Research Article

The Existence of Solution of Diffusion Equation with the General Conformable Derivative

Shanshan Li [ID],[1] Shuqin Zhang,[2] and Ruiying Liu[3]

[1]*School of Mathematics and Information Sciences, Yantai University, 264005 Yantai, China*
[2]*School of Science, China University of Mining and Technology, 100083 Beijing, China*
[3]*Lang Yue Hui Foreign Language Middle School, 450001 Zhengzhou, China*

Correspondence should be addressed to Shanshan Li; shanhuyuli@163.com

In this paper, the general conformable fractional derivative is used in the classical diffusion equations, and the corresponding maximum principle is obtained. By the maximum principle, this paper proves the uniqueness of the solution and the continuous dependence on source function and initial-boundary conditions of the solution. Furthermore, by employing the variable separation method, this paper obtains some existence results and the asymptotic behavior of the classical solution.

1. Introduction

It has been several decades since the fractional calculus has attracted extensive attention in both pure and applied fields [1–8]. Fractional calculus has many definitions, such as the Riemann-Liouville type, Caputo type, Hadamard type, Erdélyi-Kober type, and Riesz type. Most of the above fractional derivatives are defined by their corresponding fractional integrals. Compared with some integral-order partial differential equations such as [9–19], fractional derivatives have hereditary and nonlocal property so that they are much more suitable for describing long-memory processes than the classical integer-order derivatives.

However, the traditional fractional calculus have complex expressions which causes many difficulties in applying them in engineering calculation, physical application, numerical modeling, etc. In addition, the traditional fractional calculus loses some basic but important properties, such as product rule and chain rule. Khalil et al. [20] proposed a new local fractional derivative, called conformable derivative, and proved the product rule and the fractional mean value theorem. Abdeljawad supplemented the Taylor power series representation, the fractional chain rule, the Gronwall inequality, and the fractional Laplace transform

in [21]. Zhao et al. [22, 23] further extended the definition to the general conformable fractional derivative (GCFD). Zhang et al. defined the conformable variable order derivative in [24]. Owing to its well-behaved properties and the close connection with the classical derivatives, conformable derivative generated great research interest [25–29].

In [30, 31], the author considered a generalized time-fractional diffusion equation by replacing the first-order time derivative to the Caputo fractional derivative. Recently, some literatures about the fractional diffusion equations with various definitions of fractional calculus occur. Al-Refai et al. [32, 33] considered a nonlinear fractional diffusion equation with the Riemann-Liouville time-fractional derivative and obtained their corresponding maximum principle. Borikhanov et al. [34] considered a nonlinear time-fractional diffusion equation with the Atangana-Baleanu derivative. Chen et al. [35] used the finite element method to approximate the solutions to a time-fractional advection-diffusion equations with the Caputo variable order derivative. Zhang et al. [36] obtained the sharp blow-up and global existence of solutions, a time-fractional diffusion system with the Riemann-Liouville derivative. In this paper, we consider the same equation as in [30] with the GCFD on domain $D \times (0, T)$, $D \subset R^n$, i.e.,

$$\left(D_t^{\alpha,\psi}u\right)(t) = -A(\mathrm{u}) + F(x,t), 0 < \alpha \leq 1, \qquad (1)$$

with the nonhomogeneous initial-boundary conditions

$$\begin{cases} u|_{t=0} = u_0(x), x \in \bar{D}, \\ u|_{\partial D} = v(x,t), (x,t) \in \partial D \times [0, T], \end{cases} \qquad (2)$$

where $F(x,t)$, $u_0(x)$, and $v(x,t)$ are continuous functions, $D^{\alpha,\psi}$ is a GCFD of order α, ∂D is the boundary of D, and $-A$ is a linear elliptic operator

$$-A(u) := p(x)\Delta u + (\nabla p, \nabla u) - q(x)u, x \in \bar{D}, \qquad (3)$$

where Δ is the Laplace operator, ∇ is the gradient operator, $p \in C^1(\bar{D})$, $q \in C(\bar{D})$, $p(x) > 0$, $q(x) \geq 0$, and the domain of definition of operator A is

$$M_A = \{f, f \in C^2(D) \cap C^1(\bar{D}),$$

$$f \text{ satisfies the boundary condition and } A(f) \in L^2(D)\}. \qquad (4)$$

The GCFD is defined in the following sense (see [22]):

$$(D^{\alpha,\psi}f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon\psi(t,\alpha)) - f(t)}{\varepsilon}, t > 0, \qquad (5)$$

and the GCFD at 0 is defined as $(D^{\alpha,\psi}f)(0) = \lim_{t \to 0+} (D^{\alpha,\psi}f)(t)$, where $\psi(t,\alpha)$ is a continuous real function depending on t and the fractional order α and satisfying the below conditions

$$\begin{aligned} \psi(t,\alpha) &\neq 0, \\ \psi(t,1) &= 1, \end{aligned} \qquad (6)$$

and the relationship between the function $\psi(t,\alpha)$ and the order α should be one-to-one. By the definition (5), we know that the GCFD is an extension of the classical derivative ($\alpha = 1$ or $\psi(t,\alpha) = 1$) and the conformable derivative ($\psi(t,\alpha) = t^{1-\alpha}$) defined in [20, 21]. Compared with the definition of conformable derivative in [20, 21] and the Definition 2.5 in [24], the GCFD in relation (5) becomes the conformable variable order derivative defined in [24] if $\psi(t,\alpha) = t^{1-\alpha}$ and the order α is a time-dependent function $\alpha(t)$, that is

$$\left(D^{\alpha(t)}f\right)(t) = \lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon t^{1-\alpha(t)}\right) - f(t)}{\varepsilon}, t > 0. \qquad (7)$$

If the limit (5) exists, it is called that f is α-differentiable. Furthermore, if f is differentiable, then by direct calculation of definition, we can obtain that f is α-differentiable and

$$(D^{\alpha,\psi}f)(t) = f'(t)\psi(t,\alpha). \qquad (8)$$

In this paper, we introduce the fractional Taylor power series expansion (see Lemma 5) and prove the theorem of term-by-term integration and differentiation (see Lemma 7

and 8) with the general conformable fractional calculus. Then by using the above and some other properties of the general conformable factional calculus (see Section 2), we obtain the maximum principle (see Theorems 9 and 10) for the classical diffusion Equation (1) with the GCFD and get some existence (see Theorems 17 and 18) and uniqueness (see Theorem 12) results of the classical solution of (1). Finally, we get the asymptotic behavior of the classical solution (see Theorem 19). The problems (1) and (2) have a solution implies that u is α-differentiable and $(D_t^{\alpha,\psi}u)(t)$ is continuous on $(0, T)$. We define that $u(x,t)$ is called a classical solution of problems (1) and (2), if $u(x,t) \in C(\bar{D} \times [0, T]) \cap C_x^2(D) \cap C_t^1(0, T)$ and satisfies Equation (1) and the initial-boundary condition (2).

2. Preliminaries

In this section, we will recall some properties of the GCFD and its fractional integral calculus.

Definition 1 (see [22]). The integral of a function $f : (0, t) \longrightarrow R$ of order α is defined by

$$(I^{\alpha,\psi}f)(t) = \int_0^t \frac{f(\tau)}{\psi(\tau,\alpha)} d\tau, t > 0, \qquad (9)$$

where the integral is the Riemann integral.

Remark 2. To further study the properties of general conformable fractional calculus, we assume that $\psi(t,\alpha) > 0$ and $1/\psi(\cdot,\cdot): (0, T] \times (0, 1] \longrightarrow (0, +\infty)$ is Riemann integrable and this assumption is valid in the following sections.

Lemma 3 (see [20–22]). *If f, g is α-differentiable at t, then*

(1) $D^{\alpha,\psi}(c) = 0$, where c is any real constant,

(2) $(Linearity)D^{\alpha,\psi}(af(t) + bg(t)) = aD^{\alpha,\psi}f(t) + bD^{\alpha,\psi}g(t), a, b \in R$,

(3) $(Product\ rule)D^{\alpha,\psi}(f(t)g(t)) = f(t)D^{\alpha,\psi}g(t) + g(t)D^{\alpha,\psi}f(t)$,
$D^{\alpha,\psi}(f(t)/g(t)) = g(t)D^{\alpha,\psi}f(t) - f(t)D^{\alpha,\psi}g(t)/g^2(t)$,

(4) $(Chain\ rule)$ *If f is α – differentiable at g and $g \neq 0$, then*

$$D^{\alpha,\psi}(f(g(t))) = (D^{\alpha,\psi}f)(g(t)) \cdot D^{\alpha,\psi}(g(t)) \cdot \frac{1}{\psi(g(t),\alpha)} \qquad (10)$$

If f is differentiable at g, then $D^{\alpha,\psi}(f(g(t))) = f'(g(t))D^{\alpha,\psi}(g(t))$.

Remark 4. Especially, by the definition (5), the relation (8) and Lemma 3 (4), we have

$$D^{\alpha,\psi}\left(\int_0^t \frac{1}{\psi(\tau,\alpha)}d\tau\right) = 1, D^{\alpha,\psi}\left(e^{\int_0^t \frac{1}{\psi(\tau,\alpha)}d\tau}\right) = e^{\int_0^t \frac{1}{\psi(\tau,\alpha)}d\tau},$$

$$(11)$$

which coincides with the classical derivatives $(\psi(t,\alpha) = 1)$ and the conformable fractional derivatives $(\psi(t,\alpha) = t^{1-\alpha})$ defined in [20, 21].

Lemma 5 (Fractional Taylor power series expansions). *Assume that f is an infinite α-differentiable function at a neighbourhood of t_0. Then f has the fractional power series expansion at point t_0:*

$$f(t) = \sum_{k=0}^{+\infty} \frac{(D^{\alpha,\psi}f)^{(k)}(t_0)}{k!}\left(\int_{t_0}^t \frac{1}{\psi(\tau,\alpha)}d\tau\right)^k, t_0 < t < t_0 + R, R > 0,$$

$$(12)$$

where $(D^{\alpha,\psi}f)^{(k)}(t_0)$ means the application of the GCFD k times.

Proof. The proof is similar to that of Theorem 17 in [21]. And this result coincides with the classical derivatives and the conformable derivatives. Especially, the exponential function $e^{\int_{t_0}^t (1/\psi(\tau,\alpha))d\tau}$ has the factional Taylor power series expansion at point t_0:

$$e^{\int_{t_0}^t \frac{1}{\psi(\tau,\alpha)}d\tau} = \sum_{k=0}^{+\infty} \frac{1}{k!}\left(\int_{t_0}^t \frac{1}{\psi(\tau,\alpha)}d\tau\right)^k. \quad (13)$$

Lemma 6 (see [22]). *Let f be a continuous differentiable function on $(0,T)$, then*

$$I^{\alpha,\psi}(D^{\alpha,\psi}f(t)) = f(t) - f(0), D^{\alpha,\psi}(I^{\alpha,\psi}f(t)) = f(t). \quad (14)$$

Lemma 7. *Let a sequence of functions $f_i(t)$, $i = 1, 2, \cdots$ satisfying the following conditions:*

(1) for any given $\alpha(>0)$, there exists fractional integrals $(I^{\alpha,\psi}f_i)(t)$, $i = 1, 2, \cdots$

(2) both the series $\sum_{i=1}^{+\infty} f_i(t)$ and $\sum_{i=1}^{+\infty} (I^{\alpha,\psi}f_i)(t)$ are uniformly convergent for any $t \in [0, T]$ then the function $\sum_{i=1}^{+\infty} f_i(t)$ is α-integrable on $(0, T)$ and

$$I^{\alpha,\psi}\left(\sum_{i=1}^{+\infty} f_i(t)\right) = \sum_{i=1}^{+\infty} (I^{\alpha,\psi}f_i)(t). \quad (15)$$

Proof. Due to the uniformly convergence of $\sum_{i=1}^{+\infty} f_i(t)$ and $\sum_{i=1}^{+\infty} (I^{\alpha,\psi}f_i)(t)$, then for any $\varepsilon > 0$, there exists a positive integer N such that for any $n > N$ and $t \in (0, T)$,

$$\left|\sum_{i=1}^{+\infty} f_i(t) - \sum_{i=1}^{n} f_i(t)\right| < \frac{\varepsilon}{2\int_0^T (1/\psi(\tau,\alpha))d\tau}, \left|\sum_{i=n}^{+\infty} (I^{\alpha,\psi}f_i)(t)\right| < \frac{\varepsilon}{2}.$$

$$(16)$$

Therefore, by (16) and the linearity of the operator $I^{\alpha,\psi}$, we obtain

$$\left|I^{\alpha,\psi}\left(\sum_{i=1}^{+\infty} f_i(t)\right) - \sum_{i=1}^{+\infty} (I^{\alpha,\psi}f_i)(t)\right|$$

$$\leq \left|I^{\alpha,\psi}\left(\sum_{i=1}^{+\infty} f_i(t) - \sum_{i=1}^{n} f_i(t)\right)\right| + \left|\sum_{i=n+1}^{+\infty} (I^{\alpha,\psi}f_i)(t)\right|$$

$$\leq \left|\int_0^t \frac{\sum_{i=1}^{+\infty} f_i(\tau) - \sum_{i=1}^{n} f_i(\tau)}{\psi(\tau,\alpha)}d\tau\right| + \left|\sum_{i=n+1}^{+\infty} (I^{\alpha,\psi}f_i)(t)\right| < \varepsilon.$$

$$(17)$$

i.e., the function $\sum_{i=1}^{+\infty} f_i(t)$ is α-integrable and $I^{\alpha,\psi}(\sum_{i=1}^{+\infty} f_i(t)) = \sum_{i=1}^{+\infty} (I^{\alpha,\psi}f_i)(t)$, then the proof is completed.

Lemma 8. *Let a continuous differentiable sequence of functions $f_i(t)$, $i = 1, 2, \cdots$ satisfying the following conditions:*

(1) for any given $\alpha(>0)$, there exists fractional integrals $(D^{\alpha,\psi}f_i)(t)$, $i = 1, 2, \cdots$

(2) both the series $\sum_{i=1}^{+\infty} f_i(t)$ and $\sum_{i=1}^{+\infty} (D^{\alpha,\psi}f_i)(t)$ are uniformly convergent for any $t \in [0, T]$. Moreover, $\sum_{i=1}^{+\infty} (D^{\alpha,\psi}f_i)(t) \in L(0, T)$, then the function $\sum_{i=1}^{+\infty} f_i(t)$ is α-differentiable on $(0, T)$ and

$$D^{\alpha,\psi}\left(\sum_{i=1}^{+\infty} f_i(t)\right) = \sum_{i=1}^{+\infty} (D^{\alpha,\psi}f_i)(t). \quad (18)$$

Proof. Since $\sum_{i=1}^{+\infty} f_i(t)$ and $\sum_{i=1}^{+\infty} (D^{\alpha,\psi}f_i)(t)$ are uniformly convergent for any $t \in [0, T]$, then by Lemma 6 and Lemma 7.

$$I^{\alpha,\psi}\left(\sum_{i=1}^{+\infty} (D^{\alpha,\psi}f_i)(t)\right) = \sum_{i=1}^{+\infty} (I^{\alpha,\psi}((D^{\alpha,\psi}f_i)(t)))$$

$$= \sum_{i=1}^{+\infty} f_i(t) - \sum_{i=1}^{+\infty} f_i(0).$$

$$(19)$$

The left side of the above equality is α-differentiable; then by Lemma 3(1), $\sum_{i=1}^{+\infty} f_i(t)$ is α-differentiable and

$$D^{\alpha,\psi}\left(\sum_{i=1}^{+\infty} f_i(t)\right) = D^{\alpha,\psi}\left(\sum_{i=1}^{+\infty} f_i(t) - \sum_{i=1}^{+\infty} f_i(0)\right)$$

$$= \sum_{i=1}^{+\infty} (D^{\alpha,\psi}f_i)(t).$$

$$(20)$$

3. The Uniqueness of Solution

Theorem 9. *Let u be a classical solution of Equation (1) in the domain $D \times (0, T)$ and $F(x, t) \leq 0$, then*

$$u(x, t) \leq \max\left\{0, \max_{(x,t)\in S} u(x, t)\right\}, (x, t) \in \bar{D} \times [0, T], \quad (21)$$

where $S := (\bar{D} \times \{0\}) \cup (\partial D \times [0, T])$.

Proof. The proof follows by Lemma 3 (4) and setting the same auxiliary function $w(x, t) := u(x, t) + (\varepsilon/2) \cdot (T - t/T)$ as [30]. Due to u is a classical solution of the problems (1) and (2), then u is differentiable with respect to t. Therefore, we have $(D_t^{\alpha,\psi}u)(t_0) = \psi(t_0, \alpha)u_t(x_0, t_0) = 0$, where (x_0, t_0) is the maximum point of u over the domain $\bar{D} \times [0, T]$.

Theorem 10. *Let u be a classical solution of Equation (1) in the domain $D \times (0, T)$ and $F(x, t) \geq 0$, $(x, t) \in D \times (0, T)$, then*

$$u(x, t) \geq \min_{(x,t)\in S} u(x, t), (x, t) \in \bar{D} \times [0, T]. \quad (22)$$

Theorem 11. *Let u be a classical solution of problems (1) and (2) and $F \in C(\bar{D} \times [0, T])$ with the norm $\|F\|_{C(\bar{D}\times[0,T])} = \sup_{\bar{D}\times[0,T]} |F(x, t)| := M_1$.*

Denote by $M_2 = \|u_0\|_{C(\bar{D})} = \sup_{\bar{D}} |u_0(x)|$, $M_3 = \|v\|_{C(\partial D\times[0,T])} = \sup_{\partial D\times[0,T]} |v(x, t)|$, then the following estimate of the solution norm holds true:

$$\|u\|_{C(\bar{D}\times[0,T])} \leq 2CM_1 + \max\{M_2, M_3\}, \quad (23)$$

where $C := \int_0^T (1/\psi(\tau, \alpha))d\tau$.

Proof. Define the auxiliary function g:

$$g(x, t) = u(x, t) - \int_0^t \frac{M_1}{\psi(\tau, \alpha)} d\tau, (x, t) \in \bar{D} \times [0, T]. \quad (24)$$

By the continuity of ψ and Remark 4, then g is a classical solution of the following problem:

$$\begin{cases} (D_t^{\alpha,\psi}g)(t) = -A(g) + F_1(x, t), (x, t) \in D \times (0, T), \\ g|_{t=0} = u_0, x \in \bar{D}, \\ g|_{\partial D} = v(x, t) - \int_0^t \frac{M_1}{\psi(\tau, \alpha)} d\tau := v_1(x, t), (x, t) \in \partial D \times [0, T], \end{cases} \quad (25)$$

where $F_1(x, t) = F(x, t) - M_1 - q(x)\int_0^t (M_1/\psi(\tau, \alpha))d\tau \leq 0$. According to the conditions, we have

$$\|v_1\|_{C(\partial D\times[0,T])} \leq M_3 + CM_1. \quad (26)$$

Then by Theorem 9, the classical solution g satisfies the estimate,

$$g(x, t) \leq \max\{M_2, M_3 + CM_1\}, (x, t) \in \bar{D} \times [0, T], \quad (27)$$

which means that

$$u(x, t) \leq 2CM_1 + \max\{M_2, M_3\}. \quad (28)$$

Similarly, consider of the auxiliary function g_1 and using the Theorem 10

$$g_1(x, t) = u(x, t) + \int_0^t \frac{M_1}{\psi(\tau, \alpha)} d\tau, (x, t) \in \bar{D} \times [0, T], \quad (29)$$

then we have

$$g_1(x, t) \geq \max\{-M_2, -M_3 - CM_1\}, (x, t) \in \bar{D} \times [0, T], \quad (30)$$

i.e., $u(x, t) \geq -2CM_1 - \max\{M_2, M_3\}$. Combined with (28), we obtain

$$\|u\|_{C(\bar{D}\times[0,+\infty))} \leq 2CM_1 + \max\{M_2, M_3\}. \quad (31)$$

Theorem 12. *The problems (1) and (2) have at most one classical solution and the solution continuously depends on the data given in the problem in the sense that if*

$$\|F_1 - F_2\|_{C(\bar{D}\times[0,T])} \leq \varepsilon_1, \left\|u_0^{(1)} - u_0^{(2)}\right\|_{C(\bar{D})} \leq \varepsilon_2, \|v_1 - v_2\|_{C(\partial D\times[0,T])} \leq \varepsilon_3, \quad (32)$$

then for the corresponding classical solution u_1 and u_2, the estimate holds:

$$\|u_1 - u_2\|_{C(\bar{D}\times[0,T])} \leq 2C\varepsilon_1 + \max\{\varepsilon_2, \varepsilon_3\}. \quad (33)$$

Proof. Let u_1, u_2 be the classical solution of the following problem, respectively,

$$\begin{cases} (D_t^{\alpha,\psi}u)(t) = -A(u) + F_i(x, t), (x, t) \in D \times (0, T), \\ u|_{t=0} = u_0^{(i)}(t), x \in \bar{D}, \\ u|_{\partial D} = v_i(x, t), (x, t) \in \partial D \times [0, T], \end{cases} \quad (34)$$

where $i = 1, 2$. Then $u = u_1 - u_2$ is the classical solution of the corresponding problem

$$\begin{cases} (D_t^{\alpha,\psi}u)(t) = -A(u) + F_1(x,t) - F_2(x,t), (x,t) \in D \times (0,T), \\ u|_{t=0} = u_0^{(1)}(t) - u_0^{(2)}(t), x \in \bar{D}, \\ u|_{\partial D} = v_1(x,t) - v_2(x,t), (x,t) \in \partial D \times [0,T]. \end{cases}$$

(35)

By (23), the estimate (33) holds. The uniqueness of the classical solution of (1) and (2) is a direct consequence of Theorem 9, Theorem 10 and the estimate (33).

4. The Existence of Solution

In this section, we will consider some existence results of solution of problems (1) and (2) with homogenous boundary conditions

$$\begin{cases} u|_{t=0} = u_0(x), x \in \bar{D}, \\ u|_{\partial D} = 0, (x,t) \in \partial D \times [0,T]. \end{cases}$$

(36)

Firstly, we consider the homogenous equation, i.e., $F(x,t) \equiv 0$. In the following, we seek the solution of Equation (1) by using the variable separation method in the form

$$u(x,t) = X(x)T(t), (x,t) \in D \times (0,T),$$

(37)

which satisfies the boundary condition. Substituting expression (37) in Equation (1) and separating the variables, we obtain the following sense:

$$\frac{(D^{\alpha,\psi}T)(t)}{T(t)} = -\frac{A(X(x))}{X(x)} = -\lambda,$$

(38)

where λ is a constant independent on t and x. Therefore, addition to the boundary condition, we have

$$(D^{\alpha,\psi}T)(t) + \lambda T(t) = 0,$$

(39)

and the eigenvalues problem for the operator A

$$\begin{cases} A(X(x)) = \lambda X(x), (x,t) \in D \times (0,T), \\ X|_{\partial D} = 0, (x,t) \in \partial D \times [0,T]. \end{cases}$$

(40)

The properties of A is well known in [37] that A is a positive and self-adjoint linear operator. Moreover, (40) has a counted number of positive eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots$, $\lambda_k \longrightarrow +\infty (k \longrightarrow +\infty)$ with finite multiplicity, and any function $f \in M_A$ can be represented through its Fourier series:

$$f(x) = \sum_{i=1}^{+\infty} (f, X_i) X_i(x),$$

(41)

where $X_i \in M_A$ is the corresponding eigenfunction of the eigenvalue λ_i. In fact, in the following, we choose $\{X_i\}_{i=1}^{+\infty}$ to be real and orthonormal.

For Equation (39) and its inhomogenous equation

$$(D^{\alpha,\psi}T)(t) + \lambda T(t) = f(t),$$

(42)

we consider the following first-order linear differential equation for $\lambda = \lambda_i$:

$$T_i'(t) + \frac{\lambda_i}{\psi(t,\alpha)} T_i(t) = \frac{f(t)}{\psi(t,\alpha)}.$$

(43)

It is well known that the solution of Equation (43) is given by

$$T_i(t) = e^{-\int_0^t \frac{\lambda_i}{\psi(\tau,\alpha)}d\tau} \left[c_i + \int_0^t e^{\int_0^\tau \frac{\lambda_i}{\psi(s,\alpha)}ds} \frac{f(\tau)}{\psi(\tau,\alpha)} d\tau \right],$$

(44)

where c_i are some constants. Obviously, $T_i(t) \in C[0,T] \cap C^1 (0,T)$. Moreover, $T_i(t)$ in (44) satisfies the following equation:

$$\psi(\tau,\alpha)T_i'(t) + \lambda_i T_i(t) = f(t).$$

(45)

Hence, according to the relation (8) we have

$$(D^{\alpha,\psi}T_i)(t) + \lambda_i T_i(t) = \psi(\tau,\alpha)T_i'(t) + \lambda_i T_i(t) = f(t), \quad (46)$$

which means the function in (44) is the solution of Equation (42). Correspondingly, the solution of Equation (39) for $\lambda = \lambda_i$ is

$$T_i(t) = c_i e^{-\int_0^t \frac{\lambda_i}{\psi(\tau,\alpha)}d\tau}.$$

(47)

Naturally, each function

$$T_i(t)X_i(x) = c_i e^{-\int_0^t \frac{\lambda_i}{\psi(\tau,\alpha)}d\tau} X_i(x),$$

(48)

and its finite sum

$$u_k(x,t) = \sum_{i=1}^k T_i(t)X_i(x) = \sum_{i=1}^k c_i e^{-\int_0^t \frac{\lambda_i}{\psi(\tau,\alpha)}d\tau} X_i(x), \quad (49)$$

satisfy Equation (1) and the boundary condition.

In addition to the initial condition, we define the formal solution of problems (1) and (36) as the following sense:

$$u(x,t) = \sum_{i=1}^{+\infty} T_i(t)X_i(x) = \sum_{i=1}^{+\infty} (u_0, X_i) e^{-\int_0^t \frac{\lambda_i}{\psi(\tau,\alpha)}d\tau} X_i(x). \quad (50)$$

Next, we consider the case of inhomogenous equation. Similarly to the case of homogenous condition, the formal solution of inhomogenous equation also has the expression (50) with appropriate $T_i(t)$. For any i, we multiply Equation (1) by X_i, integrating over D. Then by Lemma 8 and using the

self-adjoint property of the operator A, combined by the orthogonality of $\{X_i\}_{i=1}^{+\infty}$, we obtain

$$
\begin{aligned}
(D^{\alpha,\psi}T_i)(t) &= \int_D \left(D_t^{\alpha,\psi}u\right)(t) \cdot X_i(x)dx \\
&= \int_D D_t^{\alpha,\psi}\left(\sum_{i=1}^{+\infty} T_i(t)X_i(x)\right) \cdot X_i(x)dx \\
&= \int_D \left(\sum_{i=1}^{+\infty} D_t^{\alpha,\psi}(T_i(t)X_i(x))\right) \cdot X_i(x)dx \\
&= -(Au, X_i) + (F, X_i) = -\lambda_i(u, X_i) + (F, X_i) \\
&= -\lambda_i T_i + (F, X_i).
\end{aligned}
\tag{51}
$$

That is

$$
(D^{\alpha,\psi}T_i)(t) + \lambda_i T_i(t) = (F(x,t), X_i(x)).
\tag{52}
$$

According to the relation (44), the solution of Equation (52) is

$$
T_i(t) = e^{-\int_0^t \frac{\lambda_i}{\psi(\tau,\alpha)}d\tau}\left[c_i + \int_0^t e^{\int_0^\tau \frac{\lambda_i}{\psi(s,\alpha)}ds}\frac{(F(x,\tau), X_i(x))}{\psi(\tau,\alpha)}d\tau\right],
\tag{53}
$$

where c_i are some constants. Consequently, combined with the initial condition, we define the formal solution of problems (1) and (36) as the following sense:

$$
\begin{aligned}
u(x,t) &= \sum_{i=1}^{+\infty} T_i(t)X_i(x) = \sum_{i=1}^{+\infty}(u_0, X_i)e^{-\int_0^t \frac{\lambda_i}{\psi(\tau,\alpha)}d\tau}X_i(x) \\
&+ \sum_{i=1}^{+\infty} e^{-\int_0^t \frac{\lambda_i}{\psi(\tau,\alpha)}d\tau}\int_0^t e^{\int_0^\tau \frac{\lambda_i}{\psi(s,\alpha)}ds}\frac{(F(x,\tau), X_i(x))X_i(x)}{\psi(\tau,\alpha)}d\tau.
\end{aligned}
\tag{54}
$$

Remark 13. For Equation (52), we try applying the operator $I^{\alpha,\psi}$ to its both sides, then by Definition 1 and Lemma 6, we have

$$
T_i(t) = T_i(0) - \lambda_i\int_0^t \frac{T_i(\tau)}{\psi(\tau,\alpha)}d\tau + \int_0^t \frac{(F(x,\tau), X_i(x))}{\psi(\tau,\alpha)}d\tau.
\tag{55}
$$

Next, we apply the method of successive approximations to solve this integral equation by setting

$$
T_i^{(0)}(t) = T_i(0),
\tag{56}
$$

$$
\begin{aligned}
T_i^{(m)}(t) &= T_i^{(0)}(t) - \lambda_i\int_0^t \frac{T_i^{(m-1)}(\tau)}{\psi(\tau,\alpha)}d\tau \\
&+ \int_0^t \frac{(F(x,\tau), X_i(x))}{\psi(\tau,\alpha)}d\tau, m \in N^+.
\end{aligned}
\tag{57}
$$

Taking (56) into account, then

$$
T_i^{(1)}(t) = T_i(0) - \lambda_i T_i(0)\int_0^t \frac{1}{\psi(\tau,\alpha)}d\tau + \int_0^t \frac{(F(x,\tau), X_i(x))}{\psi(\tau,\alpha)}d\tau.
\tag{58}
$$

Similarly, using (56)–(58), we have

$$
\begin{aligned}
T_i^{(2)}(t) &= T_i(0) - \lambda_i T_i(0)\int_0^t \frac{1}{\psi(\tau,\alpha)}d\tau + \lambda_i^2 T_i(0)\int_0^t\int_0^s \frac{1}{\psi(\tau,\alpha)} \\
&\cdot \frac{1}{\psi(s,\alpha)}ds d\tau + \int_0^t \frac{(F(x,\tau), X_i(x))}{\psi(\tau,\alpha)}\left[1 - \lambda_i\int_\tau^t \frac{1}{\psi(s,\alpha)}ds\right]d\tau \\
&= \sum_{k=0}^2 \frac{(-\lambda_i)^k T_i(0)}{k!}\left(\int_0^t \frac{1}{\psi(\tau,\alpha)}d\tau\right)^k \\
&+ \int_0^t \frac{(F(x,\tau), X_i(x))}{\psi(\tau,\alpha)}\left[\sum_{k=0}^1 \frac{(-\lambda_i)^k}{k!}\left(\int_\tau^t \frac{1}{\psi(\tau,\alpha)}ds\right)^k\right]d\tau.
\end{aligned}
\tag{59}
$$

Continuing this process, we obtain the following expression for $T_i^{(m)}(t)$:

$$
\begin{aligned}
T_i^{(m)}(t) &= \sum_{k=0}^m \frac{(-\lambda_i)^k T_i(0)}{k!}\left(\int_0^t \frac{1}{\psi(\tau,\alpha)}d\tau\right)^k \\
&+ \int_0^t \frac{(F(x,\tau), X_i(x))}{\psi(\tau,\alpha)}\left[\sum_{k=0}^{m-1} \frac{(-\lambda_i)^k}{k!}\left(\int_\tau^t \frac{1}{\psi(\tau,\alpha)}ds\right)^k\right]d\tau.
\end{aligned}
\tag{60}
$$

Taking the limit as $m \longrightarrow +\infty$, we obtain

$$
\begin{aligned}
T_i(t) &= \sum_{k=0}^{+\infty} \frac{(-\lambda_i)^k T_i(0)}{k!}\left(\int_0^t \frac{1}{\psi(\tau,\alpha)}d\tau\right)^k \\
&+ \int_0^t \frac{(F(x,\tau), X_i(x))}{\psi(\tau,\alpha)}\left[\sum_{k=0}^{+\infty} \frac{(-\lambda_i)^k}{k!}\left(\int_\tau^t \frac{1}{\psi(\tau,\alpha)}ds\right)^k\right]d\tau.
\end{aligned}
\tag{61}
$$

By Lemma 5, the expression (61) is the fractional Taylor power series expansion of (53).

Definition 14. A formal solution of the problems (1) and ((36)) is called the Fourier series in the form (50) ($F \equiv 0$) and (54) ($F \neq 0$).

Definition 15. Let the sequences of $u_{0k} \in C(\bar{D})$ and $v_k \in C(\partial D \times [0, T])$, $F_k \in C(\bar{D} \times [0, T])$, $k = 1, 2, \cdots$ such that for $k \longrightarrow +\infty$, $u_{0k} \longrightarrow u_0$, $v_k \longrightarrow v$, $F_k \Longrightarrow F$ in $L^2(D)$ and $t \in [0, T]$. And for any $k = 1, 2, \cdots$, there exists a classical solution u_k of the following problem

$$
\begin{cases}
\left(D_t^{\alpha,\psi}u_k\right)(t) = -A(u_k) + F_k(x,t), \\
u_k|_{t=0} = u_{0k}(x), x \in \bar{D}, \\
u_k|_{\partial D} = v_k(x,t), (x,t) \in \partial D \times [0, T].
\end{cases}
\tag{62}
$$

Suppose there exists a function $u \in C(\bar{D} \times [0, T])$ such that $u_k \longrightarrow u$ as $k \longrightarrow +\infty$, then the function u is called a generalized solution of the problems (1) and (2).

Remark 16. By Theorem 12, the convergence of u_{0k}, v_k, F_k *and the completeness of* $C(\bar{D} \times [0, T], \|\cdot\|_{C(\bar{D}\times[0,T])})$, *we know that there always exists a function* $u \in C(\bar{D} \times [0, T])$ *which is the limit of the sequence* u_k. *In fact, under the assumption* u_0, $F \in M_A$, *the formal solution is also the generalized solution of problems* (1) *and* (36). *That is the following theorem.*

Theorem 17. *Suppose that* $u_0, F \in M_A$, *then the formal solution* (54) *is the generalized solution of the problems* (1) *and* (36).

Proof. For every $k = 1, 2, \cdots$,

$$
\begin{aligned}
u_k(x, t) = {} & \sum_{i=1}^{k} (u_0, X_i) e^{-\int_0^t \frac{\lambda_i}{\psi(\tau,\alpha)} d\tau} X_i(x) \\
& + \sum_{i=1}^{k} e^{-\int_0^t \frac{\lambda_i}{\psi(\tau,\alpha)} d\tau} \int_0^t e^{\int_0^\tau \frac{\lambda_i}{\psi(s,\alpha)} ds} \frac{(F(x,\tau), X_i(x)) X_i(x)}{\psi(\tau, \alpha)} d\tau,
\end{aligned}
\tag{63}
$$

is the classical solution of the problems (1) and (36) with the initial condition

$$
u_{0k}(x) = \sum_{i=1}^{k} (u_0, X_i) X_i(x).
\tag{64}
$$

Note that $u_0 \in M_A$, $u_{0k} \longrightarrow u_0$ as $k \longrightarrow +\infty$ in $\bar{D} \times [0, T]$ and $e^{-\int_0^t (\lambda_i/\psi(\tau,\alpha))d\tau} \leq 1$; therefore, $\sum_{i=1}^{k} (u_0, X_i) e^{-\int_0^t (\lambda_i/\psi(\tau,\alpha))d\tau} X_i(x)$ uniformly converges on $\bar{D} \times [0, T]$.

In view of $F \in M_A$, then $F(x, t) = \sum_{i=1}^{+\infty} (F(x, t), X_i) X_i(x)$ uniformly converges on $\bar{D} \times [0, T]$. Therefore,

$$
\begin{aligned}
& \sum_{i=1}^{k} e^{-\int_0^t \frac{\lambda_i}{\psi(\tau,\alpha)} d\tau} \int_0^t e^{\int_0^\tau \frac{\lambda_i}{\psi(s,\alpha)} ds} \left| \frac{(F(x,\tau), X_i(x)) X_i(x)}{\psi(\tau, \alpha)} \right| d\tau \\
& \leq \int_0^t \left| \frac{1}{\psi(\tau, \alpha)} \cdot \left(\sum_{i=1}^{k} (F(x,\tau), X_i) X_i(x) \right) \right| d\tau \\
& \longrightarrow \int_0^t \left| \frac{1}{\psi(\tau, \alpha)} \cdot \left(\sum_{i=1}^{+\infty} (F(x,\tau), X_i) X_i(x) \right) \right| d\tau (ask \longrightarrow +\infty).
\end{aligned}
\tag{65}
$$

Consequently, (54) is the generalized solution of the problems (1) and (36).

Theorem 18. *Suppose that* $u_0, F \in M_A$, $A(u_0), A^2(u_0) \in M_A$ *and for any* $t \in (0, T)$, $A(F), A^2(F) \in M_A$, *then the formal solution* (54) *is the classical solution of the problems* (1) *and* (36).

Proof. By Theorem 17, the formal solution (54) is the generalized solution; it remains to prove that (54) is at least twice differentiable with respect to the spatial variable x and differentiable with respect to the time variable t.

Differentiating term-by-term (54) with respect to x, we construct a series

$$
\begin{aligned}
& \sum_{i=1}^{+\infty} (u_0, X_i) e^{-\int_0^t \frac{\lambda_i}{\psi(\tau,\alpha)} d\tau} \nabla X_i(x) \\
& + \sum_{i=1}^{+\infty} e^{-\int_0^t \frac{\lambda_i}{\psi(\tau,\alpha)} d\tau} \int_0^t e^{\int_0^\tau \frac{\lambda_i}{\psi(s,\alpha)} ds} \frac{(F(x,\tau), X_i(x)) \nabla X_i(x)}{\psi(\tau, \alpha)} d\tau.
\end{aligned}
\tag{66}
$$

Now we shall prove that the above series (66) is uniformly convergent. Since $u_0 \in M_A$, $F \in M_A$, then $A(u_0)$, $A(F) \in L^2(D)$. Moreover, $A(u_0), A(F) \in M_A$, so $A(u_0)$ and $A(F)$ can be expanded into uniformly convergent series

$$
A(u_0) = \sum_{i=1}^{+\infty} (A(u_0), X_i) X_i, A(F) = \sum_{i=1}^{+\infty} (A(F), X_i) X_i,
\tag{67}
$$

for any $t \in (0, T)$. Then by the Parseval equality,

$$
\begin{aligned}
\|A(u_0)\|_2^2 & = \sum_{i=1}^{+\infty} |(A(u_0), X_i)|^2 = \sum_{i=1}^{+\infty} |(u_0, A(X_i))|^2 \\
& = \sum_{i=1}^{+\infty} \lambda_i^2 |(u_0, X_i)|^2 < +\infty,
\end{aligned}
\tag{68}
$$

$$
\begin{aligned}
\|A(F)\|_2^2 & = \sum_{i=1}^{+\infty} |(A(F), X_i)|^2 = \sum_{i=1}^{+\infty} |(F, A(X_i))|^2 \\
& = \sum_{i=1}^{+\infty} \lambda_i^2 |(F, X_i)|^2 < +\infty.
\end{aligned}
\tag{69}
$$

Considering that the series $\sum_{i=1}^{+\infty} (|\nabla X_i|^2/\lambda_i^2)$ uniformly convergent (see [38]), then applying the Cauchy inequality we have

$$
\begin{aligned}
\sum_{i=1}^{+\infty} |(F(x, t), X_i(x)) \nabla X_i(x)| & \leq \sum_{i=1}^{+\infty} |\lambda_i(F(x, t), X_i(x))| \cdot \left| \frac{\nabla X_i(x)}{\lambda_i} \right| \\
& \leq \left(\sum_{i=1}^{+\infty} \lambda_i^2 |(F(x, t), X_i(x))|^2 \right)^{1/2} \\
& \quad \cdot \left(\sum_{i=1}^{+\infty} \left(\frac{|\nabla X_i(x)|^2}{\lambda_i^2} \right) \right)^{1/2} \\
& \leq \|A(F)\|_2 \left(\sum_{i=1}^{+\infty} \left(\frac{|\nabla X_i(x)|^2}{\lambda_i^2} \right) \right)^{1/2},
\end{aligned}
\tag{70}
$$

which implies that the series $\sum_{i=1}^{+\infty} |(F(x, t), X_i(x)) \nabla X_i(x)|$ is uniformly convergent on D for any $t \in (0, T)$. In view that $\{X_i\}_{i=1}^{+\infty}$ is a complete orthonormal system in $L^2(D)$, using

the above facts, the estimates (68)–(70) and applying the Cauchy inequality, we get

$$
\begin{aligned}
&\sum_{i=1}^{+\infty} \left| (u_0, X_i) e^{-\int_0^t \frac{\lambda_i}{\psi(\tau,\alpha)} d\tau} \nabla X_i(x) \right| \\
&\quad + \sum_{i=1}^{+\infty} \left| e^{-\int_0^t \frac{\lambda_i}{\psi(\tau,\alpha)} d\tau} \int_0^t e^{\int_0^\tau \frac{\lambda_i}{\psi(s,\alpha)} ds} \frac{(F(x,\tau), X_i(x)) \nabla X_i(x)}{\psi(\tau,\alpha)} d\tau \right| \\
&\leq \sum_{i=1}^{+\infty} |(u_0, X_i) \nabla X_i(x)| \\
&\quad + \sum_{i=1}^{+\infty} \left| \int_0^t \frac{(F(x,\tau), X_i(x)) \nabla X_i(x)}{\psi(\tau,\alpha)} d\tau \right| \\
&\leq \sum_{i=1}^{+\infty} |\lambda_i(u_0, X_i)| \cdot \left| \frac{\nabla X_i(x)}{\lambda_i} \right| \\
&\quad + \int_0^t \frac{1}{\psi(\tau,\alpha)} \left(\sum_{i=1}^{+\infty} |\lambda_i(F(x,\tau), X_i(x))| \cdot \left| \frac{\nabla X_i(x)}{\lambda_i} \right| \right) d\tau \\
&\leq \left(\sum_{i=1}^{+\infty} \lambda_i^2 (u_0, X_i)^2 \right)^{1/2} \cdot \left(\sum_{i=1}^{+\infty} \frac{|\nabla X_i(x)|^2}{\lambda_i^2} \right)^{1/2} \\
&\quad + \int_0^t \frac{1}{\psi(\tau,\alpha)} \left(\sum_{i=1}^{+\infty} \lambda_i^2 (F, X_i)^2 \right)^{1/2} \\
&\quad \cdot \left(\sum_{i=1}^{+\infty} \frac{|\nabla X_i(x)|^2}{\lambda_i^2} \right)^{1/2} d\tau \\
&\leq \left(\|A(u_0)\|_2 + C\|A(F)\|_2 \right) \left(\sum_{i=1}^{+\infty} \frac{|\nabla X_i(x)|^2}{\lambda_i^2} \right)^{1/2},
\end{aligned}
\tag{71}
$$

where $C = \int_0^T (1/\psi(\tau,\alpha)) d\tau$. Consequently, the series (66) uniformly converges to ∇u and the generalized solution (54) belongs to $C_x^1(D)$.

Applying the similarly method, we can show the generalized solution (54) belongs to $C_x^2(D)$ as well and the relation

$$
\begin{aligned}
\Delta u &= \sum_{i=1}^{+\infty} (u_0, X_i) e^{-\int_0^t \frac{\lambda_i}{\psi(\tau,\alpha)} d\tau} \Delta X_i(x) \\
&\quad + \sum_{i=1}^{+\infty} e^{-\int_0^t \frac{\lambda_i}{\psi(\tau,\alpha)} d\tau} \int_0^t e^{\int_0^\tau \frac{\lambda_i}{\psi(s,\alpha)} ds} \frac{(F(x,\tau), X_i(x)) \Delta X_i(x)}{\psi(\tau,\alpha)} d\tau
\end{aligned}
\tag{72}
$$

holds true. In fact, since $A(u_0) \in M_A$, $A(F) \in M_A$, then $A^2(u_0) \in L^2(D)$, $A^2(F) \in L^2(D)$. Moreover, due to $A^2(u_0)$, $A^2(F) \in M_A$, so $A^2(u_0)$ and $A^2(F)$ can be expanded into uniformly convergent series for any $t \in (0, T)$. Note that the series $\sum_{i=1}^{+\infty} (|\Delta X_i|^2/\lambda_i^3)$ is uniformly convergent (see [38]), then by the Cauchy inequality, we have

$$
\begin{aligned}
&\sum_{i=1}^{+\infty} |(F(x,t), X_i(x)) \Delta X_i(x)| \\
&\leq \sum_{i=1}^{+\infty} \left| \lambda_i^{\frac{3}{2}} (F(x,t), X_i(x)) \right| \cdot \left| \frac{\Delta X_i(x)}{\lambda_i^{3/2}} \right| \\
&\leq l_1 \left(\sum_{i=1}^{+\infty} \lambda_i^4 |(F(x,t), X_i(x))|^2 \right)^{1/2} \\
&\quad \cdot \left(\sum_{i=1}^{+\infty} \left(\frac{|\Delta X_i(x)|^2}{\lambda_i^3} \right) \right)^{1/2} \\
&\leq l_1 \|A^2(F)\|_2 \left(\sum_{i=1}^{+\infty} \left(\frac{|\Delta X_i(x)|^2}{\lambda_i^3} \right) \right)^{1/2}.
\end{aligned}
\tag{73}
$$

Due to $\lim_{i\to+\infty} \lambda_i = +\infty$, there exists a constant l such that $\lambda_i^3 < l\lambda_i^4$, $i = 1, 2, \cdots$, where l_1 is a constant depending on l. Then (73) implies that the series $\sum_{i=1}^{+\infty} |(F(x,t), X_i(x)) \Delta X_i(x)|$ is uniformly convergent on D for any $t \in (0, T)$. Using the above facts and applying the Parseval equality and Cauchy inequality, we get

$$
\begin{aligned}
&\sum_{i=1}^{+\infty} \left| (u_0, X_i) e^{-\int_0^t \frac{\lambda_i}{\psi(\tau,\alpha)} d\tau} \Delta X_i(x) \right| \\
&\quad + \sum_{i=1}^{+\infty} \left| e^{-\int_0^t \frac{\lambda_i}{\psi(\tau,\alpha)} d\tau} \int_0^t e^{\int_0^\tau \frac{\lambda_i}{\psi(s,\alpha)} ds} \frac{(F(x,\tau), X_i(x)) \Delta X_i(x)}{\psi(\tau,\alpha)} d\tau \right| \\
&\leq \sum_{i=1}^{+\infty} |(u_0, X_i) \Delta X_i(x)| + \sum_{i=1}^{+\infty} \left| \int_0^t \frac{(F(x,\tau), X_i(x)) \Delta X_i(x)}{\psi(\tau,\alpha)} d\tau \right| \\
&\leq \sum_{i=1}^{+\infty} \left| \lambda_i^{\frac{3}{2}} (u_0, X_i) \right| \cdot \left| \frac{\Delta X_i(x)}{\lambda_i^{3/2}} \right| \\
&\quad + \int_0^t \frac{1}{\psi(\tau,\alpha)} \left(\sum_{i=1}^{+\infty} \left| \lambda_i^{\frac{3}{2}} (F(x,\tau), X_i(x)) \right| \cdot \left| \frac{\Delta X_i(x)}{\lambda_i^{3/2}} \right| \right) d\tau \\
&\leq l_1 \left(\sum_{i=1}^{+\infty} \lambda_i^4 (u_0, X_i)^2 \right)^{1/2} \cdot \left(\sum_{i=1}^{+\infty} \frac{|\Delta X_i(x)|^2}{\lambda_i^3} \right)^{1/2} \\
&\quad + \int_0^t \frac{l_1}{\psi(\tau,\alpha)} \left(\sum_{i=1}^{+\infty} \lambda_i^4 (F, X_i)^2 \right)^{1/2} \\
&\quad \cdot \left(\sum_{i=1}^{+\infty} \frac{|\Delta X_i(x)|^2}{\lambda_i^3} \right)^{1/2} d\tau \\
&\leq \left(l_1 \|A^2(u_0)\|_2 + Cl_1 \|A^2(F)\|_2 \right) \left(\sum_{i=1}^{+\infty} \frac{|\Delta X_i(x)|^2}{\lambda_i^3} \right)^{1/2}.
\end{aligned}
\tag{74}
$$

Consequently, the relation (72) holds true and the generalized solution (54) belongs to $C_x^2(D)$. It remains to prove that (54) belongs to $C_t^1(0, T)$. We take a derivative term by term with respect to t (54) and construct the series

$$-\sum_{i=1}^{+\infty} \frac{\lambda_i (u_0, X_i)}{\psi(t, \alpha)} e^{-\int_0^t \frac{\lambda_i}{\psi(\tau, \alpha)} d\tau} X_i(x)$$

$$-\sum_{i=1}^{+\infty} \frac{\lambda_i e^{-\int_0^t (\lambda_i/\psi(\tau, \alpha)) d\tau}}{\psi(t, \alpha)} \int_0^t e^{\int_0^\tau \frac{\lambda_i}{\psi(s, \alpha)} ds} \frac{(F(x, \tau), X_i(x)) X_i(x)}{\psi(\tau, \alpha)} d\tau$$

$$+\sum_{i=1}^{+\infty} \frac{(F(x, \tau), X_i(x)) X_i(x)}{\psi(t, \alpha)}.$$

$$(75)$$

Now let $t \in [\varepsilon, T]$, $0 < \varepsilon < T$. Note that $1/\psi(t, \alpha)$ is the Riemann integrable on $[\varepsilon, T]$; there exists $M > 0$ such that $|1/\psi(t, \alpha)| \leq M$. Next we shall to prove that the above series (75) is uniformly convergent on any interval $[\varepsilon, T]$, $0 < \varepsilon < T$. By the uniformly convergence of $\sum_{i=1}^{+\infty} (|X_i|^2/\lambda_i)$ (see [38]), applying the Parseval equality and the Cauchy inequality, we get

$$\sum_{i=1}^{+\infty} \left| \frac{\lambda_i (u_0, X_i)}{\psi(t, \alpha)} e^{-\int_0^t \frac{\lambda_i}{\psi(\tau, \alpha)} d\tau} X_i(x) \right| + \sum_{i=1}^{+\infty} \left| \frac{(F(x, \tau), X_i(x)) X_i(x)}{\psi(t, \alpha)} \right|$$

$$+ \sum_{i=1}^{+\infty} \left| \frac{\lambda_i e^{-\int_0^t (\lambda_i/\psi(\tau, \alpha)) d\tau}}{\psi(t, \alpha)} \int_0^t e^{\int_0^\tau \frac{\lambda_i}{\psi(s, \alpha)} ds} \frac{(F(x, \tau), X_i(x)) X_i(x)}{\psi(\tau, \alpha)} d\tau \right|$$

$$\leq \sum_{i=1}^{+\infty} M |\lambda_i (u_0, X_i) X_i(x)| + \sum_{i=1}^{+\infty} M |(F(x, \tau), X_i(x)) X_i(x)|$$

$$+ \sum_{i=1}^{+\infty} \int_0^t \frac{M \lambda_i \, |(F(x, \tau), X_i(x)) X_i(x)|}{\psi(\tau, \alpha)} d\tau$$

$$\leq \sum_{i=1}^{+\infty} M |\lambda_i^{3/2} (u_0, X_i)| \cdot \left| \frac{X_i(x)}{\lambda_i^{1/2}} \right| + \sum_{i=1}^{+\infty} M |\lambda_i^{1/2} (F, X_i)| \cdot \left| \frac{X_i(x)}{\lambda_i^{1/2}} \right|$$

$$+ \int_0^t \frac{M}{\psi(\tau, \alpha)} \left(\sum_{i=1}^{+\infty} |\lambda_i^{3/2} (F(x, \tau), X_i(x))| \cdot \left| \frac{X_i(x)}{\lambda_i^{1/2}} \right| \right) d\tau$$

$$\leq M_1 \left(\sum_{i=1}^{+\infty} \lambda_i^4 (u_0, X_i)^2 \right)^{1/2} \cdot \left(\sum_{i=1}^{+\infty} \frac{|X_i(x)|^2}{\lambda_i} \right)^{1/2}$$

$$+ M_2 \left(\sum_{i=1}^{+\infty} \lambda_i^2 (F, X_i)^2 \right)^{1/2} \cdot \left(\sum_{i=1}^{+\infty} \frac{|X_i(x)|^2}{\lambda_i} \right)^{1/2}$$

$$+ \int_0^t \frac{M_1}{\psi(\tau, \alpha)} \left(\sum_{i=1}^{+\infty} \lambda_i^4 (F(x, \tau), X_i)^2 \right)^{1/2}$$

$$\cdot \left(\sum_{i=1}^{+\infty} \frac{|X_i(x)|^2}{\lambda_i} \right)^{1/2} d\tau$$

$$\leq \left(M_1 \|A^2(u_0)\|_2 + M_2 \|A(F)\|_2 + C M_1 \|A^2(F)\|_2 \right)$$

$$\cdot \left(\sum_{i=1}^{+\infty} \frac{|X_i(x)|^2}{\lambda_i} \right)^{1/2}.$$

$$(76)$$

Once again, due to $\lim_{i \to +\infty} \lambda_i = +\infty$, there exists a constant l such that $\lambda_i^3 < l \lambda_i^4$, $\lambda_i < l \lambda_i^2 i = 1, 2, \cdots$, where M_1, M_2 are some constants depending on l and M. Then, the series (75) uniformly convergent on any interval $[\varepsilon, T]$,

$0 < \varepsilon < T$. Therefore, the generalized solution (54) belongs to $C_t^1(0, T)$ and the proof is complete.

Theorem 19. *The classical solution (50) of (1) and (36) with $F(x, t) \equiv 0$ decay to zero with the rate $e^{-\int_0^t (\lambda_1/\psi(s, \alpha)) ds}$.*

Proof. In fact, since $u_0 \in M_A$, then $|\sum_{i=1}^{+\infty} (u_0, X_i) X_i(x)| = |u_0| < +\infty$. Then for any $t > 0$ and $x \in \bar{D}$, we have

$$\left| \sum_{i=1}^{+\infty} (u_0, X_i) e^{-\int_0^t \frac{\lambda_i}{\psi(\tau, \alpha)} d\tau} X_i(x) \right|$$

$$\leq e^{-\int_0^t \frac{\lambda_1}{\psi(s, \alpha)} ds} \left| \sum_{i=1}^{+\infty} (u_0, X_i) e^{-\int_0^t \frac{\lambda_i - \lambda_1}{\psi(\tau, \alpha)} d\tau} X_i(x) \right| \quad (77)$$

$$\leq |u_0| e^{-\int_0^t \frac{\lambda_1}{\psi(s, \alpha)} ds} \longrightarrow 0 (ast \longrightarrow +\infty).$$

5. Conclusions

The GCFD is an extension of the classical derivatives and the conformable derivatives. In this paper, we introduce its fractional Taylor power series expansions and prove the theorem of term-by-term integration and differentiation. Utilizing its properties, we obtain the uniqueness and existence of the classical solution of the classical diffusion equations with GCFD by using variable separation method.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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