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Research Article

Unification of Generalized and p-Convexity

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In the present note, we will introduce the definition of generalized p convex function. We will investigate some properties of generalized p convex function. Moreover, we will develop Jensen's type, Schur type, and Hermite Hadamard type inequalities for generalized p convex function.

1. Introduction

Convexity is a basic and common idea which can be followed back to Archimedes (around 250 B.C.), regarding his acclaimed gauge of the estimation of p (utilizing recorded and encircled standard polygons). He saw the significant reality that the parameters of a convex figure are lesser than the parameters of some other convex figure, encompassing it. As an issue of certainties, we experience convexity constantly and from numerous points of view. The most dull precedent is our standing up position, which is verified as long as the vertical projection of our focal point of gravity lies inside the convex envelope of our feet! Additionally, convexity greatly affects our regular daily existence through its various applications in industry, business, prescription, workmanship, and so forth. So are the issues on ideal assignment of assets and balance of nonagreeable recreations.

The theory of convex functions is a piece of the general subject of convexity since a raised capacity is one whose epigraph is a convex set. In any case, it is a theory significant essentially, which contacts practically all parts of science. Likely, the principal subject who makes important the experience with this theory is the graphical examination. With this event, we learn on the second derivative text of convexity, a useful asset in perceiving convexity. At that point, an issue of finding the extremal estimations of functions of several variables raises and the utilization of Hessian as a higher dimensional speculation of the second derivatives. Going to advancement issues in infinite dimensional spaces is the following stage; however, in spite of the specialized refinement in taking care of such issues, the fundamental thoughts are really comparable with one variable case. For detailed study, we refer Jensen [1, 2]. Anyway he was not the first who is dealing with such functions. We may refer [3–5]. During the entire twentieth Century, an exceptional research movement was done and critical outcomes were acquired in mathematical economics, nonlinear optimization, convex analysis, and so on. An extraordinary job in the advancement of the subject of convex functions was played by the acclaimed book of Hardy et al. [6].

Generally, there are two fundamental properties of convex functions that made them so broadly utilized in mathematics:

(1) The maximum is attained at a boundary point.
(2) Any local minimum is also a global minimum. Moreover, a strictly convex function can have at most one minimum value.

The modern perspective on convex functions involves an amazing and rich communication among geometry and analysis, which makes the reader to share a feeling of fervor. In an essential paper dedicated to the Brunn–Minkowski imbalance, Gardner [7, 8], depicted this reality in lovely expressions: "[convexity] appears like an octopus, tentacles reaching far and wide, its shape and color changing as it roams one area to the next. [And] it is quite clear that research opportunities abound". During the years various striking books are written to the theory and utilizations of convex functions. We refer here [9–14].

Convexity plays an important role in nonlinear programming and optimization theory. Although, several results have been derived under convexity assumptions, many real world problems are nonconvex in nature. So it is always appreciable to study nonconvex functions, which are close to convex function in some sense. The following is the double inequality for the convex function, $f : I \subseteq \mathbb{R} \to \mathbb{R}$ $a, b \in I$ where $a < b$ is known as Hermite Hadamard inequality,

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

The inequality (1) has been extended and generalized for various classes of convex function, see [15–18]. For further readings, we refer some books, see [19–21].

Definition 1 (p-convex set [15]). The interval I is said to be a p-convex set if $\left[\alpha x^p + (1-\alpha)y^p\right]^{1/p} \in I$ for all $x, y \in I$ and $\alpha \in [0,1]$, where $p = 2k+1$ or $p = n/m$, $n = 2r+1$, $m = 2t+1$, and $k, r, t \in N$.

Definition 2 (p-convex function [15]). Let I be a p-convex set. A function $f : I \to \mathbb{R}$ *is* said *to* be p-convex function, if

$$f\left(\left[t^p + (1-t)y^p\right]^{1/p}\right) \leq tf(x) + (1-t)f(y) \tag{2}$$

for all $x, y \in I$ and $t \in [0,1]$.

One of the novel generalization of convexity is η-convexity introduced by Delavar and Dragomir [16].

Definition 3 (η-convex function [18]). The function $f : I \to \mathbb{R}$ is called convex with respect to $\eta : A \times A \to B$ for appropriate $A, B \subseteq \mathbb{R}$, if

$$f(tx + (1-t)y) \leq f(y) + t\eta(f(x), f(y)), \tag{3}$$

for all $x, y \in I$ and $t \in [0,1]$.

Definition 4 (Nonnegatively homogeneous [18]). A function η is said to be nonnegatively homogeneous if $\eta(\lambda x, \lambda y) = \lambda\eta(x, y)$ for all $x, y \in \mathbb{R}$ and all $\lambda \geq 0$.

Now we are ready to introduce the definition of generalized p convex function.

Definition 5 (Generalized p-convex function). Let $\eta : A \times A \to B$ be a bifunction for appropriate $A, B \subseteq \mathbb{R}$ and I be a p-convex set. We say that $f : I \to \mathbb{R}$ is a generalized p convex function, if

$$f\left(\left[tx^p + (1-t)y^p\right]^{1/p}\right) \leq f(y) + t\eta(f(x), f(y)) \tag{4}$$

for all $x, y \in I$ and $t \in [0,1]$.

Of course (4) is η-convexity for $p = 1$, p-convexity for $\eta(x, y) = x - y$ and classical convex function for $p = 1$ and $\eta(x, y) = x - y$ simultaneously.

We observe that by taking $x^p = y^p$ in (4) we get

$$t\eta(f(x), f(y)) \geq 0 \tag{5}$$

for any $x \in I$ and $t \in [0,1]$,

$$\eta(f(x), f(y)) \geq 0 \tag{6}$$

for any $x \in I$. Also, if we take $t = 1$ in (4), we get

$$f(x) - f(y) \leq \eta(f(x), f(y)) \tag{7}$$

for any $x, y \in I$. The second condition obviously implies the first.

Example 6. Consider a function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} -x^p, & \text{if } x \geq 0, \\ x^p, & \text{if } x < 0, \end{cases} \tag{8}$$

and define a bifunction η as:

$$\eta(x, y) = \begin{cases} x, & \text{if } x = 0, \\ -y, & \text{if } y = 0, \\ x - y, & \text{if } x < 0, y < 0. \end{cases} \tag{9}$$

The above function is generalized p convex, but not convex function.

Example 7. Let $f : I \to \mathbb{R}$, $f(x) = x^p$ and $\eta(x, y) \geq x - y$ where $x \geq 0$. Then f is generalized p convex function.

Proof. Take,

$$\begin{aligned} f\left(tx^p + (1-t)y^p\right)^{1/p} &= tx^p + (1-t)y^p \\ &= tx^p + y^p - ty^p \\ &= y^p + t\left(x^p - y^p\right) \\ &= f(y) + t(f(x) - f(y)) \\ &\leq f(y) + t\eta(f(x), f(y)). \end{aligned} \tag{10}$$

Hence f is generalized p convex. □

The paper is organized as follows: In the next section, we will derive some basic properties of generalized p convex functions. In the last section, we will develop Hermite Hadamard, Jensen, Schur and Fejer type inequalities for generalized p convex functions.

2. Basic Results

Proposition 8 (operation which preserves generalized p convexity). Let $f, g : I \to \mathbb{R}$ be two generalized p convex functions, then the following statements hold:

(1) If η is additive then $f + g : I \to \mathbb{R}$ is generalized p convex.

(2) If η is nonnegatively homogeneous, then for any $\lambda \geq 0$, $\lambda f : I \to \mathbb{R}$ is generalized p convex.

(3) If $f : [a, b] \to \mathbb{R}$ is generalized p convex function, then

$$\max_{x\in[a,b]} f(x) \leq \max\{f(b), f(b) + t\eta(f(x), f(y))\}. \tag{11}$$

Proof. The proof of (1) and (2) is straightforward.

The proof of (3) is given as: For any p convex interval, $x \in [a, b]$, we have $x^p = ta^p + (1-t)b^p$ for some $t \in [0, 1]$ which implies that

$$\begin{aligned} f\left(\frac{a^p+b^p}{2}\right)^{1/p} &= f\left(\frac{a^p+b^p}{4} + \frac{t}{2} + \frac{a^p+b^p}{4} - \frac{t}{2}\right)^{1/p} \\ &= f\left(\frac{1}{2}\left(\frac{a^p+b^p}{2} + t\right) + \frac{1}{2}\left(\frac{a^p+b^p}{2} - t\right)\right)^{1/p} \\ &= f\left(\frac{1}{2}\left(\left(\frac{a^p+b^p}{2} + t\right)^{1/p}\right)^p + \frac{1}{2}\left(\left(\frac{a^p+b^p}{2} - t\right)^{1/p}\right)^p\right)^{1/p} \\ &\leq f\left(\frac{a^p+b^p}{2} - t\right)^{1/p} + \frac{1}{2}\eta\left(f\left(\frac{a^p+b^p}{2} + t\right)^{1/p}, f\left(\frac{a^p+b^p}{2} - t\right)^{1/p}\right) \leq f\left(\frac{a^p+b^p}{2} - t\right)^{1/p} + \frac{M_\eta}{2}. \end{aligned} \tag{15}$$

Now consider $m = \left[f((a^p+b^p)/2)^{1/p} - \left(M_\eta/2\right)\right]^p$ and we obtain the required result. □

Theorem 10. *For I be a p-convex set. A function $f : I \to \mathbb{R}$ is generalized p convex if and only if for any $x_1, x_2, x_3 \in I$ with $x_1 < x_2 < x_3$,*

$$\det\begin{pmatrix} f(x_3) - f(x_2) & -(x_2^p - x_3^p) \\ \eta(f(x_1), f(x_3)) & (x_1^p - x_3^p) \end{pmatrix} \geq 0. \tag{16}$$

Proof. Suppose that f is generalized p convex function. Consider arbitrary $x_1, x_2, x_3 \in I$ with $x_1 < x_2 < x_3$. So there is a $t \in (0, 1)$ such that ${x_2}^p = t{x_1}^p + (1-t){x_3}^p$, namely $t = ({x_2}^p - {x_3}^p)/({x_1}^p - {x_3})^p$. From generalized p convexity of f we have

$$f(x_2) \leq f(x_3) + t\eta(f(x_1), f(x_3)). \tag{17}$$

Implies that,

$$(x_1^p - x_3^p)[f(x_3) - f(x_2)] + (x_2^p - x_3^p)\eta(f(x_1), f(x_3)) \geq 0, \tag{18}$$

which is equivalent to above determinant being nonnegative.

Also, for $t = 1$

$$f(x_1) \leq f(x_3) + \eta(f(x_1), f(x_3)) \tag{19}$$

and for $t = 0$

$$f(x_3) \leq f(x_3). \tag{20}$$

$$\begin{aligned} f(x) &\leq f(b) + t\eta(f(a), f(b)) \\ &\leq \max\{f(b), f(b) + t\eta(f(a), f(b))\}. \end{aligned} \tag{12}$$

Since x is arbitrary, so

$$\max_{x\in[a,b]} f(x) \leq \max\{f(b), f(b) + t\eta(f(x), f(y))\}. \tag{13}$$

Which is requried. □

Proposition 9. If $f : [a, b] \to \mathbb{R}$ be a generalized p convex function and η is bounded above, has lower and upper bounds.

Proof. Suppose that M_η is upper bound of η on $f([a, b]) \times f([a, b])$. From proposition (8), part (3)

$$f(x) \leq \max\{f(b), f(b) + t\eta(f(a), f(b))\} \leq \max\left\{f(b), f(b) + M_\eta\right\}. \tag{14}$$

Now set $M = \max\left\{f(b), f(b) + M_\eta\right\}$.

For lower bound of f consider an arbitrary point in the form $(a^p + b^p)/2 - t$ in [a, b], then

For inverse implication, consider $x, y \in I$, where I is p-convex set with $x < y$. Choosing any $t \in (0, 1)$

we have $x < \left[tx^p + (1-t)y^p\right]^{1/p} < y$ and so

$$\det\begin{pmatrix} f(y) - f\left[tx^p + (1-t)y^p\right]^{1/p} & -f(tx^p + (1-t)y^p - y^p) \\ \eta(f(x), f(y)) & x^p - y^p \end{pmatrix} \geq 0, \tag{21}$$

by expanding the determinant we get,

$$0 \leq (x^p - y^p)\left[f(y) - f\left[tx^p + (1-t)y^p\right]^{1/p} + t\eta(f(x), f(y))\right], \tag{22}$$

which implies,

$$f(tx^p + (1-t)y^p)^{1/p} \leq f(y) + t\eta(f(x), f(y)) \tag{23}$$

for any $t \in (0, 1)$. So, f is generalized p convex function. □

Theorem 11. *Let $f_j : I \to \mathbb{R}$, $j \in J$ is non empty collection of generalized p convex functions such that*

(a) there exist $\alpha \in [0, \infty]$ and $\beta \in [-1, \infty]$ such that $\eta(x, y) = \alpha x + \beta y \; \forall x, y \in \mathbb{R}$,

(b) for each $x \in I$, $\sup_{j\in J} f_j(x)$ exists in $\mathbb{R}$,

then the function $f : I \to \mathbb{R}$ defined by $f(x) = \sup_{j\in J} f_j(x)$ for each $x \in I$ is generalized p convex.

Proof. For any $x, y \in I$ and $t \in [0,1]$, we have

$$\begin{aligned} f\left[tx^p + (1-t)y^p\right]^{1/p} &= \sup_{j\in J} f_j\left[tx^p + (1-t)y^p\right]^{1/p} \\ &\le \sup_{j\in J}\left\{f_j(y) + t\eta\left(f_j(x), f_j(y)\right)\right\} \\ &= \sup_{j\in J}\left\{f_j(y) + t\left(\alpha f_j(x) + \beta f_j(y)\right)\right\} \\ &= \sup_{j\in J}\left\{(1+\beta t)f_j(y) + \alpha t f_j(x)\right\} \\ &\le (1+\beta t)\sup_{j\in J} f_j(y) + \alpha t \sup_{j\in J} f_j(x) \\ &= (1+\beta t)f(y) + \alpha t f(x) \\ &= f(y) + t(\alpha f(x) + \beta f(y)) \\ &= f(y) + t\eta(f(x), f(y)). \end{aligned} \tag{24}$$

which is required. □

3. Main Results

Theorem 12 (Schur type inequality). *Let $\eta : A \times A \to B$ be a bifunction for appropriate $A, B \subseteq \mathbb{R}$ and let f be a function defined on interval I such that f belongs to generalized p convex function. Then $\forall x_1, x_2, x_3 \in I$ such that $x_1 < x_2 < x_3$ and $x_3^p - x_1^p, x_3^p - x_2^p, x_2^p - x_1^p \in (0,1)$ the following inequality holds:*

$$f(x_3)(x_3^p - x_1^p) - f(x_2)(x_3^p - x_1^p) + (x_3^p - x_2^p)\eta(f(x_1), f(x_3)) \ge 0. \tag{25}$$

Proof. Let f be the generalized p convex function and let $x_1, x_2, x_3 \in I$ be given. Then we can easily see that

$$\frac{x_3^p - x_2^p}{x_3^p - x_1^p}, \frac{x_2^p - x_1^p}{x_3^p - x_1^p} \in (0,1) \tag{26}$$

and

$$\frac{x_3^p - x_2^p}{x_3^p - x_1^p} + \frac{x_2^p - x_1^p}{x_3^p - x_1^p} = 1. \tag{27}$$

Setting $t = (x_3^p - x_2^p)/(x_3^p - x_1^p), x = x_1$ and $y = x_3$ in (4) we have $x_2^p = tx^p + (1-t)y^p$

$$f(x_2) \le f(x_3) + \frac{x_3^p - x_2^p}{x_3^p - x_1^p}\eta(f(x_1), f(x_3)). \tag{28}$$

Assuming $x_3^p - x_1^p > 0$ and multiplying both sides of the inequality above by $x_3^p - x_1^p$,
we obtain inequality (25). □

Remark 13. In fact, if $f(x) = x^\lambda, \lambda \in \mathbb{R}, p = 1, \eta(x,y) = x - y$ and $x_1, x_2, x_3 \in I = (0,1)$, then inequality (25) gives the schur type inequality, (see [20]).

Theorem 14 (Hermite Hadamard type inequality). *Let $f : I \to \mathbb{R}$ be generalized p convex function for $a, b \in I$ with condition $a < b$, then we obtain the inequality*

$$\begin{aligned} f\left(\frac{a^p + b^p}{2}\right)^{1/p} &- \frac{p}{2(b^p - a^p)}\int_a^b x^{p-1}\eta\left(f(a^p + b^p - x^p)^{1/p}, f(x)\right)dx \\ &\le \frac{p}{b^p - a^p}\int_a^b x^{p-1} f(x)dx \le \frac{f(a) + f(b)}{2} \\ &\quad + \frac{1}{4}[\eta(f(a), f(b)) + \eta(f(b), f(a))]. \end{aligned} \tag{29}$$

Proof. Take $u^p = ta^p + (1-t)b^p$ and $v^p = (1-t)a^p + tb^p$, implies

$$\frac{a^p + b^p}{2} = \frac{u^p + v^p}{2}. \tag{30}$$

So,

$$f\left(\frac{a^p + b^p}{2}\right)^{1/p} = f\left(\frac{u^p + v^p}{2}\right)^{1/p}. \tag{31}$$

By definition of generalized p convex functions, we have

$$\begin{aligned} f\left(\frac{a^p + b^p}{2}\right)^{1/p} &= f\left(\frac{1}{2}\left(\left(ta^p + (1-t)b^p\right)^{1/p}\right)^p \right. \\ &\quad \left. + \frac{1}{2}\left(\left((1-t)a^p + ta^p\right)^{1/p}\right)^p\right)^{1/p} \\ &\le f\left((1-t)a^p + tb^p\right)^{1/p} \\ &\quad + \frac{1}{2}\eta\left(f\left(ta^p + (1-t)b^p\right)^{1/p}, f\left((1-t)a^p + tb^p\right)^{1/p}\right). \end{aligned} \tag{32}$$

Integrating above w.r.t "x" on [0, 1], we get

$$\begin{aligned} f\left(\frac{a^p + b^p}{2}\right)^{1/p} &\le \int_0^1 f\left((1-t)a^p + tb^p\right)^{1/p}dt \\ &\quad + \frac{1}{2}\int_0^1 \eta\left(f\left(ta^p + (1-t)b^p\right)^{1/p}, f\left((1-t)a^p + tb^p\right)^{1/p}\right)dt, \end{aligned} \tag{33}$$

which implies,

$$\begin{aligned} f\left(\frac{a^p + b^p}{2}\right)^{1/p} &- \frac{p}{2(b^p - a^p)}\int_a^b x^{p-1}\eta\left(f(a^p + b^p - x^p)^{1/p}, f(x)\right)dx \\ &\le \frac{p}{b^p - a^p}\int_a^b x^{p-1} f(x)dx. \end{aligned} \tag{34}$$

Now,

$$\begin{aligned} \int_a^b x^{p-1} f(x)dx &= \frac{b^p - a^p}{p}\int_0^1 f\left(ta^p + (1-t)b^p\right)^{1/p}dt \\ &\le \frac{b^p - a^p}{p}\left(f(b) + \int_0^1 t\eta(f(a), f(b))dt\right)\frac{p}{b^p - a^p}\int_a^b x^{p-1} f(x)dx \\ &\le f(b) + \int_0^1 t\eta(f(a), f(b))dt. \end{aligned} \tag{35}$$

Similarly,

$$\frac{p}{b^p - a^p}\int_a^b x^{p-1} f(x)dx \le f(a) + \int_0^1 t\eta(f(b), f(a))dt. \tag{36}$$

Adding (35) and (36), we obtain

$$\frac{p}{b^p - a^p}\int_a^b x^{p-1} f(x)dx \le \frac{f(a) + f(b)}{2} + \frac{1}{4}[\eta(f(a), f(b)) + \eta(f(b), f(a))]. \tag{37}$$

Combining (34) and (37), we obtain the inequality (29). □

Remark 15. If we put $p = 1$ then (25) becomes Hermite Hadamard type inequality for η-convexity, (see [16]).

Remark 16. If we put $\eta(x, y) = x - y$ in (25) then we get Hermite hadamard type inequality for p-convexity, (see [19]).

Remark 17. If we put $p = 1$ and $\eta(x, y) = x - y$ in (25) then we obtain classical Hermite Hadamard type inequality for convex functions.

We will use the following relations in the proof of Theorem 19 which is Jensen's type inequality for generalized p convex functions.

Theorem 18. *Let $f : I \to \mathbb{R}$ be a η-convex function, For $x_1, x_2 \in I$ and $\alpha_1 + \alpha_2 = 1$ we have $f(\alpha_1 x_1^p + \alpha_2 x_2^p)^{1/p} \le f(x_2) + \alpha\eta(f(x_1), f(x_2))$. Also when $n > 2$, for $x_1, x_2, \ldots, x_n \in I$, $\sum_{i=1}^n \alpha_i = 1$ and $T_i = \sum_{j=1}^i \alpha_j$, we have*

$$\left(\sum_{i=1}^n \alpha_i x_i^p\right)^{1/p} = f\left(T_{n-1}\left(\left(\sum_{i=1}^{n-1} \frac{\alpha_i}{T_{n-1}} x_i^p\right)^{1/p}\right)^p + \alpha_n x_n^p\right)^{1/p} \le f(x_n) + T_{n-1}\eta\left(f\left(\sum_{i=1}^{n-1} \frac{\alpha_i}{T_{n-1}} x_i^p\right)^{1/p}, f(x_n)\right). \tag{38}$$

Theorem 19 (Jensen's Type inequality). *Let $w_1, w_2, \ldots, w_n \in \mathbb{R}^+$ with $n \ge 2$. Let $f : I \to \mathbb{R}$ generalized p convex function and η be nondecreasing, non negatively sub linear variable. Then we have following inequality*

$$f\left(\left[\frac{1}{W_n}\sum_{i=1}^n w_i x_i^p\right]^{1/p}\right) \le f(x_n) + \sum_{i=1}^n \left(\frac{W_i}{W_n}\right)\eta_f(x_i, x_{i+1}, \ldots, x_n) \tag{39}$$

where $W_n = \sum_{i=1}^n w_i$, and $\eta_f(x_i, x_{i+1}, \ldots, x_n) = \eta\ \left(\eta_f(x_i, x_{i+1}, \ldots, x_{n-1}), f(x_n)\right)$ and $\eta_f(x) = f(x)\ \forall x \in I$.

Proof. Since η be nondecreasing nonnegatively sub linear in first variable, so from Theorem 18 it follows that

$$\begin{aligned}
&f\left(\left[\frac{1}{W_n}\sum_{i=1}^n w_i x_i^p\right]^{1/p}\right)\\
&= f\left(\left[\frac{w_n}{W_n}x_n^p + \sum_{i=1}^{n-1}\frac{w_i}{W_n}x_i^p\right]^{1/p}\right)\\
&= f\left(\left[\frac{W_{n-1}}{W_n}\sum_{i=1}^{n-1}\frac{w_i}{W_{n-1}}x_i^p + \frac{w_n}{W_n}x_n^p\right]^{1/p}\right)\\
&\le f(x_n) + \frac{W_{n-1}}{W_n}\eta\left(f\left(\left[\sum_{i=1}^{n-1}\frac{w_i}{W_{n-1}}x_i^p\right]^{1/p}\right), f(x_n)\right)\\
&= f(x_n) + \frac{W_{n-1}}{W_n}\eta\left(f\left[\frac{W_{n-2}}{W_{n-1}}\sum_{i=1}^{n-2}\frac{w_i}{W_{n-2}}x_i^p + \frac{w_{n-1}}{W_{n-1}}x_{n-1}^p\right]^{1/p}, f(x_n)\right)\\
&\le f(x_n) + \frac{W_{n-1}}{W_n}\eta\left(f(x_{n-1}) + \frac{W_{n-2}}{W_{n-1}}\eta\left(f\left[\sum_{i=1}^{n-2}\frac{w_i}{W_{n-2}}x_i^p\right]^{1/p}, f(x_{n-1})\right), f(x_n)\right)\\
&\le f(x_n) + \frac{W_{n-1}}{W_n}\eta(f(x_{n-1}), f(x_n)) + \frac{W_{n-2}}{W_n}\\
&\quad\times\eta\left(\eta\left(f\left[\sum_{i=1}^{n-2}\frac{w_i}{W_{n-2}}x_i^p\right]^{1/p}, f(x_{n-1})\right), f(x_n)\right)\\
&\le \cdots\\
&\le f(x_n) + \frac{W_{n-1}}{W_n}\eta(f(x_{n-1}), f(x_n)) + \frac{W_{n-2}}{W_n}\\
&\quad\times\eta(\eta(f(x_{n-2}), f(x_n)), f(x_n))\\
&\quad+\cdots+\frac{W_1}{W_n}\eta(\eta(\cdots\eta(f(x_1), f(x_2)), f(x_3))\cdots)), f(x_{n-1})), f(x_n))\\
&= f(x_n) + \frac{W_{n-1}}{W_n}\eta_f(x_{n-1}, x_n) + \frac{W_{n-2}}{W_n}\eta_f(x_{n-2}, x_{n-1}, x_n)\\
&\quad+\cdots+\frac{W_1}{W_n}\eta_f(x_1, x_2, \ldots, x_{n-1}, x_n)\\
&= f(x_n) + \sum_{i=1}^{n-1}\left(\frac{W_i}{W_n}\right)\eta_f(x_i, x_{i+1}, \ldots, x_n).
\end{aligned} \tag{40}$$

□

Remark 20. For $p = 1$ in (39), then we have Jensen's type inequality for η-convex function, see [17].

Remark 21. For $\eta(x, y) = x - y$ in (39), then we have Jensen type inequality for p-convex function.

Remark 22. For $p = 1$ and $\eta(x, y) = x - y$ in (39), then we have the Jensen type inequality for classical convex function.

Theorem 23 (Fejer type inequality). *Let f, g be nonnegative generalized p convex functions $a, b \in I$ $a < b$ such that $fg \in L_1[a, b]$, then*

$$\frac{p}{b^p - a^p}\int_a^b x^{p-1} f(x)g(x)dx \le M(a, b) + \frac{1}{2}N(a, b), \tag{41}$$

where,

$$\begin{aligned}
M(a, b) &= f(b)g(b) + \tfrac{1}{3}\eta(f(a), f(b))\eta(g(a), g(b)),\\
N(a, b) &= f(b)\eta(g(a), g(b)) + g(b)\eta(f(a), f(b)).
\end{aligned} \tag{42}$$

Proof. Since f and g are generalized p convex functions, we have

$$f\left(\left[ta^p + (1-t)b^p\right]^{1/p}\right) \le f(b) + t\eta\left(f(a), f(b)\right)$$
$$g\left(\left[ta^p + (1-t)b^p\right]^{1/p}\right) \le g(b) + t\eta\left(g(a), g(b)\right) \quad (43)$$

for all $t \in [a, b]$. Since f and g are nonnegative, so

$$\begin{aligned} &f\left(\left[ta^p + (1-t)b^p\right]^{1/p}\right)g\left(\left[ta^p + (1-t)b^p\right]^{1/p}\right) \\ &\quad \le f(b)g(b) + tf(b)\eta\left(g(a), g(b)\right) + tg(b)\eta\left(f(a), f(b)\right) \\ &\qquad + t^2\eta\left(f(a), f(b)\right)\eta\left(g(a), g(b)\right). \end{aligned} \quad (44)$$

Integrating both sides of above inequality over (0, 1), we obtain the inequality

$$\begin{aligned} &\int_0^1 f\left(\left[ta^p + (1-t)b^p\right]^{1/p}\right)g\left(\left[ta^p + (1-t)b^p\right]^{1/p}\right)dt \\ &\quad \le \int_0^1 f(b)g(b)dt + \int_0^1 tf(b)\eta\left(g(a), g(b)\right)dt \\ &\qquad + \int_0^1 tg(b)\eta\left(f(a), f(b)\right)dt + \int_0^1 t^2\eta\left(f(a), f(b)\right)\eta\left(g(a), g(b)\right)dt. \end{aligned} \quad (45)$$

Setting $x = \left[ta^p + (1-t)b^p\right]^{1/p}$, we get

$$\begin{aligned} \frac{p}{b^p - a^p}\int_a^b x^{p-1}f(x)g(x)dx \le\ & f(b)g(b) + \frac{1}{2}f(b)\eta\left(g(a), g(b)\right) \\ & + \frac{1}{2}g(b)\eta\left(f(a), f(b)\right) \\ & + \frac{1}{3}\eta\left(f(a), f(b)\right)\eta\left(g(a), g(b)\right). \end{aligned} \quad (46)$$

Then, we obtain the (41) inequality. □

Remark 24. If we put $p = 1$ and $\eta(x, y) = x - y$ in (41), then it reduces for classical convex functions.

Data Availability

All data are included within this paper.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

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References

[1] J. L. W. V. Jensen, "Om konvexe Funktioner og Uligheder mellem Middelvaerdier," *Nyt Tidskrift for Mathematik*, vol. 16B, pp. 49–69, 1905.

[2] J. L. W. V. Jensen, "Sur les fonctions convexes et les inegalités entre les valeurs moyennes," *Acta Mathematica*, vol. 30, pp. 175–193, 1906.

[3] Ch. Hermite, "Sur deux limites dune integrale definie," *Mathesis*, vol. 3, p. 82, 1883.

[4] O. Holder, "Uber einen Mittelwertsatz," *Nachrichten von der Gesellschaft der Wissenschaften zu Gottingen*, pp. 38–47, 1889.

[5] O. Stolz, *Grunzuge der Differential und Integralrechnung*, vol. 1, Teubner, Leipzig, 1893.

[6] G. H. Hardy, J. E. Littlewood, and G. Polya, *Inequalities*, Cambridge Mathematical Library, 2nd edition, 1952, Reprinted 1988.

[7] R. J. Gardner, "The Brunn–Minkowski inequality: a survey with proofs," Preprint 2001, http://faculty.wwu.edu/gardner/gorizia12.pdf.

[8] R. J. Gardner, "The Brunn–Minkowski inequality," *Bulletin of the American Mathematical Society*, vol. 39, no. 3, pp. 355–406, 2002.

[9] L. Hormander, *Notions of Convexity*, Birkhauser, Boston, 1994.

[10] M. A. Krasnoselśkii and Ya. B. Rutickii, *Convex Functions and Orlicz Spaces*, P. Nordhoff, Groningen, 1961.

[11] J. E. Pecarić, F. Proschan, and Y. C. Tong, *Convex functions, Partial Orderings and Statistical Applications*, Academic Press, New York, 1992.

[12] R. R. Phelps, *Convex functions, Monotone Operators, and Differentiability*, *Lecture Notes in Mathematics No. 1364*, Springer-Verlag, Berlin, 2nd edition, 1993.

[13] R. R. Phelps, *Lectures on Choquet's Theorem*, *Lecture Notes in Mathematics No. 1757*, Springer-Verlag, Berlin, 2nd edition, 2001.

[14] A. W. Roberts and D. E. Varberg, *Convex Functions*, Academic Press, New York and London, 1973.

[15] K. S. Zhang and J. P. Wan, "*P*-Convex functions and their properties," *Pure and Applied Mathematics*, vol. 1, no. 23, pp. 130–133, 2007.

[16] M. R. Delavar and S. S. Dragomir, "On η-Convexity," *Journal of Mathematical Inequalities Number*, vol. 1, no. 20, pp. 203–216, 2017.

[17] Z. B. Fang and R. Shi, "On the (p, h)-convex functions and some integral inequalities," *Journal of Inequalities and Applications*, 2014.

[18] M. E. Gordji, M. Rostamian, and M. Delasen, "On ϕ-convex functions," *Journal of Mathematical Inequalities*, vol. 10, pp. 173–183, 2016.

[19] J. E. Pećaric, F. Proschan, and Y. L. Tong, *Convex Functions, Partial Orderings and Statistical Applications*, Acadmic press, Boston, 1992.

[20] D. S. Mitrinoviarić, *Analytic Inequalities*, Springer, Berlin, 1970.

[21] C. P. Nisculescu and L. E. Person, *Convex Functions and their Applications*, Springer, Berlin Heidelberg, New York, Hong Kong London, Milan Pairs Tokyo, 2004.