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Research Article

Strong Converse Results for Linking Operators and Convex Functions

Ana-Maria Acu,[1] **Margareta Heilmann**,[2] **and Ioan Rasa**[3]

[1]*Lucian Blaga University of Sibiu, Department of Mathematics and Informatics, Str. Dr. I. Ratiu, No. 5-7, RO-550012 Sibiu, Romania*
[2]*School of Mathematics and Natural Sciences, University of Wuppertal, Gaußstraße 20, D-42119 Wuppertal, Germany*
[3]*Technical University of Cluj-Napoca, Faculty of Automation and Computer Science, Department of Mathematics, Str. Memorandumului nr. 28, 400114 Cluj-Napoca, Romania*

Correspondence should be addressed to Ana-Maria Acu; anamaria.acu@ulbsibiu.ro

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We consider a family $B_{n,\rho}^{[c]}$ of operators which is a link between classical Baskakov operators (for $\rho=\infty$) and their genuine Durrmeyer type modification (for $\rho=1$). First, we prove that for fixed n, c and a fixed convex function f, $B_{n,\rho}^{[c]}f$ is decreasing with respect to ρ. We give two proofs, using various probabilistic considerations. Then, we combine this property with some existing direct and strong converse results for classical operators, in order to get such results for the operators $B_{n,\rho}^{[c]}$ applied to convex functions.

1. Introduction

The Baskakov-type operators depending on a real parameter c were introduced by Baskakov in [1]. This class of operators includes the classical Bernstein, Szász-Mirakjan, and Baskakov operators as special cases for $c=-1$, $c=0$, and $c=1$, respectively.

Let $c\in\mathbb{R}$, $n\in\mathbb{R}$, $n>c$ for $c\geq 0$, and $-n/c\in\mathbb{N}$ for $c<0$. Furthermore, let $I_c=[0,\infty)$ for $c\geq 0$ and $I_c=[0,-1/c]$ for $c<0$. Consider $f: I_c\to\mathbb{R}$ given in such a way that the corresponding integrals and series are convergent.

The Baskakov-type operators are defined as follows:

$$\left(B_{n,\infty}^{[c]}f\right)(x)=\sum_{j=0}^{\infty}p_{n,j}^{[c]}(x)f\left(\frac{j}{n}\right), \tag{1}$$

where

$$p_{n,j}(x)=\begin{cases}\dfrac{n^j}{j!}x^je^{-nx} & ,c=0,\\[2ex] \dfrac{n^{c,\bar{j}}}{j!}x^j(1+cx)^{-(n/c+j)} & ,c\neq 0,\end{cases}=\begin{cases}\dfrac{1}{n-c}\dfrac{(-c)^{j+1}}{B(j+1,-n/c-j+1)}x^j(1+cx)^{-(n/c+j)} & ,c<0,\\[2ex] \dfrac{n^j}{\Gamma(j+1)}x^je^{-nx} & ,c=0,\\[2ex] \dfrac{1}{n-c}\dfrac{c^{j+1}}{B(j+1,n/c-1)}x^j(1+cx)^{-(n/c+j)} & ,c>0,\end{cases} \tag{2}$$

and $a^{c,\bar{j}} := \prod_{l=0}^{j-1}(a+cl)$, $a^{c,\bar{0}} := 1$.

We remark that (2) is well defined also for $j \in \mathbb{R}$, $j \geq 0$, which will be considered below.

The genuine Baskakov-Durrmeyer-type operators are given by

$$\left(B_{n,1}^{[c]}f\right)(x) = f(0)p_{n,0}^{[c]}(x) + f\left(-\frac{1}{c}\right)p_{n,-n/c}^{[c]}(x) + \sum_{j=1}^{-n/c-1} p_{n,j}^{[c]}(x)(n+c)\int_0^{-1/c} p_{n+2c,j-1}^{[c]}(t)f(t)dt, for c<0,$$

$$\left(B_{n,1}^{[c]}f\right)(x) = f(0)p_{n,0}^{[c]}(x) + \sum_{j=1}^{\infty} p_{n,j}^{[c]}(x) \cdot (n+c)\int_0^{\infty} p_{n+2c,j-1}^{[c]}(t)f(t)dt, for c \geq 0. \tag{3}$$

In the last years, a nontrivial link between classical Baskakov-type operators and their genuine Durrmeyer-type modification came into the focus of research. Depending on a parameter $\rho \in \mathbb{R}^+$, the linking operators are given by

$$\left(B_{n,1}^{[c]}f\right)(x) = \sum_{j=0}^{\infty} F_{n,j,\rho}^{[c]}(f)p_{n,j}^{[c]}(x), \tag{4}$$

where

$$F_{n,j,\rho}^{[c]}(f) = \begin{cases} f(0), & j=0, c \in \mathbb{R}, \\ f\left(-\frac{1}{c}\right). & j = -\frac{n}{c}, c<0, \\ \int_{I_c} \mu_{n,j,\rho}^{[c]}(t)f(t)dt, & \text{otherwise}, \end{cases} \tag{5}$$

with $\mu_{n,j,\rho}^{[c]}(t) = (n\rho + c)p_{n\rho+2c,j\rho-1}^{[c]}(t)$.

For $c<0$, $B_{n,\rho}^{[c]}f$ is well defined if $f \in L_1[0,1]$ with finite limits at the endpoints of the interval $[0,-1/c]$, i.e., $f(0) = \lim_{x\to 0^+} f(x)$ and $f(-1/c) = \lim_{x\to -1/c^-} f(x)$.

For $c \geq 0$, the operators $B_{n,\rho}^{[c]}$ are well defined for functions $f \in W_n^\rho$ having a finite limit $f(0) = \lim_{x\to 0^+} f(x)$ where W_n^ρ denotes the space of functions $f \in L_{1,loc}[0,\infty)$ satisfying certain growth conditions, i. e., there exist constants $M>0$, $0 \leq q < n\rho + c$, such that a. e. on $[0,\infty)$.

$$\begin{aligned} |f(t)| &\leq Me^{qt} f \text{ or } c=0, \\ |f(t)| &\leq Mt^{q/c} f \text{ or } c>0. \end{aligned} \tag{6}$$

First, we prove that for fixed n, c and a fixed convex function f, $B_{n,\rho}^{[c]}f$ is decreasing with respect to ρ. We give two proofs, using various probabilistic considerations. Then, we combine this property with some existing direct and strong converse results for classical operators, in order to get such results for the operators $B_{n,\rho}^{[c]}$ applied to convex functions.

2. The Case $c = -1$

For the linking Bernstein operator, i.e., $c=-1$, Rasa and Stanila [2], (10) proved that for a convex function $f \in C[0,1]$,

$$B_{n,\rho}^{[-1]}(f;x) \geq B_{n,\sigma}^{[-1]}(f;x) \geq f(x), 1 \leq \rho < \sigma \leq \infty. \tag{7}$$

For the proof, they used that $B_{n,\rho}$ can be written as a combination of the classical Bernstein operator and Beta operator and some corresponding results for the Beta operator from Adell et al. [3], Theorem 1. For the case $\rho = 1$ and the case $\rho = \infty$, strong converse results are known [4], Theorem 1.1, [5], p.117 [6], and [7], Theorem 3.2, Theorem 5:

$$\left\|B_{n,\rho}^{[-1]}f - f\right\|_\infty \sim \omega_\varphi^2\left(f, n^{-1/2}\right), \rho \in \{1,\infty\}, \tag{8}$$

where (see [5])

$$\omega_\varphi^2(f,t) := \sup_{0<h\leq t}\left\|\Delta_{h\varphi}^2 f\right\|, \tag{9}$$

with φ^2 a weight function and $\Delta_h^2 f(x) = f(x+h) - 2f(x) + f(x-h)$.

Thus, for f convex, $1 \leq \rho \leq \infty$

$$0 \leq B_{n,\infty}^{[-1]}(f,x) - f(x) \leq B_{n,\rho}^{[-1]}(f;x) - f(x) \leq B_{n,1}^{[-1]}(f;x) - f(x), \tag{10}$$

leading to

$$C_1^{-1}\omega_\varphi^2\left(f, n^{-1/2}\right) \leq \left\|B_{n,\infty}^{[-1]}f - f\right\| \leq \left\|B_{n,\rho}^{[-1]}f - f\right\| \leq \left\|B_{n,1}^{[-1]}f - f\right\| \leq C_1\omega_\varphi^2\left(f, n^{-1/2}\right), \tag{11}$$

i.e.,

$$\left\|B_{n,\rho}^{[-1]}f - f\right\|_\infty \sim \omega_\varphi^2\left(f, n^{-1/2}\right), 1 \leq \rho \leq \infty. \tag{12}$$

3. The Case $c = 0$

Consider the classical Szász-Mirakjan operators

$$B_{n,\infty}^{[0]}(f;x) := \sum_{j=0}^{\infty} p_{n,j}^{[0]}(x) f\left(\frac{j}{n}\right) \tag{13}$$

and also the operators

$$\mathcal{S}_r(f;x) = \begin{cases} \frac{r^{rx}}{\Gamma(rx)}\int_0^\infty t^{rx-1}e^{-rt}f(t)dt, & x>0, \\ f(0), & x=0, \end{cases} \tag{14}$$

where $r>0$.

Moreover, for $r > 0$, let (see [8]).

$$G_r(f\,;x) = \begin{cases} \dfrac{(r/x)^r}{\Gamma(r)} \displaystyle\int_0^\infty t^{r-1} e^{-rt/x} f(t)\,dt, & x > 0, \\ f(0), & x = 0. \end{cases} \tag{15}$$

Theorem 1 (see [8], Theorem 5 and Remark 6). *Let f and x be fixed and f convex, such that $G_r(|f|\,;x) < \infty$ for all $r > 0$. Then, $G_r(f\,;x)$ is nonincreasing with respect to r.*

Then,

$$\mathcal{S}_r(f\,;x) = G_{rx}(f\,;x). \tag{16}$$

For $c = 0$,

$$B_{n,\rho}^{[0]} = B_{n,\infty}^{[0]} \circ \mathcal{S}_{n\rho}. \tag{17}$$

Let f be convex and n and x be fixed, such that $G_r(|f|\,;x) < \infty$, for all $r > 0$.

Let $1 \le \rho \le \sigma$. Then, by (16) and Theorem 1,

$$\mathcal{S}_{n\sigma}(f\,;x) = G_{n\sigma x}(f\,;x) \le G_{n\rho x}(f\,;x) = \mathcal{S}_{n\rho}(f\,;x). \tag{18}$$

Now by (17),

$$B_{n,\sigma}^{[0]} f = B_{n,\infty}^{[0]}(\mathcal{S}_{n\sigma}) f \le B_{n,\infty}^{[0]}(\mathcal{S}_{n\rho}) f = B_{n,\rho}^{[0]} f. \tag{19}$$

Thus,

$$B_{n,\sigma}^{[0]} f \le B_{n,\rho}^{[0]} f. \tag{20}$$

Strong converse results are known also in this case (see [6], Theorem 1.2 and [9], Theorem 5.1, Theorem 5.2):

$$\left\| B_{n,\rho}^{[0]} f - f \right\|_\infty \sim \omega_\varphi^2\left(f, n^{-1/2}\right), \rho \in \{1, \infty\}. \tag{21}$$

Thus, for f convex,

$$\left\| B_{n,\rho}^{[0]} f - f \right\|_\infty \sim \omega_\varphi^2\left(f, n^{-1/2}\right), 1 \le \rho \le \infty. \tag{22}$$

4. The Case $c = 1$

To treat this case, we need some preliminaries.

If X and Y are independent random variables with densities $f(\theta)$, $g(\theta)$, and $\theta > 0$, then X/Y has density

$$w(t) = \int_0^\infty f(t\theta) g(\theta) \theta\, d\theta. \tag{23}$$

Let $(U_\tau)_{\tau \ge 0}$ and $(V_\tau)_{\tau \ge 0}$ be two independent gamma processes (see [8], p.129), i. e., U_τ has density $1/\Gamma(\tau)\theta^{\tau-1}e^{-\theta}$, $\theta > 0$. Let $x > 0$ and $r > 0$. Then, $Z_r^x := U_{rx}/V_{r+1}$ has density

$$\begin{aligned} w_{r,x}(t) &= \int_0^\infty \frac{1}{\Gamma(rx)} (t\theta)^{rx-1} e^{-t\theta} \frac{1}{\Gamma(r+1)} \theta^{r+1} e^{-\theta} d\theta \\ &= \frac{t^{rx-1}}{\Gamma(rx)\Gamma(r+1)} \int_0^\infty \theta^{rx+r} e^{-\theta(t+1)} d\theta, \text{ for } t > 0. \end{aligned} \tag{24}$$

Substitute $s = \theta(t+1)$. Then,

$$\begin{aligned} w_{r,x}(t) &= \frac{t^{rx-1}}{\Gamma(rx)\Gamma(r+1)} \int_0^\infty \frac{s^{rx+r}}{(t+1)^{rx+r}} e^{-s} \frac{ds}{t+1} \\ &= \frac{t^{rx-1}}{(t+1)^{rx+r+1}} \frac{\Gamma(rx+r+1)}{\Gamma(rx)\Gamma(r+1)} \\ &= \frac{1}{B(rx, r+1)} \frac{t^{rx-1}}{(t+1)^{rx+r+1}}. \end{aligned} \tag{25}$$

Let $\mathcal{B}_r(f\,;x) = \int_0^\infty w_{r,x}(t) f(t)\,dt$.

Consequently,

$$\mathcal{B}_r(f\,;x) = Ef\left(\frac{U_{rx}}{V_{r+1}}\right) = Ef(Z_r^x), \tag{26}$$

compare with [8], (9).

As in [10], Proof of Lemma 2, let $1 \le r \le s$, $x > 0$. Since the random vectors (U_{rx}, U_{sx}) and (V_{r+1}, V_{s+1}) are independent, we have

$$\begin{aligned} E(Z_r^x | U_{sx}, V_{s+1}) &= E(U_{rx} \mid U_{sx}) E\left(V_{r+1}^{-1} \mid V_{s+1}\right) \\ &= \frac{r}{s} U_{sx} E\left(V_{r+1}^{-1} \mid V_{s+1}\right), \end{aligned} \tag{27}$$

with [10], Lemma 1. Moreover, as in [10], (19), we get

$$E\left(V_{r+1}^{-1} \mid V_{s+1}\right) = V_{s+1}^{-1} \cdot \frac{s}{r}. \tag{28}$$

Thus,

$$E(Z_r^x | U_{sx}, V_{s+1}) = \frac{U_{sx}}{V_{s+1}} = Z_s^x. \tag{29}$$

As at the end of [10], Proof of Lemma 2, taking here the conditional expectation w.r.t. Z_s^x, we get

$$E(Z_r^x \mid Z_s^x) = Z_s^x, a.s., 1 \le r \le s, x > 0. \tag{30}$$

Now (30) is exactly the assumption of [8], Theorem 5 (a). Accordingly, [8], Theorem 5 (a) and Remark 6 show that

$$\mathcal{B}_r f \ge \mathcal{B}_s f, 1 \le r \le s, f \text{ convex}. \tag{31}$$

This implies

$$\begin{aligned} &\mathcal{B}_{n\rho} f \ge \mathcal{B}_{n\sigma} f, 1 \le \rho \le \sigma, f \text{ convex}, \\ &B_{n,\infty}^{[1]} \mathcal{B}_{n\rho} f \ge B_{n,\infty}^{[1]} \mathcal{B}_{n\sigma} f, 1 \le \rho \le \sigma, f \text{ convex}, \end{aligned} \tag{32}$$

where $B_{n,\infty}^{[1]}$ are the classical Baskakov operators.

Since, $B_{n,\rho}^{[1]} = B_{n,\infty}^{[1]} \circ \mathscr{B}_{n\rho}$, we infer that

$$B_{n,\sigma}^{[1]} f \le B_{n,\rho}^{[1]} f, \text{fconvex}. \tag{33}$$

The direct and strong converse results are known also in this case (see [11], Theorem 1.2, Theorem 1.3 and [12], Theorem 1.1):

$$\left\| B_{n,\rho}^{[1]} f - f \right\|_\infty \sim \omega_\varphi^2 \left(f, n^{-1/2} \right), \rho \in \{1, \infty\}. \tag{34}$$

Thus, for f convex,

$$\left\| B_{n,\rho}^{[1]} f - \mathrm{f} \right\|_\infty \sim \omega_\varphi^2 \left(f, n^{-1/2} \right), 1 \le \rho \le \infty. \tag{35}$$

5. An Application of Ohlin's Lemma

For more details about the techniques used in this section, the reader is referred to [13] and the references therein.

Lemma 2 (Ohlin's Lemma) (see [14]). *Let X and Y be two random variables on the same probability space such that $E\,X = EY$. If the distribution functions F_X and F_Y cross exactly one time, i.e., for some x_0 holds*

$$F_X(x) \le F_Y(x) if x < x_0 and\ F_X(x) \ge F_Y(x) if x > x_0, \tag{36}$$

then $Ef(X) \le Ef(Y)$, for all convex functions $f : \mathbb{R} \to \mathbb{R}$.

We have $\int_{I_c} \mu_{n,j,\rho}^{[c]}(t)dt = 1$ and $\int_{I_c} t\mu_{n,j,\rho}(t)dt = j/n$.

Therefore, $\mu_{n,j,\rho}^{[c]}$ is the probability density function of a random variable $X_{n,j,\rho}^{[c]}$ with expectation $EX_{n,j,\rho}^{[c]} = j/n$ and probability distribution function $G_{n,j,\rho}^{[c]}(x) = \int_0^x \mu_{n,j,\rho}^{[c]}(t)dt$.

Let $\rho < \sigma$. We will apply Ohlin's Lemma to the random variables $X_{n,j,\rho}^{[c]}$ and $X_{n,j,\sigma}^{[c]}$. Since their expectation is equal, we have to prove that $G_{n,j,\rho}^{[c]}$ and $G_{n,j,\sigma}^{[c]}$ cross exactly ones.

Let

$$g(x) = G_{n,j,\rho}^{[c]}(x) - G_{n,j,\sigma}^{[c]}(x), x \in I_c. \tag{37}$$

Then, $g'(x) = \mu_{n,j,\rho}^{[c]}(x) - \mu_{n,j,\sigma}^{[c]}(x)$.

In what follows, we suppose $c \ne 0$; the case $c = 0$ can be treated similarly or we can consider $c \to 0$ in the computations below. For $c \ne 0$, we have

$$\begin{aligned} g'(x) &= K_1 x^{j\rho-1}(1+cx)^{-(np/c+j\rho+1)} - K_2 x^{j\sigma-1}(1+cx)^{-(n\sigma/c+j\sigma+1)} \\ &= K_2 x^{j\rho-1}(1+cx)^{-\left(np/c^{np}_{c}+j\rho+1\right)} \left\{ \frac{K_1}{K_2} - \left[x^j (1+cx)^{-(n/c+j)} \right]^{\sigma-\rho} \right\} \end{aligned} \tag{38}$$

with positive constants K_1 and K_2.

On int (I_c), the equation $g'(x) = 0$ is equivalent to $(K_1/K_2)^{1/\sigma-\rho}(1+cx)^{n/c+j} = x^j$.

First, suppose that $j > 0$ and, if $c < 0$, $j < -n/c$. Then, on int (I_c), the equation $g'(x) = 0$ is equivalent to $h(x) = x$, where

$$h(x) := \left(\frac{K_1}{K_2} \right)^{1/j(\sigma-\rho)} (1+cx)^{1+n/cj}, x \in \text{int}\ (I_c) \tag{39}$$

is a strictly convex function. The equation $h(x) = x$ has at most two roots in int (I_c), and so the derivative g' has at most two zeroes in int (I_c). If $c < 0$, $g(0) = g(-1/c) = 0$; if $c > 0$, $g(0) = \lim_{x\to\infty} g(x) = 0$. Therefore, g' has at least one zero in int (I_c).

Suppose that g' has exactly one zero in int (I_c), let it be x_0. Then, g' has opposite signs in the two intervals determined by x_0. But (38) shows that g' is positive near the endpoints of I_c. This contradiction leads us to the conclusion that g' has exactly two zeroes $x_1 < x_2$ in int (I_c); they are also roots of the equation $x = h(x)$, $h(x)$ being a strictly convex function. Moreover, g' is positive outside of (x_1, x_2) and negative inside it. Therefore, $g(x)$ is strictly increasing for $x < x_1$ and for $x > x_2$, and strictly decreasing for $x_1 < x < x_2$, with $g(x_1) > 0$ and $g(x_2) < 0$.

We conclude that there exists $x_0 \in (x_1, x_2)$ with $g(x_0) = 0$, $g(x) \ge 0$ for $x < x_0$, and $g(x) \le 0$ for $x > x_0$. Remembering that $g(x) = G_{n,j,\rho}^{[c]}(x) - G_{n,j,\sigma}^{[c]}(x)$, we see that

$$\begin{aligned} G_{n,j,\sigma}^{[c]}(x) &\le G_{n,j,\rho}^{[c]}(x),\ x < x_0, \\ G_{n,j,\sigma}^{[c]}(x) &\ge G_{n,j,\rho}^{[c]}(x),\ x > x_0. \end{aligned} \tag{40}$$

Now Ohlin's Lemma shows that

$$Ef\left(X_{n,j,\sigma}^{[c]} \right) \le Ef\left(X_{n,j,\rho}^{[c]} \right), \tag{41}$$

i.e.,

$$\int_{I_c} \mu_{n,j,\sigma}^{[c]}(t) f(t) dt \le \int_{I_c} \mu_{n,j,\rho}^{[c]}(t) f(t) dt, \tag{42}$$

for f convex, $j > 0$, and if $c < 0, j < -n/c$.

With notation from (5), this means that

$$F_{n,j,\sigma}^{[c]}(f) \le F_{n,j,\rho}^{[c]}(f), f \text{convex}. \tag{43}$$

Moreover, (5) shows that the above relation is an equality if $j = 0$ and, for $c < 0$, if $j = -n/c$. Now according to (4),

$$B_{n,\sigma}^{[c]} f \le B_{n,\rho}^{[c]} f, f \text{convex}, 0 < \rho < \sigma. \tag{44}$$

Letting $\sigma \to \infty$, we see that the above inequality is valid for $0 < \rho < \sigma \le \infty$. This was proved with other methods, for $c \in \{-1, 0, 1\}$, in the preceding sections.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no competing financial interests.

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