q_2-derivative, and q_1q_2-derivative at $(z,w)\in[\xi_1,\xi_2]\times[\xi_3,\xi_4]$ for the function Φ are defined by

$$
\begin{aligned}
\frac{{}_{\xi_1}\partial_{q_1}\Phi(z,w)}{{}_{\xi_1}\partial_{q_1}z} &= \frac{\Phi(z,w)-\Phi(q_1z+(1-q_1)\xi_1,w)}{(1-q_1)(z-\xi_1)}\,(z\ne\xi_1), \\
\frac{{}_{\xi_3}\partial_{q_2}\Phi(z,w)}{{}_{\xi_3}\partial_{q_2}w} &= \frac{\Phi(z,w)-\Phi(z,q_2w+(1-q_2)\xi_3)}{(1-q_2)(w-\xi_3)}\,(w\ne\xi_3), \\
\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Phi(z,w)}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w} &= \frac{1}{(1-q_1)(1-q_2)(z-\xi_1)(w-\xi_3)} \\
&\quad \times[\Phi(q_1z+(1-q_1)\xi_1, q_2w+(1-q_2)\xi_3) \\
&\quad -\Phi(q_1z+(1-q_1)\xi_1,w) \\
&\quad -\Phi(z,q_2w+(1-q_2)\xi_3)+\Phi(z,w)] \\
&\qquad (z\ne\xi_1, w\ne\xi_3),
\end{aligned} \tag{8}
$$

respectively. The function Φ is said to be partially q_1-, q_2-, and q_1q_2-differentiable on $[\xi_1,\xi_2]\times[\xi_3,\xi_4]$ if ${}_{\xi_1}\partial_{q1}\Phi(z,w)/{}_{\xi_1}\partial_{q1}z$, ${}_{\xi_3}\partial_{q2}\Phi(z,w)/{}_{\xi_3}\partial_{q2}w$, and ${}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Phi(z,w)/{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w$ exist for all $(z,w)\in[\xi_1,\xi_2]\times[\xi_3,\xi_4]$.

Definition 6. Let $q_1, q_2 \in (0,1)$, $\xi_1<\xi_2$, $\xi_3<\xi_4$ and $\Phi : [\xi_1,\xi_2]\times[\xi_3,\xi_4] \longrightarrow \mathbb{R}$ be a continuous function. Then the q_1q_2-integral of the function Φ on $[\xi_1,\xi_2]\times[\xi_3,\xi_4]$ is defined by

$$
\begin{aligned}
\int_{\xi_3}^{t}\int_{\xi_1}^{s}\Phi(z,w){}_{\xi_1}d_{q_1}z_{\xi_3}d_{q_2}w &= (1-q_1)(1-q_2)(s-\xi_1)(t-\xi_3) \\
&\quad \times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} q_1^n q_2^m \Phi(q_1^n s+(1-q_1^n)\xi_1, q_2^m t+(1-q_2^m)\xi_3) \\
&\quad \times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} q_1^n q_2^m \Phi(q_1^n s+(1-q_1^n)\xi_1, q_2^m t+(1-q_2^m)\xi_3),
\end{aligned} \tag{9}
$$

for $(s,t)\in[\xi_1,\xi_2]\times[\xi_3,\xi_4]$.

Theorem 7. *Let $\xi_1<\xi_2$, $\xi_3<\xi_4$ and $\Phi : [\xi_1,\xi_2]\times[\xi_3,\xi_4]\longrightarrow\mathbb{R}$ be a continuous function. Then, we have the identities*

$$
\begin{aligned}
\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}s_{\xi_3}\partial_{q_2}t}\int_{\xi_4}^{t}\int_{\xi_1}^{s}\Phi(z,w){}_{\xi_1}d_{q_1}z_{\xi_3}d_{q_2}w &= \Phi(s,t), \\
\int_{\xi_3}^{t}\int_{\xi_1}^{s}\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Phi(z,w)}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\,\xi_1 d_{q_1}z_{\xi_3}d_{q_2}w &= \Phi(s,t), \\
\int_{t_1}^{t}\int_{s_1}^{s}\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Phi(z,w)}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\,\xi_1 d_{q_1}z_{\xi_3}d_{q_2}w &= \Phi(s,t)-\Phi(s,t_1) \\
&\quad -\Phi(s_1,t)+\Phi(s_1,t_1),
\end{aligned} \tag{10}
$$

for $(s_1,t_1)\in(\xi_1,s)\times(\xi_4,t)$.

Theorem 8. *Let $a\in\mathbb{R}$, $\xi_1<\xi_2$, $<\xi_3<\xi_4$ and $\Phi_1, \Phi_2 : [\xi_1,\xi_2]\times[\xi_3,\xi_4]\longrightarrow\mathcal{R}$ be continuous functions. Then, the identities*

$$
\begin{aligned}
&\int_{\xi_3}^{t}\int_{\xi_1}^{s}[\Phi_1(z,w)+\Phi_2(z,w)]_{\xi_1}d_{q_1}z_{\xi_4}d_{q_2}w \\
&\quad = \int_{\xi_3}^{t}\int_{\xi_1}^{s}\Phi_1(z,w){}_{\xi_1}d_{q_1}z_{\xi_3}d_{q_2}w + \int_{\xi_3}^{t}\int_{\xi_1}^{s}\Phi_2(z,w){}_{\xi_1}d_{q_1}z_{\xi_3}d_{q_2}w, \\
&\int_{\xi_3}^{t}\int_{\xi_1}^{s}a\Phi(z,w){}_{\xi_1}d_{q_1}z_{\xi_3}d_{q_2}w = a\int_{\xi_3}^{t}\int_{\xi_1}^{s}\Phi(z,w){}_{\xi_1}d_{q_1}z_{\xi_3}d_{q_2}w,
\end{aligned} \tag{11}
$$

holds for $(s,t)\in[\xi_1,\xi_2]\times[\xi_3,\xi_4]$.

Very recently, Toplu et al. [74] improved the Hermite-Hadamard inequality (1) by investigating the n-polynomial convexity. The main purpose of the article is to introduce the notion of n-polynomial preinvex function, provide a new generalized quantum integral identity, establish new quantum analogues of Ostrowski-type inequalities for the n-polynomial preinvex function on coordinates, and generalize and unify the previous known results.

2. Discussions and Main Results

In the beginning of this section, we introduce the definition of n-polynomial prevexity.

Definition 9 (See [75]). Let $\eta(\cdot,\cdot)\colon \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a continuous bi-function. Then, $\Omega_\eta \subset \mathbb{R}^n$ is said to be invex if

$$\xi_1 + \gamma\eta(\xi_2, \xi_1) \in \Omega_\eta, \tag{12}$$

for all $\xi_1, \xi_2 \in \Omega_\eta$ and $\gamma \in [0, 1]$.

Definition 10 (See [67]). The function $\Phi : \Omega_\eta \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be preinvex if

$$\Phi(\xi_1 + \gamma\eta(\xi_2, \xi_1)) \le (1-\gamma)\Phi(\xi_1) + \gamma\Phi(\xi_2), \tag{13}$$

for all $\xi_1, \xi_2 \in \Omega_\eta$ and $\gamma \in [0, 1]$.

Definition 11. Let $n \in \mathbb{N}$. Then, the nonnegative function $\Phi : \Omega_\eta \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be n-polynomial preinvex if

$$\begin{aligned}\Phi(\xi_1 + \gamma\eta(\xi_2, \xi_1)) \le{}& \frac{1}{n}\sum_{p=1}^{n}\left[1-(1-\gamma)^p\right]\Phi(\xi_1) \\ &+ \frac{1}{n}\sum_{p=1}^{n}[1-(1-\gamma^p)]\Phi(\xi_2),\end{aligned} \tag{14}$$

for all $\xi_1, \xi_2 \in \Omega_\eta$ and $\gamma \in [0, 1]$.

Note that if $n = 1$, then the definition of n-polynomial preinvex function reduce to the definition of preinvex function.

If we take $n = 2$, then we have 2-polynomial preinvex function inequality

$$\Phi(\xi_1 + \gamma\eta(\xi_2, \xi_1)) \le \frac{3\gamma-\gamma^2}{2}\Phi(\xi_1) + \frac{1-\gamma-\gamma^2}{2}\Phi(\xi_2). \tag{15}$$

Proposition 12. *Let $\xi_2 > 0$ and $\Phi_\alpha : [\xi_1, \xi_1 + \eta(\xi_2, \xi_1)] \longrightarrow \mathbb{R}$ be an arbitrary family of n -polynomial preinvex functions and $\Phi(\xi) = \sup_\alpha \Phi_\alpha(\xi)$. If $J_\eta = \{u \in [\xi_1, \xi_1 + \eta(\xi_2, \xi_1)]\colon \Phi(u) < 1\}$ is nonempty, then J_η is an interval and Φ is an n-polynomial preinvex function on J_η.*

Proof. Let $\gamma \in [0, 1]$ and $\xi_1, \xi_1 + \eta(\xi_2, \xi_1) \in J_\eta$. Then, we have

$$\begin{aligned}\Phi(\xi_1 + \gamma\eta(\xi_2, \xi_1)) &= \sup_\alpha \Phi_\alpha(\xi_1 + \gamma\eta(\xi_2, \xi_1)) \\ &\le \sup_\alpha\left[\frac{1}{n}\sum_{p=1}^{n}\left[1-(1-\gamma)^p\right]\Phi_\alpha(\xi_1) + \frac{1}{n}\sum_{p=1}^{n}[1-\gamma]^p\Phi_\alpha(\xi_2)\right] \\ &\le \frac{1}{n}\sum_{p=1}^{n}\left[1-(1-\gamma)^p\right]\sup_\alpha \Phi_\alpha(\xi_1) + \frac{1}{n}\sum_{p=1}^{n}(1-\gamma)^p\sup_\alpha\Phi_\alpha(\xi_2) \\ &\le \frac{1}{n}\sum_{p=1}^{n}\left[1-(1-\gamma)^p\right]\Phi(\xi_1) + \frac{1}{n}\sum_{p=1}^{n}(1-\gamma)^p\Phi(\xi_2) < \infty.\end{aligned} \tag{16}$$

This completes the proof.

In order to establish new quantum analogues of the Ostrowski-type inequalities on coordinates for the n-polynomial preinvex function, we need a key lemma, which we present in this section.

Lemma 13. *Let $q_1, q_2 \in (0, 1), \xi_1 < \xi_2, \xi_3 < \xi_4, \mathcal{N} = [\xi_1, \xi_2] \times [\xi_3, \xi_4], \mathcal{N}^\circ$, be the interior of $\mathcal{N}$, and $\Phi : \mathcal{N} \longrightarrow \mathbb{R}$ be mixed partial q_1q_2-differentiable on $\mathcal{N}^\circ$ such that $|({}_{\xi_1,\xi_3}\partial^2_{q_1,q_2} / {}_{\xi_1}\partial_{q_1} z_{\xi_3}\partial_{q_2} w)\Phi|$ is continuous and integrable on $[\xi_1, \xi_1 + \eta_1(\xi_2, \xi_1)] \times [\xi_3, \xi_3 + \eta_2(\xi_4, \xi_3)] \subset \mathcal{N}^\circ$ for $\eta_1(\xi_2, \xi_1), \eta_2(\xi_4, \xi_3) > 0$. Then, we have the identity*

$$\begin{aligned}\Omega_{q_1q_2}(\xi_1,\xi_2,\xi_3,\xi_4)(\Phi) ={}& -\frac{q_1q_2[\eta_1(\varrho,\xi_1)]^2[\eta_2(\rho,\xi_3)]^2}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)} \\ &\times \int_0^1\int_0^1 zw \frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1} z_{\xi_3}\partial_{q_2} w}\Phi(\xi_1 + z\eta_1(\varrho,\xi_1), \xi_3 + w\eta_2(\rho,\xi_3))_0 d_{q_1} z_0 d_{q_2} w \\ &- \frac{q_1q_2[\eta_1(\varrho,\xi_1)]^2[\eta_2(\xi_4,\rho)]^2}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)} \\ &\times \int_0^1\int_0^1 zw \frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1} z_{\xi_3}\partial_{q_2} w}\Phi(\xi_1 + z\eta_1(\varrho,\xi_1), \xi_4 + w\eta_2(\rho,\xi_4))_0 d_{q_1} z_0 d_{q_2} w \\ &- \frac{q_1q_2[\eta_1(\xi_2,\varrho)]^2[\eta_2(\rho,\xi_3)]^2}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)} \\ &\times \int_0^1\int_0^1 zw \frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1} z_{\xi_3}\partial_{q_2} w}\Phi(\xi_2 + z\eta_1(\varrho,\xi_2), \xi_3 + w\eta_2(\rho,\xi_3))_0 d_{q_1} z_0 d_{q_2} w \\ &- \frac{q_1q_2[\eta_1(\xi_2,\varrho)]^2[\eta_2(\xi_4,\rho)]^2}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)} \\ &\times \int_0^1\int_0^1 zw \frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1} z_{\xi_3}\partial_{q_2} w}\Phi(\xi_2 + z\eta_1(\varrho,\xi_2), \xi_4 + w\eta_2(\rho,\xi_4))_0 d_{q_1} z_0 d_{q_2} w,\end{aligned} \tag{17}$$

where

$$\begin{aligned}
\Omega_{q_1q_2}(\xi_1,\xi_2,\xi_3,\xi_4)(\Phi) &= \Phi(\varrho,\rho) \\
&\quad + \frac{1}{\eta_2(\xi_4,\xi_3)}\left[\int_{\xi_3}^{\xi_3+\eta_2(\rho,\xi_3)} \Phi(\varrho,\nu)\,{}_0d_{q_2}\nu + \int_{\xi_4}^{\xi_4+\eta_2(\rho,\xi_4)} \Phi(\varrho,\nu)\,{}_0d_{q_2}\nu\right] \\
&\quad + \frac{1}{\eta_1(\xi_2,\xi_1)}\left[\int_{\xi_1}^{\xi_1+\eta_1(\varrho,\xi_1)} \Phi(u,\rho)\,{}_0d_{q_1}u + \int_{\xi_2}^{\xi_2+\eta_1(\varrho,\xi_2)} \Phi(u,\rho)\,{}_0d_{q_1}\nu\right] - \mathcal{Q}, \\
\mathcal{Q} &= \frac{1}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)} \int_{\xi_1}^{\xi_1+\eta_1(\varrho,\xi_1)} \int_{\xi_3}^{\xi_3+\eta_2(\rho,\xi_3)} \Phi(u,\nu)\,{}_0d_{q_1}u\,{}_0d_{q_2}\nu \\
&\quad + \frac{1}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)} \int_{\xi_1}^{\xi_1+\eta_1(\varrho,\xi_1)} \int_{\xi_4}^{\xi_4+\eta_2(\rho,\xi_4)} \Phi(u,\nu)\,{}_0d_{q_1}u\,{}_0d_{q_2}\nu \\
&\quad + \frac{1}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)} \int_{\xi_2}^{\xi_2+\eta_1(\varrho,\xi_2)} \int_{\xi_3}^{\xi_3+\eta_2(\rho,\xi_3)} \Phi(u,\nu)\,{}_0d_{q_1}u\,{}_0d_{q_2}\nu \\
&\quad + \frac{1}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)} \int_{\xi_2}^{\xi_2+\eta_1(\varrho,\xi_2)} \int_{\xi_4}^{\xi_4+\eta_2(\rho,\xi_4)} \Phi(u,\nu)\,{}_0d_{q_1}u\,{}_0d_{q_2}\nu.
\end{aligned} \tag{18}$$

Proof. Considering

$$\begin{aligned}
&-\int_0^1\int_0^1 zw \frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z\,{}_{\xi_3}\partial_{q_2}w}\Phi(\xi_1+z\eta_1(\varrho,\xi_1),\xi_3+w\eta_2(\rho,\xi_3))\,{}_0d_{q_1}z\,{}_0d_{q_2}w \\
&\quad -\int_0^1\int_0^1 zw \frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z\,{}_{\xi_3}\partial_{q_2}w}\Phi(\xi_1+z\eta_1(\varrho,\xi_1),\xi_4+w\eta_2(\rho,\xi_4))\,{}_0d_{q_1}z\,{}_0d_{q_2}w \\
&\quad -\int_0^1\int_0^1 zw \frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z\,{}_{\xi_3}\partial_{q_2}w}\Phi(\xi_2+z\eta_1(\varrho,\xi_2),\xi 3+w\eta_2(\rho,\xi_3))\,{}_0d_{q_1}z\,{}_0d_{q_2}w \\
&\quad -\int_0^1\int_0^1 zw \frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z\,{}_{\xi_3}\partial_{q_2}w}\Phi(\xi_2+z\eta_1(\varrho,\xi_2),\xi_4+w\eta_2(\rho,\xi_4))\,{}_0d_{q_1}z\,{}_0d_{q_2}w,
\end{aligned} \tag{19}$$

it follows from the definitions of partial q_1q_2-derivative and q_1q_2-integral that

$$\begin{aligned}
&\int_0^1\int_0^1 zw \frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z\,{}_{\xi_3}\partial_{q_2}w}\Phi(\xi_1+z\eta_1(\varrho,\xi_1),\xi_3+w\eta_2(\rho,\xi_3))\,{}_0d_{q_1}z\,{}_0d_{q_2}w \\
&\quad = \frac{1}{(1-q_1)(1-q_2)\eta_1(\varrho,\xi_1)\eta_2(\rho,\xi_3)} \times \left[\int_0^1\int_0^1 \Phi(\xi_1+zq_1\eta_1(\varrho,\xi_1),\xi_3\right. \\
&\quad + wq_2\eta_2(\rho,\xi_2))\,{}_0d_{q_1}z\,{}_0d_{q_2}w - \int_0^1\int_0^1 \Phi(\xi_1+zq_1\eta_1(\varrho,\xi_1),\xi_3 \\
&\quad + w\eta_2(\rho,\xi_3))\,{}_0d_{q_1}z\,{}_0d_{q_2}w - \int_0^1\int_0^1 \Phi(\xi_1+z\eta_1(\varrho,\xi_1),\xi_3+wq_2\eta_2(\rho,\xi_3))\,{}_0d_{q_1}z\,{}_0d_{q_2}w \\
&\quad \left.+ \int_0^1\int_0^1 \Phi(\xi_1+z\eta_1(\varrho,\xi_1),\xi_3+w\eta_2(\rho,\xi_3))\,{}_0d_{q_1}z\,{}_0d_{q_2}w\right] = \frac{1}{\eta_1(\varrho,\xi_1)\eta_2(\rho,\xi_3)} \\
&\quad \times \left[\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m \Phi\left(\xi_1+q_1^{n+1}\eta_1(\varrho,\xi_1),\xi_3+q_2^{m+1}\eta_2(\rho,\xi_3)\right.\right. \\
&\quad \left. - \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m \Phi(\xi_1+q_1^{n+1}\eta_1(\varrho,\xi_1),\xi_3+q_2^{m}\eta_2(\rho,\xi_3))\right) \\
&\quad - \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m \Phi(\xi_1+q_1^{n}\eta_1(\varrho,\xi_1),\xi_3+q_2^{m+1}\eta_2(\rho,\xi_3)) \\
&\quad \left.+ \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m \Phi(\xi_1+q_1^{n+1}\eta_1(\varrho,\xi_1),\xi_3+q_2^{m}\eta_2(\rho,\xi_3))\right] \\
&\quad = \frac{1}{q_1q_2\eta_1(\varrho,\xi_1)\eta_2(\rho,\xi_3)} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} q_1^n q_2^m \Phi(\xi_1+q_1^{n}\eta_1(\varrho,\xi_1),\xi_3+q_2^{m}\eta_2(\rho,\xi_3)) \\
&\quad - \frac{1}{q_1\eta_1(\varrho,\xi_1)\eta_2(\rho,\xi_3)} \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m \Phi(\xi_1+q_1^{n}\eta_1(\varrho,\xi_1),\xi_3+q_2^{m}\eta_2(\rho,\xi_3)) \\
&\quad - \frac{1}{q_2\eta_1(\varrho,\xi_1)\eta_2(\rho,\xi_3)} \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} q_1^n q_2^m \Phi(\xi_1+q_1^{n}\eta_1(\varrho,\xi_1),\xi_3+q_2^{m}\eta_2(\rho,\xi_3)) \\
&\quad + \frac{1}{\eta_1(\varrho,\xi_1)\eta_2(\rho,\xi_3)} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m \Phi(\xi_1+q_1^{n}\eta_1(\varrho,\xi_1),\xi_3+q_2^{m}\eta_2(\rho,\xi_3)).
\end{aligned} \tag{20}$$

Note that

$$\begin{aligned}
&\frac{1}{q_1q_2\eta_1(\varrho,\xi_1)\eta_2(\rho,\xi_3)} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} q_1^n q_2^m \Phi(\xi_1+q_1^{n}\eta_1(\varrho,\xi_1),\xi_3+q_2^{m}\eta_2(\rho,\xi_3)) \\
&\quad = -\frac{\Phi(\varrho,\rho)}{q_1q_2\eta_1(\varrho,\xi_1)\eta_2(\rho,\xi_3)} - \frac{1}{q_1q_2\eta_1(\varrho,\xi_1)\eta_2(\rho,\xi_3)} \sum_{n=0}^{\infty} q^n \Phi(\xi_1+q_1^n\eta_1(\varrho,\xi_1),\rho) \\
&\quad - \frac{1}{q_1q_2\eta_1(\varrho,\xi_1)\eta_2(\rho,\xi_3)} \sum_{m=0}^{\infty} q_2^m(\varrho,\xi_3+q_2^m\eta_2(\rho,\xi_3)) - \frac{1}{q_1q_2\eta_1(\varrho,\xi_1)\eta_2(\rho,\xi_3)} \\
&\quad \times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m \Phi(\xi_1+q_1^{n}\eta_1(\varrho,\xi_1),\xi_3+q_2^{m}\eta_2(\rho,\xi_3)),
\end{aligned} \tag{21}$$

$$\begin{aligned}
&-\frac{1}{q_1\eta_1(\varrho,\xi_1)\eta_2(\rho,\xi_3)} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} q_1^n q_2^m \Phi(\xi_1+q_1^{n}\eta_1(\varrho,\xi_1),\xi_3+q_2^{m}\eta_2(\rho,\xi_3)) \\
&\quad = \frac{1}{q_1\eta_1(\varrho,\xi_1)\eta_2(\rho,\xi_3)} \sum_{m=0}^{\infty} q_2^m \Phi(\varrho,\xi_3+q_2^m\eta_2(\rho,\xi_3)) - \frac{1}{q_1\eta_1(\varrho,\xi_1)\eta_2(\rho,\xi_3)} \\
&\quad \times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m \Phi(\xi_1+q_1^{n}\eta_1(\varrho,\xi_1),\xi_3+q_2^{m}\eta_2(\rho,\xi_3)),
\end{aligned} \tag{22}$$

$$
\begin{aligned}
&-\frac{1}{q_2\eta_1(\varrho,\xi_1)\eta_2(\rho,\xi_3)}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}q_1^n q_2^m \Phi(\xi_1+q_1^n\eta_1(\varrho,\xi_1),\xi_3+q_2^m\eta_2(\rho,\xi_3))\\
&\quad=\frac{1}{q_2\eta_1(\varrho,\xi_1)\eta_2(\rho,\xi_3)}\sum_{n=0}^{\infty}q_1^n\Phi(\xi_1+q_1^n\eta_1(\varrho,\xi_1),\rho)-\frac{1}{q_2\eta_1(\varrho,\xi_1)\eta_2(\rho,\xi_3)}\\
&\quad\quad\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^n q_2^m\Phi(\xi_1+q_1^n\eta_1(\varrho,\xi_1),\xi_3+q_2^m\eta_2(\rho,\xi_3)).
\end{aligned}
\tag{23}
$$

From (20)–(23) we get

$$
\begin{aligned}
&\int_0^1\int_0^1 zw\,\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\Phi(\xi_1+z\eta_1(\varrho,\xi_1),\xi_3+w\eta_2(\rho,\xi_3))\,{}_0d_{q_1}z\,{}_0d_{q_2}w\\
&\quad=-\frac{\Phi(\varrho,\rho)}{q_2q_2\eta_1(\varrho,\xi_1)\eta_2(\rho,\xi_3)}-\frac{(1-q_2)\eta_2(\rho,\xi_3)}{q_1q_2\eta_1(\varrho,\xi_1)[\eta_2(\rho,\xi_3)]^2}\sum_{m=0}^{\infty}q_2^m(\varrho,\xi_3\\
&\quad\quad+q_2^m\eta_2(\rho,\xi_3))-\frac{(1-q_1)\eta_1(\varrho,\xi_1)}{q_1q_2[\eta_1(\varrho,\xi_1)]^2\eta_2(\rho,\xi_3)}\sum_{n=0}^{\infty}q_2^n\Phi(\xi_1+q_1^m\eta_1(\varrho,\xi_1),\rho)\\
&\quad\quad+\frac{(1-q_1)(1-q_2)\eta_1(\varrho,\xi_1)\eta_2(\rho,\xi_3)}{q_1q_2[\eta_1(\varrho,\xi_1)]^2[\eta_2(\rho,\xi_3)]^2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^nq_2^m\Phi(\xi_1+q_1^n\eta_1(\varrho,\xi_1),\xi_3\\
&\quad\quad+q_2^m\eta_2(\rho,\xi_3)),
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
&\int_0^1\int_0^1 zw\,\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\Phi(\xi_1+z\eta_1(\varrho,\xi_2),\xi_3+w\eta_2(\rho,\xi_3))\,{}_0d_{q_1}z\,{}_0d_{q_2}w\\
&\quad=-\frac{\Phi(\varrho,\rho)}{q_1q_2\eta_1(\varrho,\xi_1)\eta_2(\rho,\xi_3)}-\frac{\Phi(\varrho,\rho)}{q_1q_2\eta_1(\varrho,\xi_1)[\eta_2(\rho,\xi_3)]^2}\int_{\xi_3}^{\xi_3+\eta_2(\rho,\xi_3)}\Phi(\varrho,v)\,{}_0d_{q_2}v\\
&\quad\quad-\frac{1}{q_1q_2[\eta_1(\varrho,\xi_1)]^2\eta_2(\rho,\xi_3)}\int_{\xi_1}^{\xi_3+\eta_1(\varrho,\xi_1)}\Phi(u,\rho)\,{}_0d_{q_1}v\\
&\quad\quad+\frac{1}{q_1q_2[\eta_1(\varrho,\xi_1)]^2[\eta_2(\rho,\xi_3)]^2}\int_{\xi_1}^{\xi_2+\eta_1(\varrho,\xi_2)}\int_{\xi_3}^{\xi_3+\eta_2(\varrho,\xi_3)}\Phi(u,v)\,{}_0d_{q_1}u\,{}_0d_{q_2}v.
\end{aligned}
\tag{25}
$$

Multiplying both sides of equality (25) by

$$
\frac{q_1q_2[\eta_1(\varrho,\xi_1)]^2[\eta_2(\rho,\xi_3)]^2}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)},
\tag{26}
$$

leads to

$$
\begin{aligned}
&\frac{q_1q_2[\eta_1(\varrho,\xi_1)]^2[\eta_1(\rho,\xi_3)]^2}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\int_0^1\int_0^1 zw\,\frac{\partial^2_{q_1,q_2}}{\xi_1\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\times\Phi(\xi_1+z\eta_1(\varrho,\xi_1),\xi_3+w\eta_2(\rho,\xi_3))\,{}_0d_{q_1}z\,{}_0d_{q_2}w\\
&\quad=-\frac{\eta_1(\varrho,\xi_1)\eta_2(\rho,\xi_3)}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\Phi(\varrho,\rho)-\frac{\eta_1(\varrho,\xi_1)}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\int_{\xi_3}^{\xi_3+\eta_2(\rho,\xi_3)}\Phi(\varrho,v)\,{}_0d_{q_2}v\\
&\quad\quad-\frac{\eta_2(\rho,\xi_3)}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\int_{\xi_1}^{\xi_1+\eta_1(\varrho,\xi_1)}\Phi(u,\rho)\,{}_0d_{q_1}u+\frac{1}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\int_{\xi_3}^{\xi_3+\eta_2(\rho,\xi_3)}\Phi(u,v)\,{}_0d_{q_1}u\,{}_0d_{q_2}v.
\end{aligned}
\tag{27}
$$

Similarly, we have

$$
\begin{aligned}
&\frac{q_1q_2[\eta_1(\varrho,\xi_1)]^2[\eta_2(\xi_4,\rho)]^2}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\int_0^1\int_0^1 zw\,\frac{\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\times\Phi(\xi_1+z\eta_1(\varrho,\xi_1),\xi_4+z\eta_2(\rho,\xi_4))\,{}_0d_{q_1}z\,{}_0d_{q_2}w\\
&\quad=-\frac{\eta_1(\varrho,\xi_1)\eta_2(\xi_4,\rho)}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\Phi(\varrho,\rho)-\frac{\eta_1(\varrho,\xi_1)}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\int_{\xi_4}^{\xi_4+\eta_2(\rho,\xi_4)}\Phi(\varrho,v)\,{}_0d_{q_2}v\\
&\quad\quad+\frac{\eta_2(\xi_4,\rho)}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\int_{\xi_1}^{\xi_1+\eta_1(\varrho,\xi_1)}\Phi(u,\rho)\,{}_0d_{q_1}u+\frac{1}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\int_{\xi_1}^{\xi_1+\eta_1(\varrho,\xi_1)}\int_{\xi_4}^{\xi_4+\eta_2(\rho,\xi_4)}\Phi(u,v)\,{}_0d_{q_1}u\,{}_0d_{q_2}v,\\
&\quad\quad\times\frac{q_1q_2[\eta_1(\xi_2,\varrho)]^2[\eta_2(\rho,\xi_3)]^2}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\int_0^1\int_0^1 zw\,\frac{\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\times\Phi(\xi_2+z\eta_1(\varrho,\xi_2)+w\eta_2(\rho,\xi_3))\,{}_0d_{q_1}z\,{}_0d_{q_2}w\\
&\quad=-\frac{\eta_1(\xi_2,\varrho)\eta_2(\rho,\xi_3)}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\Phi(\varrho,\rho)-\frac{\eta_1(\xi_2,\varrho)}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\int_{\xi_3}^{\xi_3+\eta_2(\rho,\xi_3)}\Phi(\varrho,v)\,{}_0d_{q_2}v\\
&\quad\quad-\frac{\eta_2(\rho,\xi_3)}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\int_{\xi_2}^{\xi_2+\eta_1(\rho,\xi_2)}\Phi(u,\rho)\,{}_0d_{q_1}u+\frac{1}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\int_{\xi_2}^{\xi_2+\eta_1(\varrho,\xi_2)}\int_{\xi_3}^{\xi_3+\eta_2(\rho,\xi_3)}\Phi(u,v)\,{}_0d_{q_1}u\,{}_0d_{q_2}v,
\end{aligned}
\tag{28}
$$

$$\begin{aligned}
&\frac{q_1q_2[\eta_1(\xi_2,\varrho)]^2[\eta_2(\rho,\xi_3)]^2}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\int_0^1\int_0^1 zw\frac{\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\times\Phi(\xi_2+z\eta_1(\varrho,\xi_2)+w\eta_2(\rho,\xi_3))_0d_{q_1}z_0d_{q_2}w\\
&=-\frac{\eta_1(\xi_2,\varrho)\eta_2(\rho,\xi_3)}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\Phi(\varrho,\rho)-\frac{\eta_1(\xi_2,\varrho)}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\int_{\xi_3}^{\xi_3+\eta_2(\rho,\xi_3)}\Phi(\varrho,v)_0d_{q_2}v\\
&\quad-\frac{\eta_2(\rho,\xi_3)}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\int_{\xi_2}^{\xi_2+\eta_1(\rho,\xi_2)}\Phi(u,\rho)_0d_{q_1}u+\frac{1}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\int_{\xi_2}^{\xi_2+\eta_1(\varrho,\xi_2)}\int_{\xi_3}^{\xi_3+\eta_2(\rho,\xi_3)}\Phi(u,v)_0d_{q_1}u_0d_{q_2}v,
\end{aligned}\tag{29}$$

$$\begin{aligned}
&\frac{q_1q_2[\eta_1(\xi_2,\varrho)]^2[(\xi_4,\rho)]^2}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\int_0^1\int_0^1 zw\frac{\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\times\Phi(\xi_2+z\eta_1(\varrho,\xi_2),\xi_4+w\eta_2(\rho,\xi_4))_0d_{q_1}z_0d_{q_2}w\\
&=-\frac{\eta_1(\xi_2,\varrho)\eta_2(\xi_4,\rho)}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\Phi(\varrho,\rho)-\frac{\eta_1(\xi_2,\varrho)}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\int_{\xi_4}^{\xi_4+\eta_2(\rho,\xi_4)}\Phi(\varrho,v)_0d_{q_2}v\\
&\quad-\frac{\eta_2(\xi_4,\rho)}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\int_{\xi_2}^{\xi_2+\eta_1(\varrho,\xi_2)}\Phi(u,\rho)_0d_{q_1}u+\frac{1}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\int_{\xi_2}^{\xi_2+\eta_1(\varrho,\xi_2)}\int_{\xi_4}^{\xi_4+\eta_2(\rho,\xi_4)}\Phi(u,v)_0d_{q_1}u_0dq_2v.
\end{aligned}\tag{30}$$

Therefore, Lemma 13 follows from (19), (27), (28), (29), and (30).

Theorem 14. *Let $q_1,q_2\in(0,1)$, $\xi_1<\xi_2$, $\xi_3<\xi_4,\Phi:\mathcal{N}\subseteq\mathbb{R}^2\longrightarrow\mathbb{R}$ be mixed partial q_1q_2 differentiable on N° (the interior of $\mathcal{N}$) such that its mixed partial q_1q_2-derivatives is continuous and integrable on $[\xi_1,\xi_1+\eta_1(\xi_2,\xi_1)]\times[\xi_3,\xi_3+\eta_2(\xi_4,\xi_3)]\subset\mathcal{N}^\circ$ for $\eta_1(\xi_2,\xi_1),\eta_2(\xi_4,\xi_3)>0$. If $|(\xi_1,\xi_3\partial^2_{q_1,q_2}/\xi_1\partial_{q_1}z_{\xi_3}\partial_{q_2}w)\Phi|$ is an n-polynomial preinvex function on the coordinates on $[\xi_1,\xi_1+\eta_1(\xi_2,\xi_1)]\times[\xi_3,\xi_3+\eta_2(\xi_4,\xi_3)]$ and $|(\xi_1,\xi_3\partial^2_{q_1,q_2}/\xi_1\partial_{q_1}z_{\xi_3}\partial_{q_2}w)\Phi(\varrho,\rho)|\le M$ for $\varrho,\rho\in\mathcal{N}$, then we have*

$$\begin{aligned}
&\left|\Phi(\varrho,\rho)+\frac{1}{\eta_2(\xi_4,\xi_1)}\left[\int_{\xi_3}^{\xi_3+\eta_2(\rho,\xi_3)}\Phi(\varrho,v)_0d_{q_2}v+\int_{\xi_4}^{\xi_3+\eta_2(\rho,\xi_3)}\Phi(\varrho,v)_0d_{q_2}v\right]\right.\\
&\left.\quad+\frac{1}{\eta_1(\xi_2,\xi_1)}\left[\int_{\xi_1}^{\xi_1+\eta_1(\varrho,\xi_1)}\Phi(u,\rho)_0d_{q_1}u+\int_{\xi_2}^{\xi_2,\eta_1(\varrho,\xi_3)}\Phi(u,\rho)_0d_{q_1}u\right]-Q\right|\\
&\le q_1q_2M\left(\mathscr{A}_{q_1}+\mathscr{B}_{q_1}\right)\left(\mathscr{A}_{q_2}+\mathscr{B}_{q_2}\right)\left[\frac{[\eta_1(\varrho,\xi_1)]^2+[\eta_1(\xi_2,\varrho)]^2}{\eta_1(\xi_2,\xi_1)}\right]\\
&\quad\times\left[\frac{[\eta_2(\rho,\xi_3)]^2+[\eta_2(\xi_4,\rho)]^2}{\eta_2(\xi_4,\xi_3)}\right],
\end{aligned}\tag{31}$$

where Q is defined in Lemma 13.

Proof. It follows from (19) and the properties of the n-polynomial preinvexity function of the function $|(\partial^2_{q_1,q_2}/\xi_1\partial_{q_1}z_{\xi_3}\partial_{q2}w)\Phi|$ on coordinates that

$$\begin{aligned}
&\left|\Phi(\varrho,\rho)+\frac{1}{\eta_2(\xi_4\xi_3)}\left[\int_{\xi_3}^{\xi_3+\eta_2(\rho,\xi_3)}\Phi(\varrho,\rho)_0d_{q_2}v+\int_{\xi_4}^{\xi_4+\eta_2(\rho,\xi_4)}\Phi(\varrho,\rho)_0d_{q_2}v\right]+\frac{1}{\eta_1(\xi_2\xi_1)}\left[\int_{\xi_1}^{\xi_1+\eta_1(\varrho,\xi_1)}\Phi(u,\rho)_0d_{q1}u+\int_{\xi_2}^{\xi_2+\eta_1(\varrho,\xi_2)}\Phi(u,\rho)_0d_{q1}u\right]-Q\right|\\
&\le\frac{q_1q_2[\eta_1(\varrho,\xi_1)]^2[\eta_2(\rho,\xi_3)]^2}{\eta_1(\xi_2\xi_1)\eta_2(\xi_4\xi_3)}\int_0^1\int_0^1 zw\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q2}w}\Phi(\xi_1+z\eta_1(\varrho,\xi_1),\xi_3+w\eta_2(\rho,\xi_3))\right|_0d_{q_1}z_0d_{q_2}w\\
&\quad+\frac{q_1q_2[\eta_1(\varrho,\xi_1)]^2[\eta_2(\xi_4,\rho)]^2}{\eta_1(\xi_2\xi_1)\eta_2(\xi_4\xi_3)}\int_0^1\int_0^1 zw\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q2}w}\Phi(\xi_1+z\eta_1(\varrho,\xi_1),\xi_4+w\eta_2(\rho,\xi_4))\right|_0d_{q_1}z_0d_{q_2}w\\
&\quad+\frac{q_1q_2[\eta_1(\xi_2,\varrho)]^2[\eta_2(\rho,\xi_3)]^2}{\eta_1(\xi_2\xi_1)\eta_2(\xi_4\xi_3)}\int_0^1\int_0^1 zw\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q2}w}\Phi(\xi_2+z\eta_1(\varrho,\xi_2),\xi_3+w\eta_2(\rho,\xi_3))\right|_0d_{q_1}z_0d_{q_2}w\\
&\quad+\frac{q_1q_2[\eta_1(\xi_2,\varrho)]^2[\eta_2(\xi_4,\rho)]^2}{\eta_1(\xi_2\xi_1)\eta_2(\xi_4\xi_3)}\int_0^1\int_0^1 zw\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q2}w}\Phi(\xi_2+z\eta_1(\varrho,\xi_2),\xi_4+w\eta_2(\rho,\xi_4))\right|_0d_{q_1}z_0d_{q_2}w.
\end{aligned}\tag{32}$$

Considering the first integral

$$\begin{aligned}&\int_0^1\int_0^1 zw\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q2}w}\Phi(\xi_1+z\eta_1(\varrho,\xi_1),\xi_3+w\eta_2(\rho,\xi_3))\right|{}_0d_{q_1}z_0d_{q_2}w\\&\le\int_0^1 w\left\{\int_0^1 z\left[\frac{1}{n}\sum_{p=1}^{n}\left[1-(1-z)^p\right]\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q2}w}\Phi(\varrho,\xi_3+w\eta_2(\rho,\xi_3))\right|\right.\right.\\&\left.\left.+\frac{1}{n}\sum_{p=1}^{n}[1-z^p]\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q2}w}\Phi(\xi_1,\xi_3+w\eta_2(\rho,\xi_3))\right|\right]{}_0d_{q_1}z\right\}\cdot{}_0d_{q_2}w,\end{aligned}\tag{33}$$

in view of the Definition 6 for $k=1,2$, we get

$$\begin{aligned}\mathscr{A}_{qk}&=\frac{1}{n}\sum_{p=1}^{n}\int_0^1 z\left[1-(1-z)^p\right]{}_0d_{qk}z=\frac{1}{1+q_k}-\frac{(1-q_k)}{n}\sum_{p=1}^{n}\sum_{e=0}^{\infty}q_k^{2e}(1-q_k^e)^p,\\\mathscr{B}_{qk}&=\frac{1}{n}\sum_{p=1}^{n}\int_0^1 z[1-z^p]{}_0d_{qk}z=\frac{1}{1+q_k}-\frac{1}{n}\sum_{p=1}^{n}\frac{1-q_k}{1-q_k^{p+2}}.\end{aligned}\tag{34}$$

From (33) and computing the q_1-integral, we get

$$\begin{aligned}&\int_0^1\int_0^1 zw\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q2}w}\Phi(\xi_1+z\eta_1(\varrho,\xi_1),\xi_3+w\eta_2(\rho,\xi_3))\right|{}_0d_{q_1}z_0d_{q_2}w\\&\le\int_0^1 w\left[\begin{array}{l}\mathscr{A}_{q_1}\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q2}w}\Phi(\varrho,\xi_3+w\eta_2(\rho,\xi_3))\right|\\+\mathscr{B}_{q_1}\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q2}w}\Phi(\xi_1,\xi_3+w\eta_2(\rho,\xi_3))\right|\end{array}\right]{}_0d_{q2}w.\end{aligned}\tag{35}$$

Computing the q_2-integral and utilizing the fact $|({}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}/{}_{\xi_1}\partial_{q_1}z\partial_{q_2}w)\Phi(\varrho,\rho)|\le M$ for $\varrho,\rho\in\mathscr{N}$, inequality (35) leads to the conclusion that

$$\begin{aligned}&\int_0^1\int_0^1 zw\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z\partial_{q_2}w}\Phi(\xi_1+z\eta_1(\varrho,\xi_1),\xi_3+w\eta_2(\rho,\xi_3))\right|{}_0d_{q_1}z_0d_{q_2}w\\&\le M\left(\mathscr{A}_{q_1}+\mathscr{B}_{q_1}\right)\left(\mathscr{A}_{q_2}+\mathscr{B}_{q_2}\right).\end{aligned}\tag{36}$$

Analogously, we also have

$$\begin{aligned}&\int_0^1\int_0^1 zw\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\Phi(\xi_1+z\eta_1(\varrho,\xi_1),\xi_4+w\eta_2(\rho,\xi_4))\right|{}_0d_{q_1}z_0d_{q_2}w\\&\le M\left(\mathscr{A}_{q_1}+\mathscr{B}_{q_1}\right)\left(\mathscr{A}_{q_2}+\mathscr{B}_{q_2}\right),\end{aligned}\tag{37}$$

$$\begin{aligned}&\int_0^1\int_0^1 zw\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\Phi(\xi_2+z\eta_1(\varrho,\xi_2),\xi_3+w\eta_2(\rho,\xi_3))\right|{}_0d_{q_1}z_0d_{q_2}w\\&\le M\left(\mathscr{A}_{q_1}+\mathscr{B}_{q_1}\right)\left(\mathscr{A}_{q_2}+\mathscr{B}_{q_2}\right),\end{aligned}\tag{38}$$

$$\begin{aligned}&\int_0^1\int_0^1 zw\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\Phi(\xi_2+z\eta_1(\varrho,\xi_2),\xi_4+w\eta_2(\rho,\xi_4))\right|{}_0d_{q_1}z_0d_{q_2}w\\&\le M\left(\mathscr{A}_{q_1}+\mathscr{B}_{q_1}\right)\left(\mathscr{A}_{q_2}+\mathscr{B}_{q_2}\right).\end{aligned}\tag{39}$$

Making use of the inequalities (36), (37), (38), and (39) and the fact that

$$\begin{aligned}&[\eta_1(\varrho,\xi_1)]^2[\eta_2(\rho,\xi_3)]^2+[\eta_1(\varrho,\xi_1)]^2[\eta_2(\xi_4,\rho)]^2\\&\quad+[\eta_1(\xi_2,\varrho)]^2[\eta_2(\rho,\xi_3)]^2+[\eta_1(\xi_2,\varrho)]^2[\eta_2(\xi_4,\rho)]^2\\&=\left[[\eta_1(\varrho,\xi_1)]^2+[\eta_1(\xi_2,\varrho)]^2\right]\left[[\eta_2(\rho,\xi_3)]^2+[\eta_2(\xi_4,\rho)]^2\right],\end{aligned}\tag{40}$$

we get the desired inequality (31).

Theorem 15. *Let $\gamma_1,\gamma_2>1$ with $(1/\gamma_2)+(1/\gamma_1)=1$, $q_1,q_2\in(0,1)$, and $\Phi:\mathscr{N}\subseteq\mathbb{R}^2\longrightarrow\mathbb{R}$ be mixed partial q_1q_2-differentiable on N° such that its mixed partial q_1q_2-derivative is continuous and integrable on $[\xi_1,\xi_1+\eta_1(\xi_2,\xi_1)]\times[\xi_3,\xi_3+\eta_2(\xi_4,\xi_3)]\subset\mathscr{N}^\circ$ for $\eta_1(\xi_2,\xi_1),\eta_2(\xi_4,\xi_3)>0$. If $|({}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}/{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w)\Phi|^{\gamma_2}$ is an n-polynomial preinvex function on the coordinates on $[\xi_1,\xi_1+\eta_1(\xi_2,\xi_1)]\times[\xi_3,\xi_3+\eta_2(\xi_4,\xi_3)]$ and $|({}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}/{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w)\Phi(\varrho,\rho)|\le M$ for $\varrho,\rho,\in\mathscr{N}$, then we have*

$$\begin{aligned}&\left|\Phi(\varrho,\rho)+\frac{1}{\eta_2(\xi_4,\xi_3)}\left[\int_{\xi_3}^{\xi_3+\eta_2(\rho,\xi_3)}\Phi(\varrho,\rho){}_0d_{q_2}v+\int_{\xi_4}^{\xi_3+\eta_2(\rho,\xi_3)}\Phi(\varrho,\rho){}_0d_{q_2}v\right]\frac{1}{\eta_1(\xi_2,\xi_1)}\right.\\&\left.\cdot\left[\int_{\xi_1}^{\xi_1+\eta_1(\varrho,\xi_1)}\Phi(u,\rho){}_0d_{q_1}u+\int_{\xi_2}^{\xi_2+\eta_1(\varrho,\xi_2)}\Phi(u,\rho){}_0d_{q_1}u\right]-\mathscr{Q}\right|\le\frac{q_1q_2M\sqrt[\gamma_2]{\left(\mathscr{C}_{q_1}+\mathscr{D}_{q_1}\right)\left(\mathscr{C}_{q_2}+\mathscr{D}_{q_2}\right)}}{\sqrt[\gamma_1]{[\gamma_1+1]_{q_1}[\gamma_1+1]_{q_2}}}\\&\cdot\left[\frac{[\eta_1(\varrho,\xi_1)]^2+[\eta_1(\xi_2,\varrho)]^2}{\eta_1(\xi_2,\xi_1)}\right]\times\left[\frac{[\eta_2(\rho,\xi_3)]^2+[\eta_2(\xi_4,\rho)]^2}{\eta_2(\xi_4,\xi_3)}\right],\end{aligned}\tag{41}$$

where $\mathcal{Q}$ is defined in Lemma 13.

Proof. It follows from (19), the Hölder inequality and the property of n-polynomial preinvexity of the function $|\partial^2_{q_1,q_2}\Phi/\xi_1\partial_{q_1}z_{\xi_3}\partial_{q2}w|^{\gamma_2}$ on coordinates that

$$\begin{aligned}
&\left|\Phi(\varrho,\rho)+\frac{1}{\eta_2(\xi_4,\xi_3)}\left[\int_{\xi_3}^{\xi_3+\eta_2(\rho,\xi_3)}\Phi(\varrho,v)_0d_{q_2}v\right.\right.\\
&\quad\left.+\int_{\xi_4}^{\xi_4+\eta_2(\rho,\xi_4)}\Phi(\varrho,v)_0d_{q_2}v\right]\\
&\quad+\frac{1}{\eta_1(\xi_2,\xi_1)}\left[\int_{\xi_1}^{\xi_1+\eta_1(\varrho,\xi_1)}\Phi(u,\rho)_0d_{q_1}u\right.\\
&\quad\left.\left.+\int_{\xi_2}^{\xi_2+\eta_1(\varrho,\xi_2)}\Phi(u,\rho)_0d_{q_1}u\right]-\mathcal{Q}\right|\\
&\le\left(\int_0^1\int_0^1 z^{\gamma_1}w^{\gamma_1}{}_0d_{q_1}z_0d_{q_2}w\right)^{1/\gamma_1}\\
&\quad\times\left[\frac{q_1q_2[\eta_1(\varrho,\xi_1)]^2[\eta_2(\rho,\xi_3)]^2}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\right.\\
&\quad\cdot\left(\int_0^1\int_0^1\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\Phi(\xi_1+z\eta_1(\varrho,\xi_1),\xi_3+w\eta_2(\rho,\xi_3))\right|^{\gamma_2}{}_0d_{q_1}z_0d_{q_2}w\right)^{1/\gamma_2}\\
&\quad+\frac{q_1q_2[\eta_1(\varrho,\xi_1)]^2[\eta_2(\xi_3,\rho)]^2}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\\
&\quad\cdot\left(\int_0^1\int_0^1\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\Phi(\xi_1+z\eta_1(\varrho,\xi_1),\xi_4+w\eta_2(\rho,\xi_4))\right|^{\gamma_2}{}_0d_{q_1}z_0d_{q_2}w\right)^{1/\gamma_2}\\
&\quad+\frac{q_1q_2[\eta_1(\xi_2,\varrho)]^2[\eta_2(\rho,\xi_3)]^2}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\\
&\quad\cdot\left(\int_0^1\int_0^1\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\Phi(\xi_2+z\eta_1(\varrho,\xi_2),\xi_3+w\eta_2(\rho,\xi_3))\right|^{\gamma_2}{}_0d_{q_1}z_0d_{q_2}w\right)^{1/\gamma_2}\\
&\quad+\frac{q_1q_2[\eta_1(\xi_2,\varrho)]^2[\eta_2(\xi_4,\rho)]^2}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\\
&\quad\left.\cdot\left(\int_0^1\int_0^1\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\Phi(\xi_2+z\eta_1(\varrho,\xi_2),\xi_4+w\eta_2(\rho,\xi_4))\right|^{\gamma_2}{}_0d_{q_1}z_0d_{q_2}w\right)^{\frac{1}{\gamma_2}}\right],
\end{aligned}\tag{42}$$

for all $\varrho,\rho\in N$.

From the n-polynomial preinvexity of the function $|({}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}/{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w)\Phi|^{\gamma_2}$, we get

$$\begin{aligned}
&\int_0^1\int_0^1\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\Phi(\xi_1+z\eta_1(\varrho,\xi_1),\xi_3+w\eta_2(\rho,\xi_3))\right|^{\gamma_2}{}_0d_{q_1}z_0d_{q_2}w\\
&\le\int_0^1\left\{\int_0^1\left[\frac{1}{n}\sum_{p=1}^{n}\left[1-(1-z)^p\right]\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\Phi(\varrho,\xi_3+w\eta_2(\rho,\xi_3))\right|^{\gamma_2}\right.\right.\\
&\quad\left.\left.+\frac{1}{n}\sum_{p=1}^{n}[1-z^p]\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\Phi(\xi_1,\xi_3+w\eta_2(\rho,\xi_3))\right|^{\gamma_2}\right]{}_0d_{q_1}z\right\}{}_0d_{q_2}w.
\end{aligned}\tag{43}$$

Computing the q_1-integral on the right-hand side of (43), we have

$$\begin{aligned}
&\int_0^1\int_0^1\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\Phi(\xi_1+z\eta_1(\varrho,\xi_1),\xi_3+w\eta_2(\rho,\xi_3))\right|^{\gamma_2}{}_0d_{q_1}z_0d_{q_2}w\\
&\le\int_0^1\left[\frac{1}{n}\sum_{p=1}^{n}\left[1-(1-z)^p\right]\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\Phi(\varrho,\xi_3+w\eta_2(\rho,\xi_3))\right|^{\gamma_2}\right.\\
&\quad\left.+\frac{1}{n}\sum_{p=1}^{n}[1-z^p]\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\Phi(\xi_1,\xi_3+w\eta_2(\rho,\xi_3))\right|^{\gamma_2}\right]{}_0d_{q_1}z.
\end{aligned}\tag{44}$$

In view of the Definition 6 for $k=1,2$, we obtain

$$\begin{aligned}
\mathcal{C}_{q_k}&=\frac{1}{n}\sum_{p=1}^{n}\int_0^1\left[1-(1-z)^p\right]{}_0d_{q_k}z=1-\frac{(1-q_k)}{n}\sum_{p=1}^{n}\sum_{e=0}^{\infty}q_k^e(1-q_k^e)^p,\\
\mathcal{D}_{q_k}&=\frac{1}{n}\sum_{p=1}^{n}\int_0^1\left[1-z^p\right]{}_0d_{q_k}z=1-\frac{1}{n}\sum_{p=1}^{n}\frac{1-q_k}{1-q_k^{p+2}}.
\end{aligned}\tag{45}$$

Therefore, we get

$$\begin{aligned}
&\int_0^1\int_0^1\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\Phi(\xi_1+z\eta_1(\varrho,\xi_1),\xi_3+w\eta_2(\rho,\xi_3))\right|^{\gamma_2}{}_0d_{q_1}z_0d_{q_2}w\\
&\le\int_0^1\left[\mathcal{C}_{q_1}\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\Phi(\varrho,\xi_3+w\eta_2(\rho,\xi_3))\right|^{\gamma_2}\right.\\
&\quad\left.+\mathcal{D}_{q_1}\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\Phi(\xi_1,\xi_3+w\eta_2(\rho,\xi_3))\right|^{\gamma_2}\right]{}_0d_{q_2}w.
\end{aligned}\tag{46}$$

Similarly, computing the q_2-integral and utilizing the fact $|({}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}/{}_{\xi_1}\partial_{q_1}z\partial_{q_2}w)\Phi(\varrho,\rho)|\le M$ for $\varrho,\rho\in\mathcal{N}$ on the right-hand side of (46), one has

$$\begin{aligned}
&\int_0^1\int_0^1\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\Phi(\xi_1+z\eta_1(\varrho,\xi_1),\xi_3+w\eta_2(\rho,\xi_3))\right|^{\gamma_2}{}_0d_{q_1}z_0d_{q_2}w\\
&\le M^{\gamma_2}\left(\mathcal{C}_{q_1}+\mathcal{D}_{q_1}\right)\left(\mathcal{C}_{q_2}+\mathcal{D}_{q_2}\right).
\end{aligned}\tag{47}$$

Analogously, we also can get

$$\begin{aligned}
&\int_0^1\int_0^1\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\Phi(\xi_1+z\eta_1(\varrho,\xi_1),\xi_4+w\eta_2(\rho,\xi_4))\right|^{\gamma_2}{}_0d_{q_1}z_0d_{q_2}w,\\
&\le M^{\gamma_2}\left(\mathcal{C}_{q_1}+\mathcal{D}_{q_1}\right)\left(\mathcal{C}_{q_2}+\mathcal{D}_{q_2}\right),
\end{aligned}\tag{48}$$

$$\int_0^1\int_0^1 \left| \frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1} z_{\xi_3}\partial_{q_2} w} \Phi(\xi_2 + z\eta_1(\varrho,\xi_2), \xi_3 + w\eta_2(\rho,\xi_3)) \right|^{\gamma_2} {}_0d_{q_1} z_0 d_{q_2} w \le M^{\gamma_2}\left(\mathscr{C}_{q_1} + \mathscr{D}_{q_1}\right)\left(\mathscr{C}_{q_2} + \mathscr{D}_{q_2}\right), \tag{49}$$

$$\int_0^1\int_0^1 \left| \frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1} z_{\xi_3}\partial_{q_2} w} \Phi(\xi_2 + z\eta_1(\varrho,\xi_2), \xi_4 + w\eta_2(\rho,\xi_4)) \right|^{\gamma_2} {}_0d_{q_1} z_0 d_{q_2} w \le M^{\gamma_2}\left(\mathscr{C}_{q_1} + \mathscr{D}_{q_1}\right)\left(\mathscr{C}_{q_2} + \mathscr{D}_{q_2}\right). \tag{50}$$

Therefore, the desired inequality (41) follows from (47), (48), (49), and (50) and the fact that

$$\int_0^1\int_0^1 z^{\gamma_1} w^{\gamma_1}{}_0d_{q_1} z_0 d_{q_2} w = \frac{1}{[\gamma_1+1]_{q_1}[\gamma_1+1]_{q_2}}, \tag{51}$$

where $[\gamma_1+1]_{q1}$ and $[\gamma_1+1]_{q2}$ are the q_1- and q_2-analogues of γ_1+1 and γ_2+1, respectively.

Theorem 16. *Let $q_1, q_2 \in (0,1)$, $\gamma > 1$ and $\Phi : \mathscr{N} \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ be mixed partial q_1q_2-differentiable over $\mathscr{N}^\circ$ such that its partial q_1q_2-derivative is continuous and integrable on $[\xi_1, \xi_1 + \eta_1(\xi_2, \xi_1)] \times [\xi_3, \xi_3 + \eta_2(\xi_4, \xi_3)] \subset \mathscr{N}^\circ$ for $\eta_1(\xi_2,\xi_1), \eta_2(\xi_4,\xi_3) > 0$. If $|({}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}/{}_{\xi_1}\partial_{q_1} z_{\xi_3}\partial_{q_2} w)\Phi|$ is an n-polynomial preinvex function on the coordinates on $[\xi_1, \xi_1 + \eta_1(\xi_2, \xi_1)] \times [\xi_3, \xi_3 + \eta_2(\xi_4, \xi_3)]$ and $|({}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}/{}_{\xi_1}\partial_{q_1} z_{\xi_3}\partial_{q_2} w)\Phi(\varrho,\rho)| \le M$ for $\varrho, \rho \in \mathscr{N}$, then we have the inequality*

$$\left|\Phi(\varrho,\rho) + \frac{1}{\eta_2(\xi_4,\xi_3)}\left[\int_{\xi_3}^{\xi_3+\eta_2(\rho,\xi_3)} \Phi(\varrho,v)_0 d_{q_2} v + \int_{\xi_4}^{\xi_3+\eta_2(\rho,\xi_3)} \Phi(\varrho,v)_0 d_{q_2} v\right] + \frac{1}{\eta_1(\xi_2,\xi_1)}\left[\int_{\xi_1}^{\xi_1+\eta_1(\varrho,\xi_1)} \Phi(u,\rho)_0 d_{q_1} u + \int_{\xi_2}^{\xi_2+\eta_1(\varrho,\xi_2)} \Phi(u,\rho)_0 d_{q_1} u\right] - \mathscr{Q}\right| \le \frac{q_1q_2M\sqrt[\gamma]{\left(\mathscr{A}_{q_1} + \mathscr{B}_{q_1}\right)\left(\mathscr{A}_{q_2} + \mathscr{B}_{q_2}\right)}}{[(1+q_1)(1+q_2)]^{1-1/\gamma}} \times \left[\frac{[\eta_1(\varrho,\xi_1)]^2 + [\eta_1(\xi_2,\varrho)]^2}{\eta_1(\xi_2,\xi_1)}\right]\left[\frac{[\eta_2(\rho,\xi_3)]^2 + [\eta_2(\xi_4,\rho)]^2}{\eta_2(\xi_4,\xi_3)}\right], \tag{52}$$

where $\mathscr{Q}$ is defined in Lemma 13.

for all $\varrho, \rho \in N$.

Proof. It follows from (19), the power mean inequality and the property of n-polynomial preinvexity of the function $|{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Phi/{}_{\xi_1}\partial_{q_1} z_{\xi_3}\partial_{q_2} w|$ on coordinates that

$$\begin{aligned}
&\left|\Phi(\varrho,\rho) + \frac{1}{\eta_2(\xi_4,\xi_3)}\left[\int_{\xi_3}^{\xi_3+\eta_2(\rho,\xi_3)} \Phi(\varrho,\rho)_0 d_{q_2} v + \int_{\xi_4}^{\xi_4+\eta_2(\rho,\xi_4)} \Phi(\varrho,v)_0 d_{q_2} v\right] + \frac{1}{\eta_1(\xi_2,\xi_1)}\left[\int_{\xi_1}^{\xi_1+\eta_1(\varrho,\xi_1)} \Phi(u,\rho)_0 d_{q_1} u + \int_{\xi_2}^{\xi_2+\eta_1(\varrho,\xi_2)} \Phi(u,\rho)_0 d_{q_1} u\right] - \mathscr{Q}\right| \\
&\le \left(\int_0^1\int_0^1 zw_0 d_{q_1} z_0 d_{q_2} w\right)^{1-(1/\gamma)} \times \left[\frac{q_1q_2[\eta_1(\varrho,\xi_1)]^2[\eta_2(\rho,\xi_3)]^2}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)} \cdot \left(\int_0^1\int_0^1 zw\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1} z_{\xi_3}\partial_{q_2} w}\Phi(\xi_1 + z\eta_1(\varrho,\xi_1), \xi_3 + w\eta_2(\rho,\xi_3))\right|^{\gamma} {}_0d_{q_1} z_0 d_{q_2} w\right)^{1/\gamma}\right] \\
&+ \left[\frac{q_1q_2[\eta_1(\varrho,\xi_1)]^2[\eta_2(\xi_4,\rho)]^2}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)} \left(\int_0^1\int_0^1 zw\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1} z_{\xi_3}\partial_{q_2} w}\Phi(\xi_1 + z\eta_1(\varrho,\xi_1), \xi_4 + w\eta_2(\rho,\xi_4))\right|^{\gamma} {}_0d_{q_1} z_0 d_{q_2} w\right)^{1/\gamma}\right] \\
&+ \left[\frac{q_1q_2[\eta_1(\xi_2,\varrho)]^2[\eta_2(\rho,\xi_3)]^2}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)} \left(\int_0^1\int_0^1 zw\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1} z_{\xi_3}\partial_{q_2} w}\Phi(\xi_2 + z\eta_1(\varrho,\xi_2), \xi_3 + w\eta_2(\rho,\xi_3))\right|^{\gamma} {}_0d_{q_1} z_0 d_{q_2} w\right)^{1/\gamma}\right] \\
&+ \left[\frac{q_1q_2[\eta_1(\xi_2,\varrho)]^2[\eta_2(\xi_4,\rho)]^2}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)} \left(\int_0^1\int_0^1 zw\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1} z_{\xi_3}\partial_{q_2} w}\Phi(\xi_2 + z\eta_1(\varrho,\xi_2), \xi_4 + w\eta_2(\rho,\xi_4))\right|^{\gamma} {}_0d_{q_1} z_0 d_{q_2} w\right)^{1/\gamma}\right],
\end{aligned} \tag{53}$$

By similar argument as in Theorem 14, we can prove that

$$\int_0^1\int_0^1 zw\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z\,{}_{\xi_3}\partial_{q_2}w}\Phi(\xi_1+z\eta_1(\varrho,\xi_1),\xi_3+w\eta_2(\rho,\xi_3))\right|^{\gamma} {}_0d_{q_1}z\,{}_0d_{q_2}w \leq M^{\gamma}\left(\mathscr{A}_{q_1}+\mathscr{B}_{q_1}\right)\left(\mathscr{A}_{q_2}+\mathscr{B}_{q_2}\right), \tag{54}$$

$$\int_0^1\int_0^1 zw\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z\,{}_{\xi_3}\partial_{q_2}w}\Phi(\xi_1+z\eta_1(\varrho,\xi_1),\xi_4+w\eta_2(\rho,\xi_4))\right|^{\gamma} {}_0d_{q_1}z\,{}_0d_{q_2}w \leq M^{\gamma}\left(\mathscr{A}_{q_1}+\mathscr{B}_{q_1}\right)\left(\mathscr{A}_{q_2}+\mathscr{B}_{q_2}\right), \tag{55}$$

$$\int_0^1\int_0^1 zw\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z\,{}_{\xi_3}\partial_{q_2}w}\Phi(\xi_2+z\eta_1(\varrho,\xi_2),\xi_3+w\eta_2(\rho,\xi_3))\right|^{\gamma} {}_0d_{q_1}z\,{}_0d_{q_2}w \leq M^{\gamma}\left(\mathscr{A}_{q_1}+\mathscr{B}_{q_1}\right)\left(\mathscr{A}_{q_2}+\mathscr{B}_{q_2}\right), \tag{56}$$

$$\int_0^1\int_0^1 zw\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z\,{}_{\xi_3}\partial_{q_2}w}\Phi(\xi_2+z\eta_1(\varrho,\xi_2),\xi_4+w\eta_2(\rho,\xi_4))\right|^{\gamma} {}_0d_{q_1}z\,{}_0d_{q_2}w \leq M^{\gamma}\left(\mathscr{A}_{q_1}+\mathscr{B}_{q_1}\right)\left(\mathscr{A}_{q_2}+\mathscr{B}_{q_2}\right). \tag{57}$$

Now by making use of the inequalities (54), (55), (56), and (57) and the fact that

$$\int_0^1\int_0^1 zw\,{}_0d_{q_1}z\,{}_0d_{q_2}w=\frac{1}{(1+q_1)(1+q_2)}, \tag{58}$$

we get the desired inequality (52).

3. Conclusion

In the article, we have introduced a new class of preinvex functions which is named n-polynomial preinvex functions and discovered a new quantum integral identity involving second-order mixed partial differentiable function. By using the obtained quantum integral identity as an auxiliary result, we have established several q_1q_2-Ostrowski-type inequalities for the class of n-polynomial preinvex functions on coordinates, which have improved and unified many previously known results. Our given ideas and approaches may lead to a lot of follow-up research for the interested readers.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

H. Kalsoom introduced the Definition 11, carried out the proof of Proposition 12 and Lemma 13, and drafted the manuscript. M. Idrees carried out the proof of Theorem 14. D. Baleanu carried out the proof of Theorem 15. Y.-M. Chu provided the main idea of the manuscript, carried out the proof of Theorem 16, completed the final revision and submitted the article. All authors read and approved the final manuscript.

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