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Research Article

Optimal Expected Utility of Dividend Payments with Proportional Reinsurance under VaR Constraints and Stochastic Interest Rate

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In this paper, we consider the problem of maximizing the expected discounted utility of dividend payments for an insurance company taking into account the time value of ruin. We assume the preference of the insurer is of the CRRA form. The discounting factor is modeled as a geometric Brownian motion. We introduce the VaR control levels for the insurer to control its loss in reinsurance strategies. By solving the corresponding Hamilton-Jacobi-Bellman equation, we obtain the value function and the corresponding optimal strategy. Finally, we provide some numerical examples to illustrate the results and analyze the VaR control levels on the optimal strategy.

1. Introduction

In recent years, dividend optimization problems for insurance company have attracted extensive attention. The problem of optimal dividend was proposed by De Finetti in 1957. He suggested that a company would seek to find a strategy in order to maximize the accumulated value of expected discounted dividends up to the ruin time. Since then, many results on optimal dividend problems have been obtained. Some literatures on the optimal dividend include Avanzi et al. [1], Avanzi et al. [2], Avram et al. [3], Bayraktar et al. [4], Dai et al. [5], Dai et al. [6], Hubalek and Schachermayer [7], Li and Wu [8], Loeffen [9], Ng [10], Thonhauser and Albrecher [11], Yao et al. [12], and Yin and Wen [13].

On the other hand, reinsurance can effectively reduce the risk, so reinsurance is an important way for an insurer to control its risk exposure. Many interesting results have been obtained under reinsurance strategy in recent years. See, for example, Azcue and Muler [14], who studied the optimal reinsurance in the framework of Cramér-Lundberg model. Zhou and Yuen [15] considered reinsurance for a diffusion model with capital injection under variance premium principle. For some related discussions, among others, we refer the reader to Li and Shen [16], Liang and Palmowski [17], Wen and Yin [18], Wu [19], and Yao et al. [20].

In practice, Value-at-Risk (VaR) is often used to measure and control the risk of an insurance company. Recently, Chen et al. [21] investigated the optimal reinsurance strategies and the minimum probability of ruin with VaR constraints. Zhang et al. [22] discussed the optimal reinsurance strategies and the survival probability under dynamic VaR constraints. Bi and Cai [23] studied the equilibrium strategies under the mean-variance criterion with VaR constraints. Other works about optimal reinsurance problems with VaR constraints can be found in Cai and Tan [24], Cai et al. [25], Cheung et al. [26], and Yiu [27].

The interest rate is the main factor of an insurance company. And, it is affected by many factors. So, we must pay attention to the current situation of interest rate and its changing trend. It is more reasonable to assume that the interest rate is a function of time. Eisenberg [28] solved the optimal dividends under a stochastic interest rate for the case of the Brownian motion. In this paper, we are going to study the problem of maximizing the expected discounted utility of dividend, taking into account both reinsurance under VaR constraints and a stochastic interest rate. The insurer's surplus process is approximated by a Brownian motion with drift. By solving the corresponding Hamilton-Jacobi-Bellman equation, we obtain the value function and the corresponding optimal strategies with and without VaR constraints.

This paper is organized as follows. In Section 2, we provide a general formulation of the optimal reinsurance-dividend problem. Then, we investigate the problem of the expected discounted utility of dividend maximization under noncheap reinsurance assumption. We give the explicit expressions for the expected discounted utility of dividend payments and the optimal strategies in Section 3. In Section 4, we consider the optimization problem with VaR constraints and solve the optimization problem using the results derived in Section 3. We also present some numerical examples to illustrate the results in Section 5.

2. Model Settings and Problem Formulations

Let $(\Omega, \mathscr{F}, P)$ be a probability space equipped with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$.

2.1. Reserve Process of an Insurer. The reserve process $\{R_t\}_{t\geq 0}$ of an insurer is modeled by

$$dR_t = (1+\theta)a dt - dY_t,\ R_0 = x \tag{1}$$

for the aggregate cumulative amount of claims counted up to time t:

$$dY_t = a dt - b dW_t, \quad Y_0 = 0, \tag{2}$$

where a and b are positive constants, $x \geq 0$ is the initial surplus, $(1+\theta)a$ is the premium rate with the safety loading $\theta > 0$, and $\{W_t\}_{t\geq 0}$ is a standard Brownian motion.

In the following, the insurer is allowed to pay dividends. The cumulated dividends are described by a process $C = (C_t)_{t\geq 0}$, which is an adapted and nondecreasing process. C_t represents the total dividends up to time t. In this paper, we assume that $(C_t)_{t\geq 0}$ is absolutely continuous with respect to the Lebesgue measure. We suppose that the process C admits almost surely a density process denoted by $c_t \geq 0$ modeling the intensity of the dividend payments in continuous time. To manage the risk, the insurer could choose a reinsurance policy q_t at time t. When $q_t \in [0,1]$, it means the insurer purchases proportional reinsurance. In this case, the insurer only pays its $(1-q_t)Y_t$, while the reinsurer pays the rest $q_t Y_t$. A strategy π is said to be admissible, if the ruin cannot be caused by the dividend payment, and (q_t, c_t) is $\mathscr{F}_t$-progressively measurable and satisfies $0 \leq q_t \leq 1$, $c_t \geq 0$ for all $t \geq 0$. We denote the set of all admissible strategies by Π. The set of admissible strategies consists of nonnegative, nondecreasing, adapted, cádlág process. Let X_t^π denote the insurers total wealth at time t under the strategy $\pi_t = (q_t, c_t)$. Then, the dynamic of X_t^π is given by

$$\begin{aligned} dX_t^\pi &= ((1+\theta)a - (1-q_t)a)dt + b(1-q_t)dW_t \\ &\quad - (1+\eta)q_t a dt - c_t d = (\theta - \eta q_t)a dt \\ &\quad + b(1-q_t)dW_t - c_t dt,\ t \geq 0, \end{aligned} \tag{3}$$

where $\eta(>\theta)$ is the safety loading of the reinsurer and the condition of $\eta > \theta$ is required for avoiding the insurers arbitrage.

As a risk measure, we consider that dividends are discounted by the geometric Brownian motion

$$e^{\{-r-mt-\delta B_t\}}, \quad \text{if} \quad m > \frac{\delta^2}{2}, \tag{4}$$

where $r > 0$, $m > 0$, and $\delta \geq 0$. Let further $\{B_t\}_{t\geq 0}$ be a standard Brownian motion independent of $\{W_t\}_{t\geq 0}$. We suppose that the underlying filtration $\{\mathscr{F}_t\}$ is generated by $\{W_t, B_t\}$. The time of ruin for this process is defined by $\tau^\pi = \inf\{t \geq 0 : X_t^\pi \leq 0\}$. We define the return function corresponding to π to be

$$V^\pi(r,x) = \mathbb{E}_x\left[\int_0^{\tau^\pi} e^{-r-ms-\delta B_s} u(c_s) ds + \int_0^{\tau^\pi} e^{-r-ms-\delta B_s} \lambda ds\right], \tag{5}$$

where u is some fixed utility function, $\lambda > 0$, $\mathbb{E}_x$ means expectation with respect to $\mathbb{P}_x(\cdot) = P(\cdot | X_0 = x)$. Usually, it is assumed that $u : \mathbb{R}_{t\geq 0} \to \mathbb{R}_{t\geq 0}$ is differentiable and nonnegative. $e^{-\beta s}\lambda$ can be interpreted as the present value of an amount which the insurer earns as long as the company is alive. In this way, the lifetime of the portfolio becomes part of the value function and is weighted according to the choice of λ. We seek to find an optimal strategy π^* so that

$$V(r,x) = \max_{\pi\in\Pi} V^\pi(r,x) = V^{\pi^*}(r,x). \tag{6}$$

In the present paper, we consider the Constant Relative Risk Aversion (CRRA) utility function:

$$u(c) = \frac{c^p}{p}, \quad p \in (0.1). \tag{7}$$

2.2. Value-at-Risk Constraints for the Proportional Reinsurance and Dividend Strategy. Under the reinsurance-dividend strategy $\pi_t = (q_t, c_t)$, the insurer's wealth process $\{X_t^\pi, t \geq 0\}$ is a risk process. The insurer may or has to use the risk measure of VaR to control its wealth for avoiding huge loss. For $h > 0$ is small enough and for s in the interval $[t, t+h)$, we approximate π_s by π_t, $s \in [t, t+h)$. This is a reasonable approximation since the insurance strategy can only be adjusted at discrete time, and the decision is made based on the surplus at time t. Thus, the loss of the insurer in time interval $[t, t+h]$ can be expressed as $\Delta X^{\pi}_{t,h} := X^{\pi}_t - X^{\pi}_{t+h}$. Then, the SDE (3) admits a solution

$$X^{\pi}_{t+h} = X^{\pi}_t + \int_t^{t+h} (\theta - \eta q_s) a ds + \int_t^{t+h} b(1-q_s) dW_s - \int_t^{t+h} c_s d = X^{\pi}_t + (\theta - \eta q_t) ah - c_t h + b(1-q_t) \int_t^{t+h} dW_s. \tag{8}$$

Thus,

$$\Delta X^{\pi}_{t,h} = -(\theta - \eta q_t) ah + c_t h - b(1-q_t) \int_t^{t+h} dW_s. \tag{9}$$

For a given risk level $k \in (0,1)$ and a given horizon $h > 0$, we denote the conditional VaR of $\Delta X^{\pi}_{t,h}$ conditioning on $\mathcal{F}_t$ by $VaR^{k,h,\pi}_t$, namely,

$$\mathrm{VaR}^{k,h,\pi}_t := \inf \left\{ L \in \mathbb{R}; \quad \mathbb{P}\left(\Delta X^{\pi}_{t,h} \geq L \mid \mathcal{F}_t\right) \leq k \right\}. \tag{10}$$

Proposition 1. *Given risk level $k \in (0, 1)$ and time length $h > 0$. We have*

$$VaR^{k,h,\pi}_t = c_t h - (\theta - \eta q_t) ah - \Phi^{-1}(k) b(1-q_t)\sqrt{h}, \tag{11}$$

where $\Phi^{-1}(\cdot) = \inf\{x \in \mathbb{R} : \Phi(x) \geq k\}$ is the inverse function of the cumulative standard normal distribution function $\Phi(x) = \int_{-\infty}^{x} (1/\sqrt{2\pi}) e^{-z^2/2} dz$.

Proof. We have

$$\begin{aligned} \mathbb{P}\left(\Delta X^{\pi}_{t,h} \geq L \mid \mathcal{F}_t\right) &= \mathbb{P}\left(-(\theta - \eta q_t) ah + c_t h - b(1-q_t)\int_t^{t+h} dW_s \geq L \mid \mathcal{F}_t\right) \\ &= \mathbb{P}\left(b(1-q_t)\int_t^{t+h} dW_s \leq c_t h - (\theta - \eta q_t) ah - L \mid \mathcal{F}_t\right) \\ &= \mathbb{P}\left(\frac{1}{\sqrt{h}}\int_t^{t+h} dW_s \leq \frac{c_t h - (\theta - \eta q_t) ah - L}{\sqrt{h} b(1-q_t)} \mid \mathcal{F}_t\right) \\ &= \Phi\left(\frac{c_t h - (\theta - \eta q_t) ah - L}{\sqrt{h} b(1-q_t)}\right), \end{aligned} \tag{12}$$

where the last equality follows from the fact that $1/\sqrt{h}\int_t^{t+h} dW_s$ conditional on the filtration $\mathcal{F}_t$ is a standard normal random variable. Then, we have

$$\begin{aligned} \mathbb{P}\left(\Delta X^{\pi}_{t,h} \geq L \mid \mathcal{F}_t\right) \leq k &\Leftrightarrow \Phi\left(\frac{c_t h - (\theta - \eta q_t) ah - L}{\sqrt{h} b(1-q_t)}\right) \leq k \\ &\Leftrightarrow L \geq c_t h - (\theta - \eta q_t) ah - \Phi^{-1}(k) b(1-q_t)\sqrt{h}. \end{aligned} \tag{13}$$

Thus, we obtain (11).

In this paper, we will derive the optimal strategy under the constraint

$$\mathrm{VaR}^{k,h,\pi}_t \leq \bar{\mathrm{VaR}}, \quad t \geq 0. \tag{14}$$

3. Solution to the Optimization Problem without VaR Constraints

In this section, we will solve the optimal reinsurance-dividend problem without VaR constraints. The HJB equation corresponding to the problem (6) is given by

$$\max_{0 \leq q \leq 1, c \geq 0} \left\{ [(\theta - \eta q) a - c] V_x + \frac{1}{2} b^2 (1-q)^2 V_{xx} + m V_r + \frac{\delta^2}{2} V_{rr} + e^{-r}(u(c) + \lambda) \right\} = 0, \tag{15}$$

with the boundary condition

$$V(0,0) = 0. \tag{16}$$

We conjecture that the value function can be written in the form $V(r,x) = e^{-r} F(x)$. In this case, the HJB equation becomes

$$\max_{0 \leq q \leq 1, c \geq 0} \left\{ [(\theta - \eta q) a - c] F'(x) + \frac{1}{2} b^2 (1-q)^2 F''(x) + \left(\frac{\delta^2}{2} - m\right) F(x) + u(c) + \lambda \right\} = 0. \tag{17}$$

Note that supremum in (17) without constraints is realized for

$$\begin{aligned} q(x) &= 1 + \frac{\eta a}{b^2} \frac{F'(x)}{F''(x)}, \\ c(x) &= \left(F'(x)\right)^{-\frac{1}{1-p}}. \end{aligned} \tag{18}$$

Denote $\alpha = 1 + (2b^2(m - \delta^2/2)/\eta^2 a^2)$. Let

$$B = -\frac{2b^2}{\eta^2 a^2} \frac{1-p}{1-\alpha+\alpha p} > 0, \tag{19}$$

$$D = \frac{2b^2}{\alpha \eta^2 a} (\eta - \theta) > 0. \tag{20}$$

We will solve our optimization problem under the assumption:

Assumption 2. For noncheap reinsurance, assume that $\eta > \theta$, $\alpha(1-p) > 1$ and $(2-\alpha(1-p))\eta - 2\theta < 0$.

For the optimal value function $V(r,x)$, we can derive the following property.

Lemma 3. *The optimal value function $V(r,x)$ satisfies $V_x \geq 0$, $V_r \leq 0$, and $V_{xx} < 0$.*

Proof. It is easy to see that $V(r,x)$ is increasing in x and decreasing in r. We prove the property of $V_{xx} < 0$. For $0 < y < x$, denote $\tau_y^\pi = \inf\{t : X_t^\pi = y\}$. From the dynamic programming principle, we claim that $V(r,x)$ satisfies the following equality:

$$V(r,x) = \max_{\pi\in\Pi} \mathbb{E}_x \left[\int_0^{\tau_y^\pi} e^{-r-ms-\delta B_s}(u(c_s)+\lambda)ds + e^{-m\tau_y^\pi} V(r,y) \right]. \tag{21}$$

We first show the $\leq$ part. Let $\varepsilon > 0$. There exists $\tilde{\pi} \in \Pi$ such that $V^{\tilde{\pi}}(r,x) > V(r,x) - \varepsilon$. We choose the strategy $\bar{\pi} = \pi \mathbf{1}_{s\in[0,\tau_y)} + \tilde{\pi}\mathbf{1}_{s\in[\tau_y,\tau)}$ where π is an admissible strategy. Then,

$$\begin{aligned}
V(r,x) &\geq V^{\bar{\pi}}(r,x) \\
&= \mathbb{E}_x \left[\int_0^{\tau_y^\pi} e^{-r-ms-\delta B_s}(u(c_s)+\lambda)d + \int_{\tau_y^\pi}^{\tau^\pi} e^{-r-ms-\delta B_s}(u(c_s)+\lambda)ds \right] \\
&= \mathbb{E}_x \left[\int_0^{\tau_y^\pi} e^{-r-ms-\delta B_s}(u(c_s)+\lambda)ds + \int_0^{\tau^\pi-\tau_y^\pi} e^{-r-m(s+\tau_y^\pi)-\delta B_{s+\tau_y^\pi}} \right. \\
&\quad \left. \cdot \left(u\left(c_{s+\tau_y^\pi}\right) + \lambda \right) ds \right] \\
&= \mathbb{E}_x \left[\int_0^{\tau_y^\pi} e^{-r-ms-\delta B_s}(u(c_s)+\lambda)ds + e^{-m\tau_y^\pi} V^{\tilde{\pi}}(r,y) \right] \\
&\geq \mathbb{E}_x \left[\int_0^{\tau_y^\pi} e^{-r-ms-\delta B_s}(u(c_s)+\lambda)ds + e^{-m\tau_y^\pi} V(r,y) \right] - \varepsilon,
\end{aligned} \tag{22}$$

and therefore, we have

$$V(r,x) \geq \max_{\pi\in\Pi} \mathbb{E}_x \left[\int_0^{\tau_y^\pi} e^{-r-ms-\delta B_s}(u(c_s)+\lambda)ds + e^{-m\tau_y^\pi} V(r,y) \right]. \tag{23}$$

For the other direction, since

$$\begin{aligned}
V^\pi(r,x) &= \mathbb{E}_x \left[\int_0^{\tau_y^\pi} e^{-r-ms-\delta B_s}(u(c_s)+\lambda)ds + e^{-m\tau_y^\pi} V^\pi(r,y) \right] \\
&\leq \mathbb{E}_x \left[\int_0^{\tau_y^\pi} e^{-r-ms-\delta B_s}(u(c_s)+\lambda)ds + e^{-m\tau_y^\pi} V(r,y) \right],
\end{aligned} \tag{24}$$

we have

$$V(r,x) \leq \max_{\pi\in\Pi} \mathbb{E}_x \left[\int_0^{\tau_y^\pi} e^{-r-ms-\delta B_s}(u(c_s)+\lambda)ds + e^{-m\tau_y^\pi} V(r,y) \right]. \tag{25}$$

Thus, (21) is proved.

For $h > 0$, let Π^h be the set of strategies π such that

$$\int_0^\zeta (\theta - \eta q_s)ads + \int_0^\zeta b(1-q_s)dW_s - \int_0^\zeta c_s ds = -h, \tag{26}$$

on the set $\zeta < \infty$, where ζ is a stopping time defined by

$$\zeta = \inf\left\{ t \geq 0 : \int_0^t (\theta - \eta q_s)ads + \int_0^t b(1-q_s)dW_s - \int_0^t c_s ds = -h \right\}. \tag{27}$$

By putting $h = x - y$ from (21), we obtain that

$$V(r,x) = \max_{\pi\in\Pi} \mathbb{E}_x \left[\int_0^\zeta e^{-r-ms-\delta B_s}(u(c_s)+\lambda)ds + e^{-m\zeta} V(r,y) \right]. \tag{28}$$

Then,

$$\begin{aligned}
V(r,x) - V(r,x-h) &= \max_{\pi\in\Pi} \mathbb{E}_x \left[\int_0^\zeta e^{-r-ms-\delta B_s}(u(c_s)+\lambda)ds + \left(e^{-m\zeta}-1\right) V(r,x-h) \right] \\
&= \max_{\pi\in\Pi} \left\{ \mathbb{E}_x \left[\int_0^\zeta e^{-r-ms-\delta B_s}(u(c_s)+\lambda)ds \right] \right. \\
&\quad \left. + \mathbb{E}_x \left[e^{-m\zeta} - 1 \right] V(r,x-h) \right\}.
\end{aligned} \tag{29}$$

Since $\mathbb{E}_x[e^{-m\zeta}] < 1$ and $V(r,x)$ is a nondecreasing function of x, the right-hand side of (29) is a decreasing function of x. Thus, $V(r,x) - V(r,x-h)$ is decreasing in x. Hence, V_x is also decreasing and $V_{xx} < 0$. This finishes the proof.

Proposition 4. *If Assumption 2 is true, then on $(0, x^*]$, the function v solving (15) is given by*

$$v(r,x) = \frac{e^{-r}}{m - \delta^2/2} \left\{ \left[\left(\frac{1}{\alpha} - 1 \right)(\eta - \theta)a + \frac{(1-p)^2(1-\alpha)}{p(1-\alpha+\alpha p)} e^{\frac{\xi}{1-p}} \right] e^{-\xi} + \lambda \right\}, \tag{30}$$

where B and D are given in (19) and (20) and $\xi = g^{-1}(x)$, where g^{-1} is the inverse of function g:

$$g(\xi) = (1-p)Be^{\frac{\xi}{1-p}} + D\xi + Q_1, \tag{31}$$

for $Q_1 = -(1-p)Be^{\xi_0/1-p} - D\xi_0$ *and* ξ_0 *satisfies*

$$e^{-\xi_0}\frac{1-\alpha}{\alpha}(\eta-\theta)a + \frac{(1-p)^2(1-\alpha)}{p(1-\alpha-\alpha p)}e^{\frac{p\xi_0}{1-p}} + \lambda = 0. \quad (32)$$

Above

$$x^* = g(\xi^*), \quad (33)$$

where

$$\xi^* = (1-p)\ln\left(\frac{b^2-\eta a D}{\eta a B}\right). \quad (34)$$

The maximizers $q^*(x)$ and $c^*(x)$ are given by

$$\begin{aligned} q^*(x) &= 1 - \frac{\eta a}{b^2}g'(\xi), \\ c^*(x) &= (e^r v_x)^{-\frac{1}{1-p}}. \end{aligned} \quad (35)$$

Proof. From the expression of $q(x) = 1 + (\eta a/b^2)(F'(x)/F''(x))$ and $c(x) = (F'(x))^{-1/1-p}$, we have that the HJB equation of (17) becomes

$$\begin{aligned} &(\theta-\eta)aF'(x) - \frac{\eta^2a^2}{2b^2}\frac{(F'(x))^2}{F''(x)} + \left(\frac{\delta^2}{2}-m\right)F(x) \\ &+ \frac{1-p}{p}\left(F'(x)\right)^{-\frac{p}{1-p}} + \lambda = 0. \end{aligned} \quad (36)$$

Letting $x = g(\xi)$ and $F'(g(\xi)) = e^{-\xi}$, then we have $F''(g(\xi)) = -e^{-\xi}/g'(\xi)$. Plugging it into (36) and taking derivatives with ξ produce

$$(\theta-\eta)ae^{-\xi} + \frac{\eta^2a^2}{2b^2}g'(\xi)e^{-\xi} + \left(\frac{\delta^2}{2}-m\right)F(g(\xi)) + \frac{1-p}{p}e^{\frac{p}{1-p}\xi} + \lambda = 0. \quad (37)$$

We take derivatives with ξ, and we get

$$\frac{\eta^2a^2}{2b^2}g''(\xi) - \left(\frac{\eta^2a^2}{2b^2} + m - \frac{\delta^2}{2}\right)g'(\xi) - (\theta-\eta)a + e^{\frac{1}{1-p}\xi} = 0. \quad (38)$$

Denote $h(\xi) = g'(\xi)$, we have

$$h(\xi) = g'(\xi) = Ae^{\alpha\xi} + Be^{\frac{\xi}{1-p}} + D, \quad (39)$$

where $A = h(0) - (2b^2/\alpha\eta^2a)(\eta-\theta) + (2b^2/\eta^2a^2)(1-p/1-\alpha+\alpha p)$. In view of $F''(x) < 0$, the value of A should be nonnegative. Then, $h(\xi) > 0$. If A is strictly positive, from (37) and (38), we get $F(x) \sim K_0 x^{\alpha-1/\alpha}$, where K_0 is some fixed constant. Note that

$$\begin{aligned} F(x) &\le \mathbb{E}\left\{\int_0^\infty \left[e^{r-ms-\delta B_s}u(x+(\theta+1)as+bB_s)+\lambda\right]ds\right\} \\ &\le Kx^p + \frac{\lambda e^{-r}}{m-\delta^2/2}, \end{aligned} \quad (40)$$

where $K > 0$ is some fixed constant. Therefore, by Assumption 2, we must choose A to be equal zero. Recall that $h(\xi) = g'(\xi)$ and (37), we obtain the representation of the function

$$\begin{aligned} F(x) &= \frac{\lambda}{m-\delta^2/2} + \frac{e^{-\xi}}{m-\delta^2/2}\left[-(\eta-\theta)a + \frac{\eta^2a^2}{2b^2}g'(\xi) + \frac{1-p}{p}e^{\frac{\xi}{1-p}}\right] \\ &= \frac{\lambda}{m-\delta^2/2} + \frac{e^{-\xi}}{m-\delta^2/2}\left[-(\eta-\theta)a + \frac{\eta^2a^2}{2b^2}Be^{\frac{\xi}{1-p}} + \frac{\eta^2a^2}{2b^2}D + \frac{1-p}{p}e^{\frac{\xi}{1-p}}\right]. \end{aligned} \quad (41)$$

Thus, we have $v(r,x)$ given in (30). Since $F(0) = 0$ and $g'(\xi_0) = Be^{\xi_0/1-p} + D$, we get (31) and (32).

Since

$$\begin{cases} q(x) = 1 - \dfrac{\eta a}{b^2}g'(\xi), \\ g'(\xi) > 0, \end{cases} \quad (42)$$

we get $q(x) < 1$. Since $g''(\xi) = (B/1-p)e^{\xi/1-p} > 0$, to satisfy the required condition $q(x) > 0$, we need to show that the inequality $(\eta a/b^2)g'(\xi) < 1$ is true. It is easy to verify that ξ^* given in (34) is the unique solution of the equation $(\eta a/b^2)g'(\xi) = 1$. In view of Assumption 2 and $(\eta a/b^2)g''(\xi) = 2(\eta-\theta)/\alpha\eta(1-p)$, we have $(\eta a/b^2)g'(\xi_0) < 1$, so we have $0 < g'(\xi) < b^2/\eta a$ for $\xi_0 < \xi < \xi^*$, i.e., $0 < q(x) < 1$ on $(0, x^*]$.

It follows from the facts $F'(x) = e^{-\xi} > 0$ and $F''(x) = -e^{-\xi}/g'(\xi) < 0$ that $F(x)$ is increasing and strictly concave. Substituting the values of F' and F'' into the left of (17), we have

$$\begin{aligned} &\max_{0\le q\le 1, c\ge 0}\left\{(\theta-\eta q)aF'(x) + \frac{1}{2}b^2(1-q)^2F''(x) + \left(\frac{\delta^2}{2}-m\right)F(x) - cF'(x) + u(c) + \lambda\right\} \\ &= \max_{0\le q\le 1, c\ge 0}\left\{(\theta-\eta q)ae^{-\xi} + \frac{1}{2}b^2(1-q)^2\frac{e^{-\xi}}{g'(\xi)} + \left(\frac{\delta^2}{2}-m\right)F(x) - ce^{-\xi} + u(c) + \lambda\right\}. \end{aligned} \quad (43)$$

Since $u(\cdot)$ is an increasing and concave function and $0 < g'(\xi) < b^2/\eta a$ for $\xi_0 < \xi < \xi^*$, therefore, the maximizers

q^* and c^* are given by $q^*(x) = 1 - (\eta a/b^2)g'(\xi)$ and $c(x) = e^{\xi/1-p}$ with $x = g(\xi)$. Hence, Equation (43) can be rewritten as

$$(\theta - \eta)ae^{-\xi} + \frac{\eta^2 a^2}{2b^2} g'(\xi)e^{-\xi} + \left(\frac{\delta^2}{2} - m\right)F(x) + \frac{1-p}{p} e^{\frac{p}{1-p}\xi} + \lambda = 0. \tag{44}$$

It follows from (30) and (31) that (44) is true. This finishes the proof.

Since $q^*(x) \le 0$ on the interval (x^*, ∞), it is reasonable to let $q^*(x) = 0$; then, the HJB equation is equivalent to the following equation:

$$\theta a v_x + \frac{1}{2} b^2 v_{xx} + m v_r + \frac{\delta^2}{2} v_{rr} - c v_x + e^{-r} u(c) + \lambda = 0. \tag{45}$$

We also conjecture that the value function can be written in the form $V(x, r) = e^{-r}F(x)$. In this case, the HJB equation of (45) becomes

$$\theta a F'(x) + \frac{1}{2} b^2 F''(x) - \left(m - \frac{\delta^2}{2}\right)F(x) + \frac{1-p}{p}\left(F'(x)\right)^{-\frac{p}{1-p}} + \lambda = 0. \tag{46}$$

with the boundary conditions

$$F'(x^*) = e^{-\xi^*}, \tag{47}$$

$$\frac{F'(x^*)}{F''(x^*)} = -\frac{b^2}{\eta a}. \tag{48}$$

If we set $F(x) = G(x) + (\lambda/m - \delta^2/2)$, then the HJB equation of (46) becomes

$$\theta a G'(x) + \frac{1}{2} b^2 G''(x) - \left(m - \frac{\delta^2}{2}\right)G(x) + \frac{1-p}{p}\left(G'(x)\right)^{-\frac{p}{1-p}} = 0, \tag{49}$$

with the boundary conditions

$$G'(x^*) = e^{-\xi^*} \text{ and } \frac{G'(x^*)}{G''(x^*)} = -\frac{b^2}{\eta a}. \tag{50}$$

Similar to the proof of Proposition 5 in Liang and Palmowski [17], we have the following results whose proof is omitted.Summarizing Propositions 4 and 5 gives the following result.

Proposition 5. *Assume that Assumption 2 holds. Suppose also that F on (x^*, ∞) solves (46)–(48). Then $F'(x) > 0$, $F''(x) < 0$ on (x^*, ∞), and the the maximum in*

$$\max_{0\le q\le 1}\left\{(\theta - \eta q)aF'(x) + \frac{1}{2}b^2(1-q)^2F''(x)\right\} = 0 \tag{51}$$

is attained when $q = 0$ in this case.

Theorem 6. *Assume that Assumption 2 holds. Then the value function $V(r, x)$ is given by*

$$V(r,x) = \begin{cases} \dfrac{e^{-r}}{m - \delta^2/2}\left\{\left[\left(\dfrac{1}{\alpha} - 1\right)(\eta - \theta)a + \dfrac{(1-p)^2(1-\alpha)}{p(1-\alpha+\alpha p)} e^{\frac{\xi}{1-p}}\right]e^{-\xi} + \lambda\right\}, & \text{if } 0 < x \le x^*, \\ e^{-r}F(x), & \text{if } x > x^*, \end{cases} \tag{52}$$

with B and D given in (19) and (20) and $x = g(\xi)$, where g is defined in (31). Moreover, x^* is given in (33), and $F(x)$ solves (46). The corresponding optimal dividend strategy is

$$c^*(x) = (e^r v_x)^{-\frac{1}{1-p}}, \tag{53}$$

and the optimal reinsurance proportion is

$$q^*(x) = \begin{cases} 1 - \dfrac{\eta a}{b^2} g'(\xi), & \text{if } 0 < x \le x^*, \\ 0, & \text{if } x > x^*. \end{cases} \tag{54}$$

Proof. If $0 < x \le x^*$, denote $V(r, x) = e^{-r}F(x)$, where

$$F(x) = \frac{\lambda}{m - \delta^2/2} + \frac{e^{-\xi}}{m - \delta^2/2}\left[-(\eta - \theta)a + \frac{\eta^2 a^2}{2b^2} B e^{\frac{\xi}{1-p}} + \frac{\eta^2 a^2}{2b^2} D + \frac{1-p}{p} e^{\frac{\xi}{1-p}}\right]. \tag{55}$$

Since $F(x)$ is a continuously differentiable solution of Equation (17), it is easy to obtain that $e^{-r}F(x)$ is a solution of (15). If $x > x^*$, denote $V(r, x) = e^{-r}F(x)$, where $F(x)$ is a continuously differentiable solution of Equation (36); it is easy to obtain that $e^{-r}F(x)$ is a solution of (15).

Table 1: Parameters of the insurance market and the geometric Brownian motion.

a	b	p	η	θ	r	m	δ	λ
0.03	0.5	0.01	0.8	0.4	1	2	1	0.5

From now on, we prove the optimality of strategy

$$\pi^* = (q_t^*, c_t^*) = \left(q\left(X_t^{\pi^*}\right), c\left(X_t^{\pi^*}\right)\right). \tag{56}$$

Let π be an admissible strategy with dividend and reinsurance strategy. Select $0 < \xi_1 < x < \xi_2$. Define

$$\tau_x = \inf\{t \ge 0 : X_t^\pi = x\}. \tag{57}$$

It follows from the Itô's formula that we have

$$\begin{aligned}
e^{-r-m\left(t\wedge\tau_{\xi_1}\wedge\tau_{\xi_2}\right)-\delta B_{t\wedge\tau_{\xi_1}\wedge\tau_{\xi_2}}} F\left(X^\pi_{t\wedge\tau_{\xi_1}\wedge\tau_{\xi_2}}\right) &= \int_0^{t\wedge\tau_{\xi_1}\wedge\tau_{\xi_2}} e^{-r-ms-\delta B_s}\left\{F'(X_s^\pi)(\theta-\eta q_s)a + \frac{1}{2}b^2(1-q_s)^2F''(X_s^\pi) - c_sF'(X_s^\pi)\right\}ds \\
&\quad - \int_0^{t\wedge\tau_{\xi_1}\wedge\tau_{\xi_2}} e^{-r-ms-\delta B_s}\left(m-\frac{\delta^2}{2}\right)F(X_s^\pi)d + \delta\int_0^{t\wedge\tau_{\xi_1}\wedge\tau_{\xi_2}} e^{-r-ms-\delta B_s}F(X_s^\pi)dB_s \\
&\quad + \int_0^{t\wedge\tau_{\xi_1}\wedge\tau_{\xi_2}} e^{-r-ms-\delta B_s}F'(X_s^\pi)b(-q_s)dW_s \le e^{-r}F(x) + \int_0^{t\wedge\tau_{\xi_1}\wedge\tau_{\xi_2}} e^{-r-ms-\delta B_s}F'(X_s^\pi)b(1-q_s)dW_s \\
&\quad + \delta\int_0^{t\wedge\tau_{\xi_1}\wedge\tau_{\xi_2}} e^{-r-ms-\delta B_s}F(X_s^\pi)dB_s - \int_0^{t\wedge\tau_{\xi_1}\wedge\tau_{\xi_2}} e^{-r-ms-\delta B_s}(u(c_s)+\lambda)ds.
\end{aligned} \tag{58}$$

Taking expectations on the both sides of the above inequality yields

$$\begin{aligned}
&\mathbb{E}_x\left[e^{-r-m\left(t\wedge\tau_{\xi_1}\wedge\tau_{\xi_2}\right)-\delta B_{t\wedge\tau_{\xi_1}\wedge\tau_{\xi_2}}} F\left(X^\pi_{t\wedge\tau_{\xi_1}\wedge\tau_{\xi_2}}\right)\right] \\
&\quad \le e^{-r}F(x) - \mathbb{E}_x\left[\int_0^{t\wedge\tau_{\xi_1}\wedge\tau_{\xi_2}} e^{-r-ms-\delta B_s}(u(c_s)+\lambda)ds\right].
\end{aligned} \tag{59}$$

Note that the $\pi = \pi^*$ equality holds. We claim that $\{e^{-r-mt-\delta B_t}F(X_t^\pi)\}_{t\ge0}$ is uniformly integrable. Then, letting $\xi_1 \downarrow 0$, $\xi_2 \to \infty$, and by applying the dominated convergence theorem and the monotone convergence theorem, we have

$$\mathbb{E}_x\left[\int_0^\tau e^{-r-ms-\delta B_s}(u(c_s^*)+\lambda)ds\right] = e^{-r}F(x). \tag{60}$$

Thus,

$$e^{-r}F(x)\begin{cases} \ge V^\pi(r,x), & \text{if } \pi \text{ arbitrary}, \\ = V^{\pi^*}(r,x), & \text{if } \pi = \pi^*. \end{cases} \tag{61}$$

And, we can conclude that $V(r,x) \ge V^{\pi^*}(r,x) = e^{-r}F(x) \ge \sup_\pi V^\pi(r,x) = V(r,x)$. By the concavity of F, we have

$$F(x) \le F'(0)x. \tag{62}$$

Thus, for each $T \ge 0$,

$$\begin{aligned}
\mathbb{E}_x\left[e^{-r-mT-\delta B_T}F\left(X_T^{\pi^*}\right)\right] &\le \mathbb{E}_x\left[e^{-r-mT-\delta B_T}\sup_{0\le t\le T}F\left(X_t^{\pi^*}\right)\right] \\
&\le \mathbb{E}_x\left[e^{-r-mT-\delta B_T}F'(0)\sup_{0\le t\le T}F\left(X_t^{\pi^*}\right)\right] \\
&\le e^{-r-\left(m-\frac{\delta^2}{2}\right)T}F'(0)\mathbb{E}_x\left[\sup_{0\le t\le T}X_t^{\pi^*}\right] \\
&\le F'(0)\mathbb{E}_x\left[\sup_{0\le t\le T}X_t^{\pi^*}\right].
\end{aligned} \tag{63}$$

We introduce now a new stochastic process Y_t having the following dynamics:

$$dY_t = \theta a dt + b(1-q_t^*)dW_t, \quad Y_0 = x. \tag{64}$$

Then, one can observe that $\{Y_t\}_{t\ge0}$ is a submartingale and $X_t^{\pi^*} \le Y_t$ for any $t \ge 0$. We denote $Y_T^* = \sup_{0\le t\le T}Y_t$. Then, by the Cauchy-Swartz inequality and Doob's maximal inequality for submartingale, we have

$$\begin{aligned}
\mathbb{E}_x\left[\sup_{0\le t\le T}X_t^{\pi^*}\right] &\le \mathbb{E}_x(Y_T^*) \le \sqrt{\mathbb{E}_x(Y_T^*)^2} \le 2\sqrt{\mathbb{E}_x(Y_T)^2} \\
&\le K_1T + K_2\sqrt{T} < \infty,
\end{aligned} \tag{65}$$

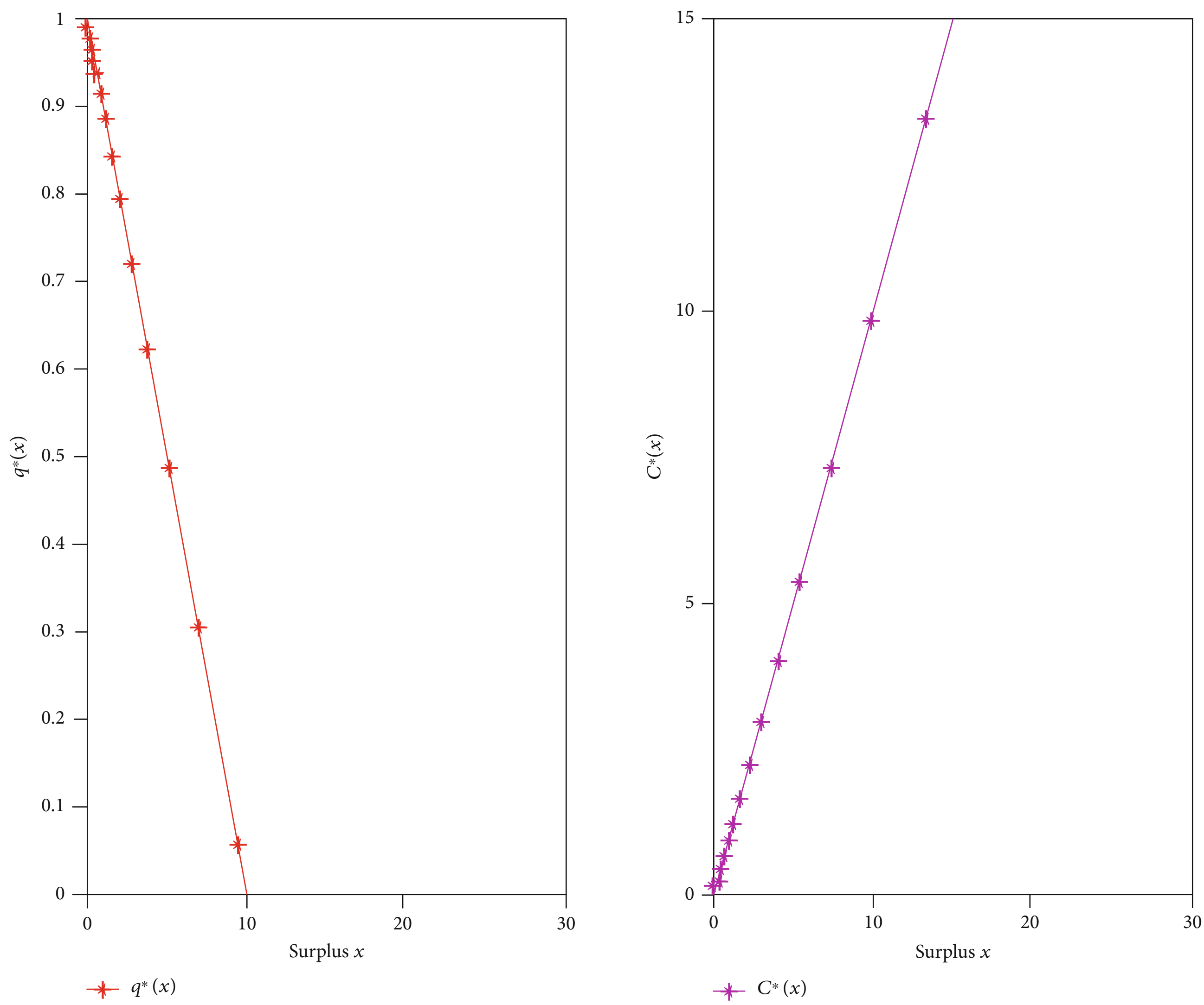

Figure 1: $q^*(x)$ and $c^*(x)$ change with x without VaR constraint.

where K_1 and K_2 are some fixed constants. Therefore, in view of (64) and (65), we can conclude that $e^{-r-mt-\delta B_t} F(X_t^*)$ is uniformly integrable. This completes our proof.

Remark 7. Letting $m = \beta$ and $r = \delta = 0$ and $\lambda \to 0$ in Theorem 6, the result reduces to the result of Theorem 7 in Liang and Palmowski [17].

4. Solution to the Optimization Problem under VaR Constraints

In this section, we will use the results in Section 3 to solve the optimal problem with VaR constraints (14). To do so, we make the following assumption.

Assumption 8. We assume that $\overline{\mathrm{VaR}} \ge -\Phi^{-1}(k)b\sqrt{h} - \theta ah$ and $\Phi^{-1}(k) < 0$ or equivalently $k < 1/2$.

Under Assumption 8, the VaR constraint $\mathrm{VaR}_t^{k,h,\pi} \le \overline{\mathrm{VaR}}$ is equivalent to

$$c(x)h + \left(\Phi^{-1}(k)b\sqrt{h} + \eta ah\right)q(x) - \Phi^{-1}(k)b\sqrt{h} - \theta ah - \overline{\mathrm{VaR}} \le 0. \tag{66}$$

Note that $-\Phi^{-1}(k)b\sqrt{h} - \theta ah - \overline{\mathrm{VaR}} \le 0$, so there exists at least one strategy $(q(x), c(x)) \equiv (0, 0)$ that satisfies (66). Then, the control space defined in Assumption 8 is not empty. The strategy under VaR constraints should satisfy

$$\begin{cases} 0 \le q(x) \le 1, \\ c(x) \ge 0, \\ c(x)h + \left(\Phi^{-1}(k)b\sqrt{h} + \eta ah\right)q(x) - \Phi^{-1}(k)b\sqrt{h} - \theta ah - \overline{\mathrm{VaR}} \le 0. \end{cases} \tag{67}$$

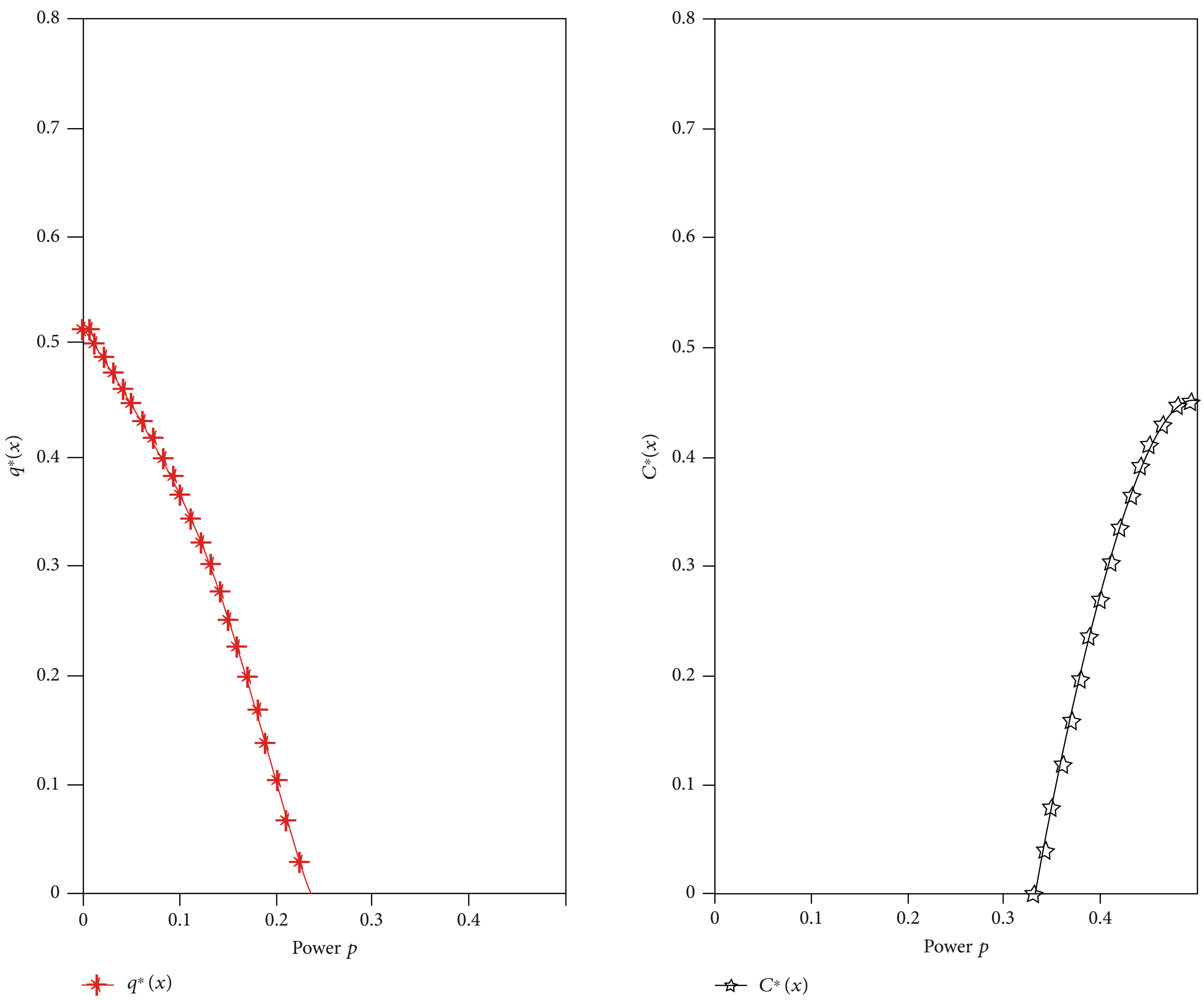

Figure 2: $q^*(x)$ and $c^*(x)$ change with p without VaR constraint.

Table 2: VaR control levels and levels with time interval $h = 1/250$.

	Case 1	Case 2	Case 3
$\bar{\text{VaR}}$	0.08	0.1	0.1
k	0.01	0.01	0.03

If the strategy $(q(x), c(x))$ defined in Theorem 6 satisfies the last inequality in (67), then the strategy $(q^*(x), c^*(x))$ is also a solution with the constraint, namely $(q^*(x), c^*(x)) = (q^*_{VaR}(x), c^*_{VaR}(x))$. Otherwise, if the strategy $(q(x), c(x))$ defined in the first and second equations in (67) is outside the control space at the initial point x, then the optimal strategy is just the boundary of the control space. If the strategy $(q(x), c(x))$ defined in the first and second equations in (67) is inside the control space at the initial point x, but it leaves the control space at sometime, we define the first exit point $\bar{x}$ of the control space as

$$\bar{x} = \inf \{x > 0 : \mathrm{c}(x)h + Eq(x) - F > 0, \text{or } q(x) < 0, \text{or } c(x) < 0\}, \tag{68}$$

where $E = \Phi^{-1}(k) b\sqrt{h} + \eta a h$ and $F = V\bar{a}R + \Phi^{-1}(k) b\sqrt{h} + \theta a h$. Then, under the VaR constraints, the optimal strategy is $\pi^*_{\text{VaR}} = (q^*_{\text{VaR}}(x), c^*_{\text{VaR}}(x))$ with

$$q^*_{\text{VaR}}(x) = \begin{cases} q^*(x), & \text{if } x \in [0, \bar{x} \wedge x^*], \\ q^*(\bar{x} \wedge x^*), & \text{if } x > \bar{x} \wedge x^*, \end{cases} \tag{69}$$

$$c^*_{\text{VaR}}(x) = \begin{cases} c^*(x), & \text{if } x \in [0, \bar{x}], \\ c^*(\bar{x}), & \text{if } x > \bar{x}. \end{cases} \tag{70}$$

5. Numerical Examples

In this section, we illustrate the results obtained in Sections 3 and 4 by numerical examples. We set the model parameters

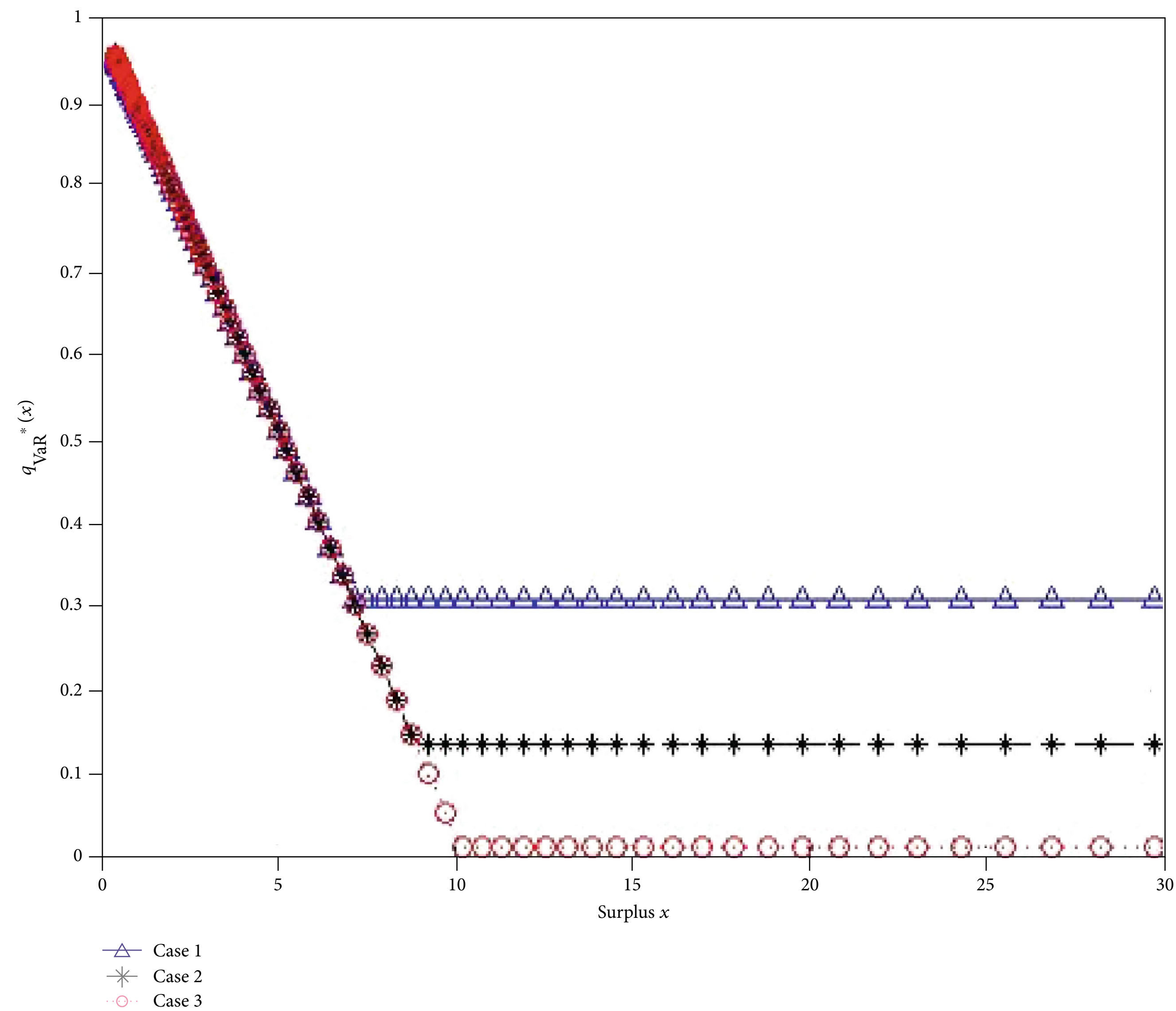

FIGURE 3: $q^*(x)$ change with x under the VaR constraints.

of the insurers reserve process and the financial market in Table 1.

First, we plot the optimal dividend and optimal reinsurance proportion for surplus x ranging from 0 to 30 without VaR constraint. The strategies are shown in Figure 1. From Figure 1, we find the optimal threshold x^* equals to 10.3337 according to Theorem 6, which implies that the insurer will take all the claims without buying any reinsurance when the surplus is larger than 10.3337. From this graph, we can also observe that the optimal dividend rate $c^*(x)$ is nearly linear increasing with the surplus x, while the optimal reinsurance proportion is decreasing with x. When its surplus is sufficiently small, the insurer will divert most of its risk incurred by claims to the reinsurer, which seems intuitively reasonable. It is a reasonable result because when the insurer has a bigger initial wealth x, the insurer would like to retain a bigger proportion $(1-q(x))$ of its insurance risks and increase the dividend rate.

Second, we plot the optimal dividend and optimal reinsurance proportion as the functions of $p \in [0,1]$ for fixed $x=5$ without VaR constraint. The strategies are shown in Figure 2. From Figure 2, we find that the optimal dividend rate increases with increasing p, while the optimal reinsurance proportion decreases. This phenomenon can be explained by noting that the parameter $1-p$ represents the risk aversion of the insurer, so the insurer will retain more proportion of the insurance risks and give more dividend payment.

Third, we use the model parameters of the insurers reserve process market as in Table 2 and consider the VaR constraints for three different cases as in Table 2, where the time interval h is equal to 1/250, and the VaR control levels of $\bar{\mathrm{VaR}}$ are set so that (66) holds. Cases 1 of Table 2: for the VaR control level $\bar{\mathrm{VaR}}$ and risk level k given in Case 1 of Table 2, time interval $h = 1/250$, and the model parameter values given in Table 1, by using (69) and (70), we obtain

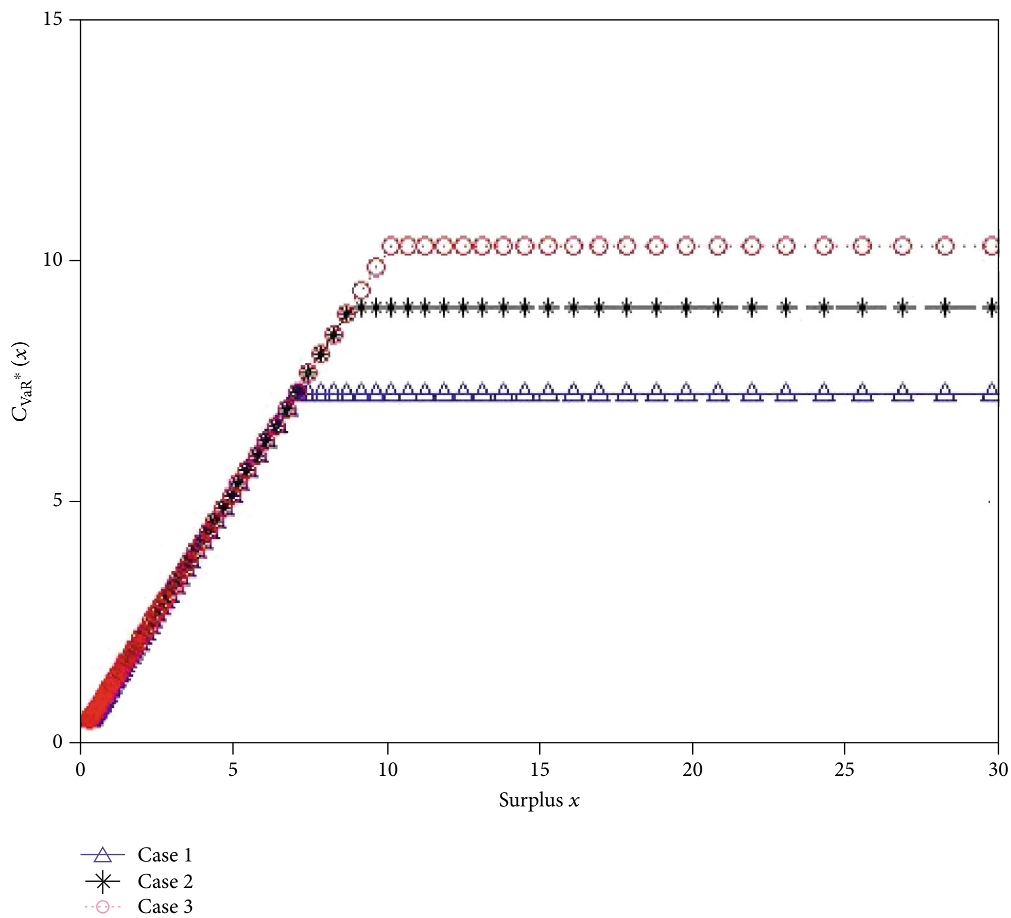

Figure 4: $c^*(x)$ change with x under the VaR constraints.

the optimal reinsurance proportion and the optimal dividend strategy as follows:

$$q^*_{\mathrm{VaR}}(x) = \begin{cases} q^*(x), & \text{if } x \in [0,7.1249], \\ 0.3085, & \text{if } x > 7.1249, \end{cases} \quad c^*_{\mathrm{VaR}}(x) = \begin{cases} c^*(x), & \text{if } x \in [0,7.1249], \\ 7.2076, & \text{if } x > 7.1249. \end{cases} \tag{71}$$

Cases 2 of Table 2: for the VaR control level $\overline{\mathrm{VaR}}$ and risk level k given in Case 2 of Table 2, time interval $h = 1/250$, and the model parameter values given in Table 1, by using (69) and (70), we obtain the optimal reinsurance proportion and the optimal dividend strategy as follows:

$$q^*_{\mathrm{VaR}}(x) = \begin{cases} q^*(x), & \text{if } x \in [0,8.9208], \\ 0.1353, & \text{if } x > 8.9208, \end{cases} \quad c^*_{\mathrm{VaR}}(x) = \begin{cases} c^*(x), & \text{if } x \in [0,8.9208], \\ 9.0151, & \text{if } x > 8.9208. \end{cases} \tag{72}$$

Cases 3 of Table 2: for the VaR control level $\overline{\mathrm{VaR}}$ and risk level k given in Case 3 of Table 2, time interval $h = 1/250$, and the model parameter values given in Table 1, by using (69) and (70), we obtain the optimal reinsurance proportion and the optimal dividend strategy as follows:

$$q_{\mathrm{VaR}}^{*}(x)=\begin{cases} q^{*}(x), & \text{if } x\in[0,10.2729],\\ 0.0141, & \text{if } x>10.2729,\end{cases}\qquad c_{\mathrm{VaR}}^{*}(x)=\begin{cases} c^{*}(x), & \text{if } x\in[0,10.2729],\\ 10.2791, & \text{if } x>10.2729.\end{cases}\tag{73}$$

Under the VaR constraints, both the reinsurance strategy $1-q_{\mathrm{VaR}}^{*}(x)$ and the $c_{\mathrm{VaR}}^{*}(x)$ investment strategy are increasing while the initial capital x increases, which are presented in Figures 3 and 4. It is a reasonable result because when the insurer has a bigger initial wealth x, the insurer would like to retain a bigger proportion of its insurance risks and a larger dividend rate. The upper bounds represent the effect of the VaR constraints on the strategies, which implies that to limit the loss of the insurer at the VaR control level $\bar{\mathrm{VaR}}$, the insurer has to limit both the amounts of dividends and the retained insurance risks within certain scope. And, we can see that the upper bounds of the reinsurance strategy $1-q_{\mathrm{VaR}}^{*}(x)$ and the rate of dividend in case 2 are larger than those in case 1, which means that the insurer can retain more insurance risks and pay more dividend payment in Case 2 than in Case 1. This finding is consistent with the fact that the insurer has a tougher VaR control level of 0.08 in case 1 than that of the VaR control level 0.1 in case 2.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflict of interests.

Authors' Contributions

All authors contributed equally to the manuscript. All authors read and approved the final manuscript.

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