Research Article

A Nonlinear Fractional Problem with Mixed Volterra-Fredholm Integro-Differential Equation: Existence, Uniqueness, H-U-R Stability, and Regularity of Solutions

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This paper considers nonlinear fractional mixed Volterra-Fredholm integro-differential equation with a nonlocal initial condition. We propose a fixed-point approach to investigate the existence, uniqueness, and Hyers-Ulam-Rassias stability of solutions. Results of this paper are based on nonstandard assumptions and hypothesis and provide a supplementary result concerning the regularity of solutions. We show and illustrate the wide validity field of our findings by an example of problem with nonlocal neutral pantograph equation, involving functional derivative and ψ -Caputo fractional derivative.

1. Introduction

Over the last few decades, there has been significant development in the area of ordinary and partial fractional differential equations with boundary integral conditions. This is reflected by various state-of-the-art papers, e.g., [1–15]. Problems with initial integral conditions have many important applications. An example is when a direct measurement quantity is impossible but their mean values are known. The fractional derivatives and integrals appear to be a very efficient tool to model various physical phenomena: kinetic theories, statistical mechanics, dynamics in complex media, control theory, signal processing, bioengineering and biomedical applications, and many others. We refer the reader to [16–18].

In an attempt to formulate different problems, distinct definitions of the fractional derivative and fractional integral are available in the literature. We mention some of them.

Let $\alpha > 0$, f an integrable function defined on $[a, b]$, and $\psi \in C^1([a, b])$ an increasing function with $\psi'(t) \neq 0$ for all $t \in [a, b]$.

(i) The Riemann–Liouville fractional derivative:

$$(D^\alpha f)(x) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{dx}\right)^m \int_a^x (x-t)^{m-\alpha-1} f(t) dt & \text{if } m-1 < \alpha < m \\ \left(\frac{d}{dx}\right)^{m-1} f(x) & \text{if } \alpha = m-1 \end{cases} \quad (1)$$

(ii) The Caputo fractional derivative:

$$({}^C D^\alpha f)(x) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-t)^{m-\alpha-1} f^{(m)}(t) dt & \text{if } m-1 < \alpha < m \\ \left(\frac{d}{dx}\right)^{m-1} f(x) & \text{if } \alpha = m-1 \end{cases} \quad (2)$$

(iii) The ψ -Riemann–Liouville fractional integral:

$$(I^{\alpha,\psi}f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} f(t) dt \quad (3)$$

(iv) The ψ -Riemann–Liouville fractional derivative:

$$(D^{\alpha,\psi}f)(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{1}{\psi'(t)} \frac{d}{dx} \right)^m \int_a^x \psi'(t)(\psi(x) - \psi(t))^{m-\alpha-1} f(t) dt, \quad (4)$$

where $m = [\alpha] + 1$

Very recently, the author introduced in [10] the so-called ψ -Caputo-type fractional integral and derivative with respect to another function as follows:

Definition 1 (ψ -Caputo fractional derivative). Let $n-1 < \alpha < n$, $f \in C^n([a, b])$, and $\psi \in C^1([a, b])$ be an increasing function with $\psi'(t) \neq 0$ for all $t \in [a, b]$. The ψ -Caputo fractional derivative operator of order α of a function f is defined by

$$\begin{aligned} ({}^c D^{\alpha,\psi}f)(x) &= \left({}_a I_x^{n-\alpha} f^{[n]} \right)(x) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_a^x \psi'(t)(\psi(x) - \psi(t))^{n-\alpha-1} \left(\frac{1}{\psi'(t)} \frac{d}{dx} \right)^n f(t) dt, \end{aligned} \quad (5)$$

where $n = [\alpha] + 1$. In particular, when $\alpha \in (0, 1)$, we have ${}^c D^{\alpha,\psi}$

$${}^c D^{\alpha,\psi}f(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x (\psi(x) - \psi(t))^{-\alpha} f'(t) dt. \quad (6)$$

For some special cases of ψ , we obtain the Caputo fractional derivative, the Caputo–Hadamard fractional derivative, and the Caputo–Erdélyi–Kober fractional derivative.

The relationship between the ψ -Caputo and the ψ -Riemann–Liouville integrals can be written as follows:

Lemma 2 [10]. Let $f \in C^n([a, b])$ and $\alpha > 0$. Then, we have

$$I^{\alpha,\psi} {}^c D^{\alpha,\psi}f(t) = f(t) - \sum_{k=1}^{n-1} \frac{f^{[k]}(a^+)}{k!} (\psi(t) - \psi(a))^k. \quad (7)$$

In particular, given $\alpha \in (0, 1)$, we have

$$I^{\alpha,\psi} {}^c D^{\alpha,\psi}f(t) = f(t) - f(a). \quad (8)$$

In this paper, we state and prove a weak form of Banach contraction principal. Together with the Schauder's fixed-point theorem, we discuss the existence, uniqueness, Hyers–Ulam–Rassias stability, and regularity of solutions for the following nonlinear fractional mixed Volterra–Fredholm integro-differential equation with nonlocal initial condition:

$${}^c D^{\alpha,\psi}u(t) = F\left(t, u, \int_a^t k(t, s, u(s))ds, \int_a^\beta h(t, s, u(s))ds\right), \quad (9)$$

$$u(a) = \int_a^\lambda g(s)u(s)ds, \quad (10)$$

where ${}^c D^{\alpha,\psi}$ is the ψ -Caputo derivative of order α , $0 < \alpha < 1$, $t \in I = [a, b]$, $\beta \leq \lambda \in (a, b]$, $g \in L^1(I, \mathbb{R}_+)$, $\int_a^\lambda g(t)dt < 1$, $u \in Y = C(I, X)$ is a continuous function on I with values in Banach space X , $\|u\|_Y = \max_{t \in I} \|u(t)\|_X$, $F : I \times Y \times X^2 \rightarrow X$, and $k, h : I^2 \times X \rightarrow X$ are continuous X valued functions. For the sake of simplicity, we denote

$$Ku(t) := \int_a^t k(t, s, u(s))ds, Hu(t) := \int_a^\beta h(t, s, u(s))ds. \quad (11)$$

The main contributions in this paper reside in the following:

- (i) This work extends, improves, and generalizes several recent state-of-the-art results, including [19–24]
- (ii) Assumptions (A2)–(A6) are not standard; they have been considerably weakened. In fact, they are not supposed to hold on the overall space, but only on a subspace. Example in last section shows the importance of this new form of assumptions
- (iii) Results in Section 2 are obtained on the basis of a weak form of the Banach contraction principal, which is stated and proved in this paper
- (iv) We discuss the Hyers–Ulam–Rassias stability of problems (9) and (10) with respect to a function φ . Generally, in the existing literature, function φ is supposed to be bounded, continuous, and verifies $\inf \varphi(t) > 0$. In our result, function φ is supposed to be neither continuous nor bounded nor positive. It is just an integrable locally bounded function
- (v) In addition to the existence, uniqueness, and stability of solutions, the main theorem in Section 3 provides a supplementary result concerning the regularity of solutions
- (vi) We give an Hyers–Ulam–Rassias stability result on unbounded intervals

- (vii) Last, in this paper, we prove the existence, uniqueness, Ulam-Hyers-Rassias stability, and regularity of solutions for an example of problems with nonlocal neutral pantograph equation to show the large validity field of our results

The remainder of this paper is structured as follows. In Section 2, we briefly recall some basic definitions and some preliminary concepts about fractional calculus and auxiliary results used in the following sections. In Section 3, we discuss the existence and uniqueness of mild solutions for problems (9) and (10), using a weak form of Banach contraction principal and Schauder's fixed-point theorem. Section 4 is devoted to present an Ulam-Hyers-Rassias stability result for problems (9) and (10), and we give a regularity result of solutions for this problem. Finally, we supply in Section 5 a well-suited example to illustrate the application and the validity of our results. In this example, we discuss the existence, uniqueness, Ulam-Hyers-Rassias stability, and regularity of solutions for a problem with nonlocal neutral pantograph equation involving a functional derivative and ψ -Caputo fractional derivative.

2. Preliminaries

Throughout this paper, we will use the following notations:

For the sake of simplicity, we used the same symbol, $\|\cdot\|$, for all norms. The closure of a set C will be denoted by $\bar{C}$. F, k, h, g are given functions (problems (9) and (10)), and L_2, N_2, C_2, μ denote the constants:

$$\begin{aligned} L_2 &= \max_{t \in I} \|F(t, 0, 0, 0)\|, N_2 \\ &= \max_{(t,s) \in I \times I} \|k(t, s, 0)\|, C_2 \\ &= \max_{(t,s) \in I \times I} \|h(t, s, 0)\|, \mu = \int_a^\lambda g(s) ds. \end{aligned} \quad (12)$$

Define operators $T : Y \rightarrow Y$ and $G : Y \times Y \rightarrow Y$ by

$$\begin{aligned} Tu(t) &= \frac{1}{(1-\mu)\Gamma(\alpha)} \int_a^\lambda \psi'(\tau) F(\tau, u, Ku(\tau), Hu(\tau)) \\ &\quad \cdot \int_\tau^\lambda g(s)(\psi(s) - \psi(\tau))^{\alpha-1} ds d\tau + \frac{1}{\Gamma(\alpha)} \\ &\quad \cdot \int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-1} F(\tau, u, Ku(\tau), Hu(\tau)) d\tau, \end{aligned} \quad (13)$$

$$\begin{aligned} G(v, u)(t) &= \frac{1}{(1-\mu)\Gamma(\alpha)} \int_a^\lambda \psi'(\tau) F(\tau, v, Kv(\tau), Hv(\tau)) \\ &\quad \cdot \int_\tau^\lambda g(s)(\psi(s) - \psi(\tau))^{\alpha-1} ds d\tau + \frac{1}{\Gamma(\alpha)} \\ &\quad \cdot \int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-1} F(\tau, u, Ku(\tau), Hu(\tau)) d\tau. \end{aligned} \quad (14)$$

For any $u \in Y$, we set:

$$W_u = \{T^n u; n \in \mathbb{N}\} \quad \text{and} \quad V_u = \{G(v_1, v_2); (v_1, v_2) \in Y \times \bar{W}_u \cup \bar{W}_u \times Y\}, \quad (15)$$

where T^n denotes T composed with itself n times.

Now, we present some important theorems and lemmas in obtaining the main results.

Theorem 3 (Weak form of the Banach contraction principal). *Let (X, d) and (Y, d) be two complete metric spaces, C a nonempty subset of $X \cap Y$, and $T : X \rightarrow Y$ such that $T(C) \subset C$. Let d_C ($d_{\bar{C}}$, resp.) denotes the induced metric on C ($\bar{C}$, resp.). If $T : (\bar{C}, d_{\bar{C}}) \rightarrow (Y, d)$ is continuous and $T : (C, d_C) \rightarrow (C, d_C)$ is a contraction mapping, i.e., there exists $0 < \lambda < 1$ such that*

$$d(Tu, Tv) \leq \lambda d(u, v), \quad (16)$$

for all $u, v \in C$. Then, T has a unique fixed point u^* in the closure of C . Further, the sequence $\{T^n u\}_n$ converges to u^* for all $u \in C$.

Proof. For all $u, v \in C$ and $n \in \mathbb{N}$, using (16) and the triangle inequality, we get on one hand that

$$d(T^n u, T^n v) \leq \lambda^n d(u, v), \quad (17)$$

and on the other that

$$d(u, v) \leq \frac{1}{1-\lambda} (d(u, Tu) + d(v, T(v))). \quad (18)$$

Now, for all $u \in C$ and $n, m \in \mathbb{N}$, by virtue of inequalities (17) and (18), we obtain

$$\begin{aligned} d(T^n u, T^m u) &\leq \frac{1}{1-\lambda} (d(T^n u, T^{n+1} u) + d(T^m u, T^{m+1} u)) \\ &\leq \frac{\lambda^n + \lambda^m}{1-\lambda} d(u, Tu) \xrightarrow{n,m \rightarrow \infty} 0. \end{aligned} \quad (19)$$

This means that $\{T^n u\}_n$ is a Cauchy sequence in the complete metric space X . Then, it converges to a point u^* of $\bar{C}$. Reusing (19), we obtain

$$d(T(T^n u), T^m u) \leq \frac{\lambda^{n+1} + \lambda^m}{1-\lambda} d(u, Tu). \quad (20)$$

Letting n, m tend to infinity, taking into account continuity of mapping T , we deduce that u^* is a fixed point of

T . If $v^* \in \bar{C}$ is a fixed point of T , there exists a sequence $\{v_n\}_n$ in C that converges to v^* . By (16), we get for all $u \in C$ and $n \in \mathbb{N}$,

$$d(Tv_n, T^{n+1}u) \leq \lambda d(v_n, T^n u). \quad (21)$$

Letting n tends to infinity, by the continuity of T in $(\bar{C}, d_C)$, we deduce that $d(Tv^*, Tu^*) \leq \lambda d(v^*, u^*)$. But u^*, v^* are two fixed points of T so $v^* = u^*$. This achieves the proof.

Theorem 4 (Schauder's fixed-point theorem). *Let F be a closed convex set in a Banach space X and assume that $T : F \rightarrow F$ is a continuous mapping such that $T(F)$ is a relatively compact subset of F . Then, T has a fixed point.*

Theorem 5 (Arzela-Ascoli theorem). *Assume that K is a compact set in $\mathbb{R}^n$, $n \geq 1$. Then, a set $S \subset C(K)$ is relatively compact in $C(K)$ if the functions in S are uniformly bounded and equicontinuous on K .*

Lemma 6. *Let $f : [a, b] \rightarrow \mathbb{R}_+$ be a nondecreasing function and $\psi \in C^1([a, b])$ an increasing function with $\psi'(t) \neq 0$ for all $t \in [a, b]$, then*

- (i) *Function f is Riemann integrable and bounded*
- (ii) *There exist positive constants P_1 and P_2 such that*

$$\begin{aligned} \int_a^t f(\xi) d\xi &\leq P_1 f(t), \\ \int_a^t \psi'(\xi)(\psi(t) - \psi(\xi))^{\alpha-1} f(\xi) d\xi &\leq P_2 f(t), \end{aligned} \quad (22)$$

for all $t \in [a, b]$.

Proof. f is a nondecreasing function on $[a, b]$; then, it is Riemann integrable, and it attains its maximum and minimum at points b and a , respectively. Let $t \in [a, b]$, and clearly,

$$\begin{aligned} \int_a^t f(\xi) d\xi &\leq (b-a)f(t), \\ \int_a^t \psi'(\xi)(\psi(t) - \psi(\xi))^{\alpha-1} f(\xi) d\xi &\leq \frac{1}{\alpha}(\psi(b) - \psi(a))^\alpha f(t). \end{aligned} \quad (23)$$

This achieves the proof.

Lemma 7. *Let $f : [a, b] \rightarrow \mathbb{R}_+$ a continuous function, $\varphi : [a, b] \rightarrow \mathbb{R}_+$ an integrable function, and P_1, P_2 two positive constants. Suppose that*

- (i) *There exists $t_0 \in [a, b]$; $\varphi(t_0) = 0$*

- (ii) *Inequality $\int_a^t \varphi(\xi) d\xi \leq P_1 \varphi(t)$ holds for all $t \in [a, b]$*
- (iii) *Inequality $f(t) \leq P_2 \varphi(t)$ holds for all $t \in [a, b]$*

Then, $f(t) = 0$ for all $t \in [a, t_0]$.

Proof. Let $t_1 \in [a, t_0]$, and by items (5) and (9), we have $\int_a^{t_1} \varphi(\xi) d\xi \leq \int_a^{t_0} \varphi(\xi) d\xi \leq P_1 \varphi(t_0) = 0$. Then, $\varphi(t) = 0$ a.a. $t \in [a, t_1]$. So, there exists a sequence $\{t_n\}_n \subset [a, b]$ converges to t_1 and verifies $\varphi(t_n) = 0$, for all $n \in \mathbb{N}$. But, by item (10), we deduce that $f(t_n) \leq P_2 \varphi(t_n)$. Keeping in mind that f is continuous, it yields $f(t_1) = 0$, and this concludes the proof.

Lemma 8. *Let $u_0 \in Y$. The following statements hold true:*

- (i) *Operators $T, G(u_0, \cdot) : Y \rightarrow Y$ are continuous*
- (ii) *Find a solution of (9) and (10) is equivalent to find a fixed point of T , that is, an element $u \in Y$ such that*

$$\begin{aligned} u(t) = & \frac{1}{(1-\mu)\Gamma(\alpha)} \int_a^\lambda \psi'(\tau) F(\tau, u, Ku(\tau), Hu(\tau)) \\ & \cdot \int_\tau^\lambda g(s)(\psi(s) - \psi(\tau))^{\alpha-1} ds d\tau + \frac{1}{\Gamma(\alpha)} \\ & \cdot \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} F(s, u, Ku(s), Hu(s)) ds \end{aligned} \quad (24)$$

- (i) *If u_0 is a solution of problems (9) and (10), then for all function $f \in Y$, we have*

$$\begin{aligned} \int_a^\lambda g(s) T f(s) ds &= T f(a) \\ \int_a^\lambda g(s) G(u_0, f)(s) ds &= G(u_0, f)(a) \end{aligned} \quad (25)$$

Proof. To prove that T is continuous, let u_n be a sequence such that $u_n \rightarrow u$ in Y . For each $t \in I$, we have

$$\begin{aligned} \|Tu_n(t) - Tu(t)\| &\leq \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \\ &\quad \times \sup_{s \in I} \|F(s, u_n, Ku_n(s), Hu_n(s)) \\ &\quad - F(s, u, Ku(s), Hu(s))\| ds \\ &\quad + \frac{1}{(1-\mu)\Gamma(\alpha)} \int_a^\lambda \psi'(\tau) \sup_{\tau \in I} \|F \\ &\quad \cdot (\tau, u_n, Ku_n(\tau), Hu_n(\tau)) \\ &\quad - F(\tau, u, Ku(\tau), Hu(\tau))\| \\ &\quad \times \int_\tau^\lambda g(s)(\psi(s) - \psi(\tau))^{\alpha-1} ds d\tau. \end{aligned} \quad (26)$$

But F, u, u_n, k, h are the continuous functions on the compact I ; hence, there exists $t_0 \in I$ verifying the following:

$$\begin{aligned} & \|F(t_0, u_n, Ku_n(t_0), Hu_n(t_0)) - F(t_0, u, Ku(t_0), Hu(t_0))\| \\ &= \sup_{\xi \in I} \|F(\xi, u_n, Ku_n(\xi), Hu_n(\xi)) - F(\xi, u, Ku(\xi), Hu(\xi))\|. \end{aligned} \quad (27)$$

So, by (26),

$$\begin{aligned} \|Tu_n(t) - Tu(t)\| &\leq \frac{(\psi(b) - \psi(a))^\alpha}{(1 - \mu)\Gamma(\alpha + 1)} \|F(t_0, u_n, Ku_n(t_0), Hu_n(t_0)) \\ &\quad - F(t_0, u, Ku(t_0), Hu(t_0))\|. \end{aligned} \quad (28)$$

Therefore, since $F : I \times Y \times X^2 \rightarrow X$ is continuous, operator T is continuous. Similarly, we prove that T_0 is continuous.

For point 2, let u be a solution of ψ -fractional integro-differential problems (9) and (10). Applying the fractional integral operator $I^{\alpha, \psi}$ on both sides of (9), we get

$$u(t) = u(a) + \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} F(s, u, Ku(s), Hu(s)) ds. \quad (29)$$

On the other hand, keeping in mind initial condition (10), we have

$$\begin{aligned} u(a) &= \int_a^\lambda g(s)u(s)ds \\ &= \int_a^\lambda g(s) \left[u(a) + \frac{1}{\Gamma(\alpha)} \int_a^s \psi'(\tau)(\psi(s) - \psi(\tau))^{\alpha-1} F \right. \\ &\quad \cdot (\tau, u, Ku(\tau), Hu(\tau)) d\tau \Big] ds \\ &= \int_a^\lambda g(s)u(a)ds + \frac{1}{\Gamma(\alpha)} \int_a^\lambda \int_a^s g(s)\psi'(\tau)(\psi(s) \\ &\quad - \psi(\tau))^{\alpha-1} F(\tau, u, Ku(\tau), Hu(\tau)) d\tau ds, \end{aligned} \quad (30)$$

by Dirichlet's formula, and it follows that

$$\begin{aligned} u(a) \left[1 - \int_a^\lambda g(s)ds \right] &= \frac{1}{\Gamma(\alpha)} \int_a^\lambda \psi'(\tau) F(\tau, u, Ku(\tau), Hu(\tau)) \\ &\quad \cdot \int_\tau^\lambda g(s)(\psi(s) - \psi(\tau))^{\alpha-1} ds d\tau. \end{aligned} \quad (31)$$

Substituting this latter in (29) and putting $\int_a^\lambda g(s)ds = \mu$, we obtain

$$\begin{aligned} u(t) &= \frac{1}{(1 - \mu)\Gamma(\alpha)} \int_a^\lambda \psi'(\tau) F(\tau, u, Ku(\tau), Hu(\tau)) \\ &\quad \cdot \int_\tau^\lambda g(s)(\psi(s) - \psi(\tau))^{\alpha-1} ds d\tau + \frac{1}{\Gamma(\alpha)} \\ &\quad \cdot \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} F(s, u, Ku(s), Hu(s)) ds. \end{aligned} \quad (32)$$

Conversely, let u in X verifying $Tu = u$. We have ${}^c D^{\alpha, \psi} u(t) = 1/(1 - \mu)\Gamma(\alpha) {}^c D^{\alpha, \psi} (\int_a^\lambda \psi'(\tau) F(\tau, u, Ku(\tau), Hu(\tau)) \int_\tau^\lambda g(s)(\psi(s) - \psi(\tau))^{\alpha-1} ds d\tau + {}^c D^{\alpha, \psi} (1/\Gamma(\alpha) \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} F(s, u, Ku(s), Hu(s)) ds) = {}^c D^{\alpha, \psi} (I^{\alpha, \psi} F(t, u, Ku(t), Hu(t))) = F(t, u, Ku(t), Hu(t))$.

Now, reusing the Dirichlet's formula, (24) gives $u(a) = \int_a^\lambda g(s)u(s)ds$, and this achieves the proof of point 2. For point 3, let $f \in Y$, and we have

$$\begin{aligned} (1 - \mu)Tf(a) &= \frac{1}{\Gamma(\alpha)} \int_a^\lambda g(s) \int_a^s \psi'(\tau)(\psi(s) \\ &\quad - \psi(\tau))^{\alpha-1} F(\tau, f, Kf(\tau), Hf(\tau)) d\tau ds \\ &= \int_a^\lambda g(s)(Tf(s) - Tf(a))ds. \end{aligned} \quad (33)$$

Therefore,

$$Tf(a) = \int_a^\lambda g(s)Tf(s)ds. \quad (34)$$

Similarly, we get

$$T_0 f(a) = \int_a^\lambda g(s)T_0 f(s)ds. \quad (35)$$

QED.

Let $u \in Y$, and the following assumptions are also used:

(A1) There exists a positive constant M such that

$$\sup_{(t, u) \in I \times Y} \|F(t, u(t), Ku(t), Hu(t))\| := M < \infty. \quad (36)$$

(A₂^u) There exists a positive constant L_1 such that, for all $u_1, u_2 \in V_u$, $x_1, y_1, x_2, y_2 \in X$, and $t \in I$:

$$\begin{aligned} & \|F(t, u_1, x_1, y_1) - F(t, u_2, x_2, y_2)\| \\ & \leq L_1(\|u_1(t) - u_2(t)\| + \|x_1 - x_2\| + \|y_1 - y_2\|). \end{aligned} \quad (37)$$

(A_3^u) There exists a positive constant N_1 such that, for all $u_1, u_2 \in V_u$ and $t, s \in I$:

$$\|k(t, s, u_1(s)) - k(t, s, u_2(s))\| \leq N_1 \|u_1(t) - u_2(t)\|. \quad (38)$$

(A_4^u) There exists a positive constant C_1 such that, for all $u_1, u_2 \in V_u$ and $t, s \in I$:

$$\|h(t, s, u_1(s)) - h(t, s, u_2(s))\| \leq C_1 \|u_1(t) - u_2(t)\|. \quad (39)$$

(A_5^u) With $M_1 := L_1(1 + (b - a)(N_1 + C_1))$, we have

$$0 \leq \frac{M_1(\psi(b) - \psi(a))^\alpha}{\Gamma(\alpha + 1)(1 - \mu)} < 1. \quad (40)$$

(A_6^u) For all $u_1, u_2 \in V_u$, if $F(a, u_1, Ku_1(a), Hu_1(a)) = F(a, u_2, Ku_2(a), Hu_2(a)) = 0$, then $u_1(a) = u_2(a)$.

Lemma 9. Let $u \in Y$, and if (A_2^u) and (A_3^u) are satisfied, then the following statements hold true.

$$\begin{aligned} \|Ku_1(t)\| & \leq (t - a)(N_1\|u_1\| + N_2), \|Ku_1(t) - Ku_2(t)\| \\ & \leq N_1(t - a)\|u_1 - u_2\|, \|Hu_1(t)\| \\ & \leq (\beta - a)(C_1\|u_1\| + C_2), \|Hu_1(t) - Hu_2(t)\| \\ & \leq C_1(\beta - a)\|u_1 - u_2\|, \end{aligned} \quad (41)$$

for any $t \in I$, and $u_1, u_2 \in V_u$. *Proof.* Immediate.

3. Existence and Uniqueness Results

Our first result is based on the weak form of the Banach contraction principle.

Theorem 10. Let $u \in Y$. If assumptions (A_2^u) – (A_5^u) are satisfied, then the fractional integro-differential problems (9)-(10) have a unique solution u^* continuous on I . Furthermore, the sequence $\{T^n u\}_n$ converges to u^* in Y .

Proof. Let $u \in Y$ where assumptions (A_2^u) – (A_5^u) hold. Clearly, $TW_u \subset W_u$. We shall prove that T is a contraction on V_u . For this, let $u_1, u_2 \in V_u$. By (13), we get

$$\begin{aligned} \|Tu_1(t) - Tu_2(t)\| & \leq \frac{1}{(1 - \mu)\Gamma(\alpha)} \int_a^\lambda \psi'(\tau) \|f \\ & \quad \cdot (\tau, u_1, Ku_1(\tau), Hu_1(\tau)) \\ & \quad - f(\tau, u_2, Ku_2(\tau), Hu_2(\tau))\| \\ & \quad \times \int_\tau^\lambda g(s)(\psi(s) - \psi(\tau))^{\alpha-1} ds d\tau \\ & \quad + \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \\ & \quad \times \|f(s, u_1, Ku_1(s), Hu_1(s)) \\ & \quad - f(s, u_2, Ku_2(s), Hu_2(s))\| ds. \end{aligned} \quad (42)$$

By means of assumption (A_2^u), we obtain

$$\begin{aligned} \|Tu_1(t) - Tu_2(t)\| & \leq \frac{1}{(1 - \mu)\Gamma(\alpha)} \int_a^\lambda \psi'(\tau) L_1(\|u_1(\tau) \\ & \quad - u_2(\tau)\| + \|Ku_1(\tau) - Ku_2(\tau)\| \\ & \quad + \|Hu_1(\tau) - Hu_2(\tau)\|) \\ & \quad \times \int_\tau^\lambda g(s)(\psi(s) - \psi(\tau))^{\alpha-1} ds d\tau \\ & \quad + \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} L_1 \\ & \quad \cdot (\|u_1(s) - u_2(s)\| + \|Ku_1(s) \\ & \quad - Ku_2(s)\| + \|Hu_1(s) - Hu_2(s)\|) ds, \end{aligned} \quad (43)$$

and by virtue of Lemma 9, it follows that

$$\begin{aligned} \|Tu_1(t) - Tu_2(t)\| & \leq \frac{L_1((b - a)(C_1 + N_1) + 1)}{(1 - \mu)\Gamma(\alpha)} \|u_1 - u_2\|_\infty \\ & \quad \cdot \int_a^\lambda \psi'(\tau) \int_\tau^\lambda g(s)(\psi(s) - \psi(\tau))^{\alpha-1} ds d\tau \\ & \quad + \frac{L_1((b - a)(C_1 + N_1) + 1)}{\Gamma(\alpha)} \|u_1 - u_2\|_\infty \\ & \quad \cdot \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} ds. \end{aligned} \quad (44)$$

On the other hand, a simple calculation shows that

$$\int_a^\lambda \psi'(\tau) \int_\tau^\lambda g(s)(\psi(s) - \psi(\tau))^{\alpha-1} ds d\tau \leq \frac{\mu}{\alpha} (\psi(\lambda) - \psi(a))^\alpha, \quad (45)$$

$$\int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} ds \leq \frac{1}{\alpha} (\psi(b) - \psi(a))^\alpha. \quad (46)$$

Substituting estimates (45) and (46) in relation (44), we deduce that

$$\|Tu_1(t) - Tu_2(t)\| \leq \frac{M_1(\psi(b) - \psi(a))^\alpha}{\Gamma(\alpha+1)(1-\mu)} \|u_1 - u_2\|_\infty, \quad (47)$$

or, keeping in mind condition (A_5^u) , operator T is a contraction on V_u and consequently continuous on $(\bar{W}_u, d_{\bar{W}_u})$. Therefore, by the weak form of the Banach contraction principal, T has a unique fixed point $u^* = Tu^*$ in the closure of W_u , which is a solution of (24) and hence, by item 2 in Lemma 8, a solution of problems (9) and (10). Note that if v^* is a solution of (9) and (10), by (9), we get $f(a, v^*, Kv^*(a), Hv^*(a)) = f(a, u^*, Ku^*(a), Hu^*(a)) = 0$. Assumption (A_6) implies that $v^*(a) = u^*(a)$; then, $G(u^*, v^*) = G(v^*, v^*) = v^*$, so $v^* \in V_u$. By (47), we have

$$\begin{aligned} \|u^*(t) - v^*(t)\| &= \|Tu^*(t) - Tv^*(t)\| \\ &\leq \frac{M_1(\psi(b) - \psi(a))^\alpha}{\Gamma(\alpha+1)(1-\mu)} \|u^* - v^*\|_\infty, \end{aligned} \quad (48)$$

and this leads to the uniqueness of solutions. As a recap,

- (i) Problems (9) and (10) have a unique solution u^*
- (ii) The sequence $\{T^n u\}_n$ converges to u^*

This achieves the proof.

In the next theorem, we shall use Schauder's fixed-point theorem to establish the existence of solutions for problems (9) and (10), with less conditions.

Theorem 11. *Under assumption (A1), problems (9) and (10) have at least one solution on I .*

Proof. Let B_r denotes the closed ball in Y of radius r : $B_r = \{u \in Y : \|u\| \leq r\} \subseteq Y$, with

$$r \geq \frac{(\psi(b) - \psi(a))^\alpha}{\Gamma(\alpha+1)(1-\mu)} \sup_{(t,u) \in I \times Y} F(t, u(t), Ku(t), Hu(t)), \quad (49)$$

and defines the operator T on the Banach space Y by

$$\begin{aligned} Tu(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} F(s, u, Ku(s), Hu(s)) ds \\ &\quad + \frac{1}{(1-\mu)\Gamma(\alpha)} \int_a^\lambda \psi'(\tau) F(\tau, u, Ku(\tau), Hu(\tau)) \\ &\quad \cdot \int_\tau^\lambda g(s)(\psi(s) - \psi(\tau))^{\alpha-1} ds d\tau. \end{aligned} \quad (50)$$

Clearly, $Tu \in B_r$ whenever $u \in B_r$, i.e., $T : B_r \rightarrow B_r$, and item 2 in Lemma 8 assures that T is continuous. Now, we shall prove that TB_r is an equicontinuous set of Y . Let $t_1, t_2 \in I$ and $u \in B_r$, we have

$$\begin{aligned} \|Tu(t_1) - Tu(t_2)\| &= \frac{1}{\Gamma(\alpha)} \left\| \int_a^{t_1} \psi'(s)(\psi(t_1) - \psi(s))^{\alpha-1} F(s, u, Ku(s), Hu(s)) ds \right. \\ &\quad \left. - \int_a^{t_2} \psi'(s)(\psi(t_2) - \psi(s))^{\alpha-1} F(s, u, Ku(s), Hu(s)) ds \right\| \\ &\leq \frac{1}{\Gamma(\alpha)} \left\| \int_{t_2}^{t_1} \psi'(s)(\psi(t_1) - \psi(s))^{\alpha-1} F(s, u, Ku(s), Hu(s)) ds \right\| \\ &\quad + \frac{1}{\Gamma(\alpha)} \left\| \int_a^{t_2} \psi'(s)[(\psi(t_2) - \psi(s))^{\alpha-1} - (\psi(t_1) - \psi(s))^{\alpha-1}] F(s, u, Ku(s), Hu(s)) ds \right\|. \end{aligned} \quad (51)$$

Taking into account assumption (A1), we obtain

$$\begin{aligned} \|Tu(t_1) - Tu(t_2)\| &\leq \frac{M}{\Gamma(\alpha)} \left(\left\| \int_{t_2}^{t_1} \psi'(s)(\psi(t_1) - \psi(s))^{\alpha-1} ds \right\| \right. \\ &\quad \left. + \left\| \int_a^{t_2} \psi'(s)[(\psi(t_2) - \psi(s))^{\alpha-1} - (\psi(t_1) - \psi(s))^{\alpha-1}] ds \right\| \right). \end{aligned} \quad (52)$$

But,

$$\begin{aligned} \int_{t_2}^{t_1} \psi'(s)(\psi(t_1) - \psi(s))^{\alpha-1} ds &= \frac{1}{\alpha} (\psi(t_1) - \psi(t_2))^\alpha \\ \int_a^{t_2} \psi'(s)[(\psi(t_2) - \psi(s))^{\alpha-1} - (\psi(t_1) - \psi(s))^{\alpha-1}] ds \\ &= \frac{1}{\alpha} [-(\psi(t_2) - \psi(a))^\alpha - (\psi(t_1) - \psi(t_2))^\alpha + (\psi(t_1) - \psi(a))^\alpha]. \end{aligned} \quad (53)$$

Therefore,

$$\|Tu(t_1) - Tu(t_2)\| \leq \frac{M}{\Gamma(\alpha+1)} [(\psi(t_1) - \psi(a))^\alpha - (\psi(t_2) - \psi(a))^\alpha], \quad (54)$$

and consequently, $\sup_{u \in B_r} \|Tu(t_1) - Tu(t_2)\| \rightarrow 0$ as $t_1 \rightarrow t_2$, which means that TB_r is an equicontinuous set of Y .

But $T : B_r \subset C(I, X) \rightarrow B_r$, so TB_r is uniformly bounded, and the Arzela-Ascoli theorem implies that TB_r is relatively

compact. Therefore, $T : B_r \rightarrow B_r$ is continuous on the closed convex set B_r , and TB_r is relatively compact. According to the Schauder's fixed-point theorem, mapping T has at least a fixed point. Item 2 in Lemma 8 achieves the proof.

4. Ulam-Hyers-Rassias Stability

In this section, we discuss the Ulam-Hyers-Rassias stability of problems (9) and (10). First, we introduce a basic definition and some notations and hypotheses for this section.

Definition 12 (Ulam-Hyers-Rassias stability). If for each function u satisfying

$$\| {}^c D^{\alpha, \psi} u(t) - F(t, u, Ku(t), Hu(t)) \| \leq \psi(t), \quad (55)$$

where ψ is a nonnegative function, there exists a solution u^* of the fractional differential problems (9) and (10) and a constant $C > 0$ independent of u and u^* such that

$$\| u(t) - u^*(t) \| \leq C\psi(t), \quad (56)$$

for all t ; then, we say that problems (9) and (10) are the Hyers-Ulam-Rassias stability.

Let (H) denotes the following hypothesis:

We denote by Φ the set of all functions $\varphi \in L^1(I, \mathbb{R}_+)$ verifying φ is locally bounded, $\inf_{t \in I} \varphi(t) = 0$, and there exist two positive constants P_1, P_2 such that, for all $t \in I$

$$\int_a^t \varphi(\tau) d\tau \leq P_1 \varphi(t), \quad \int_a^t \psi'(\tau) (\psi(t) - \psi(\tau))^{\alpha-1} \varphi(\tau) d\tau \leq P_2 \varphi(t). \quad (57)$$

Let $\varphi \in \Phi$, $P_3 \in \mathbb{R}_+$, and $C : I \rightarrow \mathbb{R}_+$ a function.

Let $u : I \rightarrow X$ be a continuously differential function. Assume that assumptions $(A_2^u) - (A_6^u)$ hold together with one of the following:

(i) For all $t \in I$ we have

$$\frac{1}{\alpha} (\psi(t) - \psi(a))^\alpha \leq P_3 \varphi(t) \quad (58)$$

(ii) For all $u_1, u_2 \in V_u$ and $t \in I$ we have

$$\| Hu_1(t) - Hu_2(t) \| \leq C(t) \left\| \int_a^\lambda g(s) (u_1(s) - u_2(s)) ds \right\| \quad (59)$$

Finally, we denote by γ the constant

$$\gamma = \begin{cases} 0; & \text{If inequality (18) holds} \\ 1; & \text{Otherwise} \end{cases}, \quad (60)$$

and we assume that

$$\Omega := L_1 \frac{P_2(1 + N_1 P_1) + P_3 \gamma C_1 \int_a^\beta \varphi(\tau) d\tau}{\Gamma(\alpha)} \in (0, 1). \quad (61)$$

Now, we state and prove the main result of this section.

Theorem 13. If the function u satisfies

$$\| {}^c D^{\alpha, \psi} u(t) - F(t, u, Ku(t), Hu(t)) \| \leq \varphi(t), \quad (62)$$

$$u(a) = \int_a^\lambda g(s) u(s) ds, \quad (63)$$

for all $t \in I$. Then, there exists a unique function $u^* : I \rightarrow \mathbb{R}$ solution of problems (9) and (10), with

$$\| u(t) - u^*(t) \| \leq \frac{P_2}{(1 - \Omega)\Gamma(\alpha)} \varphi(t), \quad \text{for all } t \in I. \quad (64)$$

Remark 14. Here, we give some important remarks:

(i) If function $t \mapsto \| F(t, 0, \int_a^t k(t, s, 0) ds, \int_a^\beta h(t, s, 0) ds) \|$ is member of Φ we get

$$\| u^*(t) \| \leq \frac{P_2}{(1 - \Omega)\Gamma(\alpha)} \left\| F\left(t, 0, \int_a^t k(t, s, 0) ds, \int_a^\beta h(t, s, 0) ds\right) \right\|, \quad (65)$$

for all $t \in I$

(ii) Using Lemma 6, we can substitute space Φ by the space of nondecreasing functions

(iii) The case $\inf_{\xi \in I} \varphi(\xi) > 0$ is obvious, so we have omitted this case, supposing that $\inf_{\xi \in I} \varphi(\xi) = 0$

Proof. For all $f_1, f_2 \in Y$, we denote by Ω_{f_1, f_2} the set

$$\Omega_{f_1, f_2} := \{ C \in [0, \infty) \mid \| f_1(t) - f_2(t) \| \leq C\varphi(t), \quad \text{for all } t \in I \}. \quad (66)$$

Easily, we can prove that $d : Y \times Y \longrightarrow \mathbb{R}$ defined by

$$d(f_1, f_2) = \begin{cases} \inf \Omega_{f_1, f_2} & \text{if } \Omega_{f_1, f_2} \neq \Phi \\ +\infty & \text{if } \Omega_{f_1, f_2} = \Phi \end{cases} \quad (67)$$

is a generalized metric on Y verifying the following assumptions:

- (i) If there exists a sequence $t_n \rightarrow t_0$ in I such that $\lim_{n \rightarrow \infty} \varphi(t_n) = 0$, then $d(f_1, f_2) = +\infty$ for all $f_1, f_2 \in Y$ satisfying $f_1(t_0) \neq f_2(t_0)$
- (ii) For all $f_1, f_2 \in Y$, we have

$$\|f_1(t) - f_2(t)\| \leq d(f_1, f_2)\varphi(t), \quad (68)$$

for any $t \in I$

We claim that (Y, d) is complete. In fact, let $\{v_n\}$ be a Cauchy sequence in (Y, d) , and having in mind definition (67), we have

$$\forall \varepsilon > 0, \exists N_\varepsilon > 0, \forall n, m \geq N_\varepsilon \forall t \in I : \|v_m(t) - v_n(t)\| \leq \varepsilon \varphi(t). \quad (69)$$

Since X is complete, (69) implies that $\{v_n(t)\}$ converges for each $t \in I$. Let us show that function $v : I \longrightarrow X$ defined by $v(t) = \lim_{n \rightarrow \infty} v_n(t)$ is continuous or belongs to Y . Passing to the limit with respect to m in the latter inequality, it follows that

$$\forall \varepsilon > 0, \exists N_\varepsilon > 0, \forall n \geq N_\varepsilon \forall t \in I : \|v(t) - v_n(t)\| \leq \varepsilon \varphi(t). \quad (70)$$

But, φ must be bounded because it is locally bounded on the compact I . So,

$$\forall \varepsilon > 0, \exists N_\varepsilon > 0, \forall n \geq N_\varepsilon \sup_{t \in I} \|v(t) - v_n(t)\| \leq \varepsilon \sup_{t \in I} \varphi(t). \quad (71)$$

Hence, $\{v_n\}$ converges uniformly to v , and consequently, v is continuous. Further, if we consider (67) and (69), we may conclude that

$$\forall \varepsilon > 0, \exists N_\varepsilon > 0, \forall n \geq N_\varepsilon d(v, v_n) \leq \varepsilon. \quad (72)$$

This means, the Cauchy sequence $\{v_n\}$ converges to v in (Y, d) ; thus, (Y, d) is complete. On the other hand, Theorem 10 assures the existence of a unique function $u_0 \in Y$ solution

of problems (9) and (10), verifying $\lim_{n \rightarrow \infty} T^n u = u_0$. We define an operator $T_0 : Y \rightarrow Y$ by $T_0 f = G(u_0, f)$, i.e.,

$$\begin{aligned} T_0 f(t) = & \frac{1}{(1-\mu)\Gamma(\alpha)} \int_a^\lambda \psi'(\tau) F(\tau, u_0, Ku_0(\tau), Hu_0(\tau)) \\ & \cdot \int_\tau^\lambda g(s)(\psi(s) - \psi(\tau))^{\alpha-1} ds d\tau + \frac{1}{\Gamma(\alpha)} \\ & \cdot \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} F(s, f, Kf(s), Hf(s)) ds, \end{aligned} \quad (73)$$

for all $f \in Y$, and we discuss its contractivity. Note that $T_0(W_u) \subset V_u$ for all $u \in Y$. Let $u : I \rightarrow X$ be a continuously differential function satisfies (62) and (63). By virtue of Lemma 7 and relation (62), we obtain

$$\|{}^c D^{\alpha, \psi} u(a) - F(a, u, Ku(a), Hu(a))\| = 0, \quad (74)$$

and by (5), we deduce that

$${}^c D^{\alpha, \psi} u(a) = {}^c D^{\alpha, \psi} u_0(a) = F(a, u_0, Ku_0(a), Hu_0(a)) = 0. \quad (75)$$

Thus, keeping assumption (A_6^u) , we deduce that $u(a) = u_0(a)$, and consequently, by item 3 in Lemma 8,

$$\int_a^\lambda g(s)f(s)ds = u_0(a) \text{ for all } f \in \Gamma_u, \quad (76)$$

where $\Gamma_u := \{T_0^n u; n \in \mathbb{N}\}$. By definition, $\Gamma_u \subset V_u$. So, taking into account assumption (A_2^u) , for all $f_1, f_2 \in \Gamma_u$, we obtain

$$\begin{aligned} \|T_0 f_1(t) - T_0 f_2(t)\| & \leq \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \|F \\ & \quad \cdot (s, f_1, Kf_1(s), Hf_1(s)) \\ & \quad - F(s, f_2, Kf_2(s), Hf_2(s))\| ds \\ & \leq \frac{L_1}{\Gamma(\alpha)} \int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-1} \\ & \quad \cdot (\|f_1(\tau) - f_2(\tau)\| + \|Kf_1(\tau) \\ & \quad - Kf_2(\tau)\| \|Hf_1(\tau) - Hf_2(\tau)\|) d\tau. \end{aligned} \quad (77)$$

By virtue of assumption (A_3^u) and relations (57) and (68), we immediately get

$$\|Kf_1(s) - Kf_2(s)\| \leq N_1 P_1 \varphi(s) d(f_1, f_2), \quad \text{for all } s \in I, \quad (78)$$

and by relation (76), assumption (A_4^u) , and hypotheses (H), we obtain

$$\begin{aligned} \|Hf_1(s) - Hf_2(s)\| &\leq \max \left\{ \gamma C_1 \int_a^\beta \|f_1(\tau) - f_2(\tau)\| d\tau, \right. \\ &\quad \cdot \left. \left\| \int_a^\lambda g(\tau)(f_1(\tau) - f_2(\tau)) d\tau \right\| \right\} \\ &= \gamma C_1 d(f_1, f_2) \int_a^\beta \varphi(\tau) d\tau. \end{aligned} \quad (79)$$

Substituting estimates (78) and (79) in (77), it yields

$$\begin{aligned} \|T_0 f_1(t) - T_0 f_2(t)\| &\leq \frac{L_1(1 + NP_1)}{\Gamma(\alpha)} d(f_1, f_2) \\ &\quad \cdot \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \varphi(s) ds \\ &\quad + \frac{L_1 \gamma C_1 \int_a^\beta \varphi(\tau) d\tau}{\Gamma(\alpha)} d(f_1, f_2) \\ &\quad \cdot \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} ds \\ &\leq \frac{L_1 P_2(1 + N_1 P_1)}{\Gamma(\alpha)} d(f_1, f_2) \varphi(t) \\ &\quad + \frac{L_1 \gamma P_3 C_1 \int_a^\beta \varphi(\tau) d\tau}{\Gamma(\alpha)} d(f_1, f_2) \varphi(t). \end{aligned} \quad (80)$$

Or, with $\Omega = L_1(P_2(1 + N_1 P_1) + P_3 \gamma C_1 \int_a^\beta \varphi(\tau) d\tau / \Gamma(\alpha))$ and using the continuity of T_0 ,

$$d(T_0 f_1, T_0 f_2) \leq \Omega d(f_1, f_2), \quad (81)$$

for all $f_1, f_2 \in \bar{\Gamma}_u$. Obviously, $T_0(\Gamma_u) \subset \Gamma_u$, and by induction, we get

$$d(T_0^m f_1, T_0^m f_2) \leq \Omega^m d(f_1, f_2). \quad (82)$$

Keeping in mind assumption (A_5^u) , from (81) we get

$$d(f_1, f_2) \leq \frac{1}{1 - \Omega} (d(f_1, T_0 f_1) + d(f_2, T_0 f_2)). \quad (83)$$

By the two last inequalities, these yield

$$d(T_0^n u, T_0^m u) \leq \frac{\Omega^n + \Omega^m}{1 - \Omega} d(u, T_0 u), \quad (84)$$

for all $n, m \in \mathbb{N}$. Applying the fractional integration operator $I^{\alpha, \psi}$ to both sides of (62), taking into account the fact that $u(a) = T_0 u(a)$, we obtain

$$\|u(t) - T_0 u(t)\| \leq \frac{P_2}{\Gamma(\alpha)} \varphi(t), \quad (85)$$

for all $t \in I$. This implies that $d(u, T_0 u) < \infty$, and consequently, $d(T_0^n u, T_0^m u) \rightarrow 0$ as n and m tend to infinity. Space Y is complete; then, this Cauchy sequence converges to a point u^* of Y . By Lemma 8, operator T_0 is continuous; so, using inequality (84), we deduce that u^* is a fixed point of T_0 . But u_0 is a fixed point for T_0 , and inequality (81) assures that u^* is none other than u_0 . Letting m tends to infinity in (84), with $n = 0$, we get

$$d(u, u^*) \leq \frac{1}{1 - \Omega} d(u, T_0 u), \quad (86)$$

and consequently,

$$\|u(t) - u^*(t)\| \leq \frac{P_2}{(1 - \Omega)\Gamma(\alpha)} \varphi(t), \quad (87)$$

for all $t \in I$. This achieves the proof.

Corollary 15. Assume that $I = [a, +\infty)$ is an unbounded interval. If the function u satisfies

$$\|{}^c D^{\alpha, \psi} u(t) - F(t, u, Ku(t), Hu(t))\| \leq \varphi(t), \quad (88)$$

$$u(a) = \int_a^\lambda g(s)u(s)ds, \quad (89)$$

for all $t \in I$. Then, there exists a unique function $u^* : I \rightarrow \mathbb{R}$ solution of problems (9) and (10), with

$$\|u(t) - u^*(t)\| \leq \frac{P_2}{(1 - \Omega)\Gamma(\alpha)} \varphi(t), \quad (90)$$

for all $t \in I$.

Proof. Let $u : I \rightarrow \mathbb{R}$ be a continuously differential function satisfies (88) and (89) and $n_0 > \lambda$ a positive integer. For all $n \geq n_0$, by Theorem 13., there exists a unique continuous function $u_n : I_n \rightarrow \mathbb{R}$ satisfying

$$\begin{aligned}
u_n(t) = & \frac{1}{(1-\mu)\Gamma(\alpha)} \int_{a_n}^{\lambda} \psi'(\tau) F(\tau, u_n, Ku_n(\tau), Hu_n(\tau)) \\
& \cdot \int_{\tau}^{\lambda} g(s)(\psi(s) - \psi(\tau))^{\alpha-1} ds d\tau + \frac{1}{\Gamma(\alpha)} \\
& \cdot \int_{a_n}^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} F(s, u_n, Ku_n(s), Hu_n(s)) ds,
\end{aligned} \tag{91}$$

$$\|u(t) - u_n(t)\| \leq \frac{P_2}{(1-\Omega)\Gamma(\alpha)} \varphi(t), \tag{92}$$

for all $t \in I_n$. By virtue of the uniqueness of the solution u_n , we deduce that $u_n(t) = u_{n+j}(t)$ for all $j \in \mathbb{N}$ and all $t \in I_n$. We define $n(t) \in \mathbb{N}$ as

$$n(t) = \min \{n \in \mathbb{N} \mid t \in I_n\} \tag{93}$$

and the function $v_0 : I \rightarrow \mathbb{R}$ by

$$v_0(t) = u_{n(t)}(t). \tag{94}$$

Easily, by (91)-(94), we can prove that

$$\begin{aligned}
v_0(t) = & \frac{1}{(1-\mu)\Gamma(\alpha)} \int_a^{\lambda} \psi'(\tau) F(\tau, v_0, Kv_0(\tau), Hv_0(\tau)) \\
& \cdot \int_{\tau}^{\lambda} g(s)(\psi(s) - \psi(\tau))^{\alpha-1} ds d\tau + \frac{1}{\Gamma(\alpha)} \\
& \cdot \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} F(s, v_0, Kv_0(s), Hv_0(s)) ds,
\end{aligned}$$

$$\|v_0(t) - u(t)\| \leq \frac{P_2}{(1-\Omega)\Gamma(\alpha)} \varphi(t), \quad \text{for all } t \in I. \tag{95}$$

This achieves the proof.

5. Example

We consider the following nonlocal problem with neutral pantograph equation involving functional derivative and ψ -Caputo fractional derivative:

$$\begin{cases} {}^C D^{\alpha, \psi} u(t) = \frac{3 \ln(t+1) + |u(t)| + \left| \int_0^1 u(2/3s) ds \right|}{\left(3\pi + \left| \int_0^t \ln(1+s)(su^2)'(s) ds \right| \right) \left(\exp(t) + |u(t)| + \left| \int_0^1 u(2/3s) ds \right| \right)}, \\ u(0) = \frac{1}{2} \int_0^1 \frac{u(s)}{s + \sqrt{s}} ds, \end{cases} \tag{96}$$

where $\alpha = 2/3$ and $\psi : I = [0, 1] \rightarrow \mathbb{R}_+$ such that $\psi(t) = \ln^2(t+1)$.

We claim that problem (96) has a unique solution u^* verifying the following:

$$|u^*(t)| \leq \frac{10\sqrt{\pi}}{10\sqrt{\pi} - 3 \ln(2)} \ln(1 + \sqrt{t}), \tag{97}$$

for all $t \in [0, 1]$.

We shall prove that if a continuously differential function $u : I \rightarrow \mathbb{R}$ satisfies

$$\left| {}^C D^{\alpha, \psi} u(t) - \frac{3 \ln(t+1) + |u(t)| + \left| \int_0^1 u(2/3s) ds \right|}{\left(3\pi + \left| \int_0^t \ln(1+s)(su^2)'(s) ds \right| \right) \left(\exp(t) + |u(t)| + \left| \int_0^1 u(2/3s) ds \right| \right)} \right| \leq \ln(1 + \sqrt{t}), \tag{98}$$

$$u(0) = \frac{1}{4} \int_0^1 \frac{u(s)}{s + \sqrt{s}} ds, \tag{99}$$

for all $t \in I$. Then, there exists a unique continuous function $u^* : I \rightarrow \mathbb{R}$ solution of problem (96) with

$$|u(t) - u^*(t)| \leq \frac{10\sqrt{\pi}}{10\sqrt{\pi} - 3 \ln(2)} \varphi(t), \quad (100)$$

for all $t \in I$.

Let $u_0 \in Y := C^1([0, 1], \mathbb{R})$ verifying (98) and (99), and we put $\varphi(t) := \ln(1 + \sqrt{t})$. It is seen that problem (96) is equivalent to the following:

$$\begin{cases} {}^C D^{\alpha, \psi} u(t) = f\left(t, u, \int_0^t k(t, s, u(s)) ds, \int_0^{2/3} h(t, s, u(s)) ds\right) \\ u(0) = \int_0^1 g(s) u(s) ds, \end{cases} \quad (101)$$

where

$$I = [0, 1], \quad k(t, s, x) = 0, \quad h(t, s, x) = \frac{3}{2}x, \quad g(t) = \frac{1}{2(t + \sqrt{t})},$$

$$f(t, u, x, y) = \frac{3 \ln(t+1) + |u(t)| + |y|}{\left(3\pi + \left|\int_0^t \ln(1+s)(su^2)''(s) ds\right|\right)(\exp(t) + |u(t)| + |y|)}, \quad (102)$$

for all $t, s \in I$, $u \in C^1(I, \mathbb{R})$, and $x, y \in \mathbb{R}$. Further, we have the following:

(B_1) f is uniformly bounded, in fact

$$\sup_{t \in I, u \in Y, x, y \in \mathbb{R}} |f(t, u, x, y)| < \frac{1}{3\pi}. \quad (103)$$

(B_2) For all $u_1, u_2 \in Y$, if $f(0, u_1, Ku_1(0), Hu_1(0)) = f(0, u_2, Ku_2(0), Hu_2(0)) = 0$, then $u_1(0) = u_2(0) = 0$.

(B_3) $f : I \times V_{u_0} \times \mathbb{R}^2$ satisfies

$$\begin{aligned} & |f(t, u_1, x_1, y_1) - f(t, u_2, x_2, y_2)| \\ & \leq \frac{3}{25} (|u_1(t) - u_2(t)| + |x_1 - x_2| + |y_1 - y_2|). \end{aligned} \quad (104)$$

In fact, for all $t \in I$, $x_1, x_2, y_1, y_2 \in \mathbb{R}$ and $u_1, u_2 \in V_{u_0}$, we have

$$\begin{aligned} & |f(t, u_1, x_1, y_1) - f(t, u_2, x_2, y_2)| \\ & \leq \frac{1}{3\pi + \left|\int_0^t \ln(1+s)(su_1^2)''(s) ds\right|} \\ & \quad \times \left| \frac{3 \ln(t+1) + |u_1(t)| + |y_1|}{\exp(t) + |u_1(t)| + |y_1|} - \frac{3 \ln(t+1) + |u_2(t)| + |y_2|}{\exp(t) + |u_2(t)| + |y_2|} \right| \\ & \quad + \left| \frac{1}{3\pi + \left|\int_0^t \ln(1+s)(su_1^2)''(s) ds\right|} - \frac{1}{3\pi + \left|\int_0^t \ln(1+s)(su_2^2)''(s) ds\right|} \right| \\ & \quad \times \frac{3 \ln(t+1) + |u_2(t)| + |y_2|}{\exp(t) + |u_2(t)| + |y_2|}. \end{aligned} \quad (105)$$

On the other hand, we have

$$\begin{aligned} & \left| \frac{3 \ln(t+1) + |u_1(t)| + |y_1|}{\exp(t) + |u_1(t)| + |y_1|} - \frac{3 \ln(t+1) + |u_2(t)| + |y_2|}{\exp(t) + |u_2(t)| + |y_2|} \right| \\ & \leq \sup_{\xi \in I} |3 \ln(\xi+1) - \exp(\xi)| \\ & \quad \times \left| \frac{1}{\exp(t) + |u_1(t)| + |y_1|} - \frac{1}{\exp(t) + |u_2(t)| + |y_2|} \right| \\ & \leq (|u_1(t) - u_2(t)| + |y_1 - y_2|), \end{aligned} \quad (106)$$

$$\begin{aligned} & \left| \frac{1}{3\pi + \left|\int_0^t \ln(1+s)(su_1^2)''(s) ds\right|} - \frac{1}{3\pi + \left|\int_0^t \ln(1+s)(su_2^2)''(s) ds\right|} \right| \\ & \leq \frac{1}{9\pi^2} \left| \int_0^t \ln(1+s)(su_1^2 - su_2^2)''(s) ds \right| \leq \frac{\ln(2)}{9\pi^2} t |u_1^2(t) - u_2^2(t)|. \end{aligned} \quad (107)$$

But, using (B_1) , for all $u, v \in V_{u_0}$, gets

$$|G(u, v)(t)| \leq \frac{2 \ln(2)}{3\pi\sqrt{\pi}(1 - \ln(2))}. \quad (108)$$

Therefore, remembering that $u_1, u_2 \in V_{u_0}$, we obtain

$$\begin{aligned} & \left| \frac{1}{3\pi + \left|\int_0^t \ln(1+s)u_1'(s) ds\right|} - \frac{1}{\pi + \left|\int_0^t \ln(1+s)u_2'(s) ds\right|} \right| \\ & \leq \frac{2 \ln(2)}{27\pi^2} |u_1(t) - u_2(t)| \end{aligned} \quad (109)$$

Substituting (106) and (109) in (105), we get, immediately, the desired inequality.

$(B_4)k : I \times I \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and satisfies the Lipschitz condition:

$$|k(t, s, x) - k(t, s, y)| \leq 0|x - y|, \quad (110)$$

for all $t, s \in I$ and $f, g \in \mathbb{R}$.

$(B_5)h : I \times I \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and satisfies the Lipschitz condition:

$$\begin{aligned} |H(u_1(s)) - H(u_2(s))| &= \left| \int_0^{2/3} h(t, s, u_1(s)) - h(t, s, u_2(s)) ds \right| \\ &= \frac{3}{2} \left| \int_0^{2/3} (u_1(s) - u_2(s)) ds \right|. \end{aligned} \quad (111)$$

for all $t, s \in I$ and $x, y, x_1, y_1, z_1, x_2, y_2, z_2 \in \mathbb{R}$. $(B_6) \varphi \in L^1(I, \mathbb{R}_+)$ verifying φ is bounded, $\inf_{t \in I} \varphi(t) = 0$, and for

all $t \in I$, we have $\int_0^t \ln(1 + \sqrt{s}) ds \leq P_1 \ln(1 + \sqrt{t})$, $\int_0^t \psi'(s) (\psi(t) - \psi(s))^{-1/2} \ln(1 + \sqrt{s}) ds \leq P_2 \ln(1 + \sqrt{t})$, and $(\psi(t) - \psi(0))^{1/2} \leq P_3 \ln(1 + \sqrt{t})$, with $P_1 = 1$, $P_2 = 2 \ln(2)$ and $P_3 = 1$.

(B_7) With $M_1 := L_1(1 + (b - a)(N_1 + C_1))$, $L_1 = 3/25$, $N_1 = 0$, $C_1 = 3/2$, $\alpha = 1/2$, $a = 0$, $b = \lambda = 1$, $P_1 = 1$, $P_2 = 2 \ln(2)$, and $P_3 = 1$, we have

$$\begin{aligned} 0 &\leq \frac{M_1(\psi(b) - \psi(a))^\alpha}{\Gamma(\alpha + 1)(1 - \mu)} < 1, \\ 0 < \Omega &:= L_1 \frac{P_2(1 + N_1 P_1) + P_3 C_1 \int_a^\lambda \varphi(\tau) d\tau}{\Gamma(\alpha)} < 1. \end{aligned} \quad (112)$$

By assumptions $(B_1) - (B_7)$ and Theorems 10 and 13., we deduce that problem (96) has, exactly, one solution u^* and if a continuously differential function $u : I \rightarrow \mathbb{R}$ satisfies

$$\begin{aligned} \left| {}^c D^{\alpha, \psi} u(t) - \frac{3 \ln(t+1) + |u(t)| + \left| \int_0^1 u(2/3s) ds \right|}{\left(3\pi + \left| \int_0^t \ln(1+s) u'(s) ds \right| \right) \left(\exp(t) + |u(t)| + \left| \int_0^1 u(2/3s) ds \right| \right)} \right| &\leq \ln(1 + \sqrt{t}), \\ u(0) &= \frac{1}{2} \int_0^1 \frac{u(s)}{s + \sqrt{s}} ds, \end{aligned} \quad (113)$$

for all $t \in I$. Then,

$$|u(t) - u^*(t)| \leq \frac{10\sqrt{\pi}}{10\sqrt{\pi} - 3 \ln(2)} \ln(1 + \sqrt{t}) \quad \text{for all } t \in I. \quad (114)$$

With function $u \equiv 0$, we get

$$\left| {}^c D^{\alpha, \psi} u(t) - \frac{3 \ln(t+1) + |u(t)| + \left| \int_0^1 u(2/3s) ds \right|}{\left(3\pi + \left| \int_0^t \ln(1+s) u'(s) ds \right| \right) \left(\exp(t) + |u(t)| + \left| \int_0^1 u(2/3s) ds \right| \right)} \right| = \frac{\ln(1+t)}{\pi \exp(t)} \leq \ln(1 + \sqrt{t}), \quad (115)$$

and

$$u(0) = \frac{1}{2} \int_0^1 \frac{u(s)}{s + \sqrt{s}} ds = 0. \quad (116)$$

Therefore, by (109),

$$|u^*(t)| \leq \frac{10\sqrt{\pi}}{10\sqrt{\pi} - 3 \ln(2)} \ln(1 + \sqrt{t}), \quad (117)$$

for all $t \in [0, 1]$.

6. Conclusions

In this paper, using a fixed-point approach and by means of the ψ -Caputo fractional derivative, we have studied the stability of solutions of a nonlinear fractional mixed Volterra-Fredholm integro-differential equation with an integral initial condition.

In the first part of this paper, a weak form of the Banach contraction principal and some useful lemmas are stated and proved. In the second part, we established conditions that assure the existence and uniqueness of solutions for the considered problem, basing on the weak form of the Banach

contraction principal and the Schauder's fixed-point theorem. The study of the Ulam-Hyers-Rassias stability is with respect to a function φ . When we impose more restrictions on φ , it will make the result less interesting in the reality world and vice versa. In the third part of this work, we have discussed of Ulam-Hyers-Rassias with respect to a function with very less restrictions. Two tough restrictions were omitted, and the continuity and that $\inf \varphi(t)$ must be positive. Results in this paper provide a supplementary result concerning the regularity of solutions. Finally, the whole analysis has been demonstrated by a suitable example.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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