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Research Article

On Janowski Analytic (p, q)-Starlike Functions in Symmetric Circular Domain

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The main object of the present paper is to apply the concepts of (p, q)-derivative by establishing a new subclass of analytic functions connected with symmetric circular domain. Further, we investigate necessary and sufficient conditions for functions belonging to this class. Convex combination, weighted mean, arithmetic mean, growth theorem, and convolution property are also determined.

1. Introduction and Definitions

Quantum calculus or q-calculus is a generalization of classical calculus without the notation of limits. The theory of q-calculus is established by Jackson, for details see [1, 2]. Due to its numerous applications in various branches of applied sciences and mathematics, for example, physics, operator theory, numerical analysis, and differential equations, attracted researchers to this field. A detailed study on applications of q-calculus in operator theory may be found in [3]. The geometric interpretation of q-calculus has been recognized through studies on quantum groups. Starlikeness and convexity are two major properties of analytic functions. Ismail et al. [4] investigated the generalized starlike function $\mathcal{S}^*$, and certain subclasses close-to-convex functions of q-Mittag-Leffler functions were studied by Srivastava and Bansal [5], also the reader is referred to [6–12] for more details.

The foundation of quantum calculus is on one parameter, while the postquantum calculus or simply (p, q)-calculus is the generalization of q-calculus based on two parameters. By setting $p = 1$ in (p, q)-calculus, the q-calculus is obtained. The (p, q)-integer was considered by Chakrabarti and Jagannathan [13], also see the work [14–18]. The idea of q-starlike is extended to (p, q)-stalikeness by Raza et al. [19]. Before we define our new class in this field, we give some basics for a better understanding of the work to follow.

Let $\mathcal{A}$ represent the family of function f that are analytic in the open unit disc $\mathfrak{D} = \{z \in \mathbb{C} : |z| < 1\}$ having the series expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, (z \in \mathfrak{D}). \tag{1}$$

A function $f(z)$ of the form (1) is subordinate to function $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, symbolically represented $f(z) \prec g(z)$, if there occur a Schwarz function $w(z)$ with limitation that $w(0) = 0$, and $|w(z)| \leq 1$, then $f(z) = g(w(z))$. While the convolution of these functions can be defined by

$$f(z) * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, (z \in \mathfrak{D}). \tag{2}$$

For $0 < q < 1$, the q-derivative of a function f is defined by

$$\partial_q f(z) = \frac{f(z) - f(qz)}{z(1-q)}, (z \neq 0, q \neq 1), \tag{3}$$

where

$$[n]_q = \frac{1-q^n}{1-q} = 1 + \sum_{l=1}^{n-1} q^l, \quad [0, q] = 0, \tag{4}$$

see [13] for details.

Also for $0 < p < q < 1$, the (p, q)-derivative of a function f is defined in [2] as

$$\partial_{p,q} f(z) = \frac{f(pz) - f(qz)}{z(p-q)}, (z \neq 0, p \neq q). \tag{5}$$

It can easily be seen that for $n \in \mathbb{N} := \{1, 2, 3, \cdots\}$ and $z \in \mathfrak{D}, \partial_{p,q}(\sum_{n=1}^{\infty} a_n z^n) = \sum_{n=1}^{\infty} [n]_{p,q} a_n z^{n-1}$,

where

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}. \tag{6}$$

We note that $\partial_{1,q} f(z) = \partial_q f(z)$ (for more on this topic one should read [20–22]).

Sakaguchi [23], in year 1956, established the class of starlike functions with respect to symmetrical points denoted by $\mathcal{S}_s^*$ of holomorphic univalent functions in $\mathfrak{D}$ if the below condition is satisfies

$$\operatorname{Re} \frac{2zf'(z)}{f(z) - f(-z)} > 0, (z \in \mathfrak{A}). \tag{7}$$

Motivated by the work of [19, 23, 24], we now define $\mathcal{S}_{p,q}^*(l, m, \mathscr{C}, \mathscr{D})$ given below.

Definition 1. Let $-\mathscr{D} \leq \mathscr{C} < \mathscr{D} \leq 1, 0 < p < q \leq 1$ and $-1 \leq m < l \leq 1$, then the function $f \in \mathscr{A}$ is in the class $\mathcal{S}_{p,q}^*(l, m, \mathscr{C}, \mathscr{D})$ if it satisfies

$$\frac{(l-m)z\partial_{p,q} f(z)}{f(lz) - f(mz)} \prec \frac{1 + \mathscr{C}z}{1 + \mathscr{D}z}, (z \in \mathfrak{D}), \tag{8}$$

where the symbol "$\prec$" indicates the well-known subordination.

We note that $\mathcal{S}_{1,q}^*(l, m, \mathscr{C}, \mathscr{D}) = \mathcal{S}_q^*(l, m, \mathscr{C}, \mathscr{D})$, where

$$\mathcal{S}_q^*(l, m, \mathscr{C}, \mathscr{D}) = \left\{ f \in \mathscr{A} : \frac{(l-m)z\partial_q f(z)}{f(lz) - f(mz)} \prec \frac{1 + \mathscr{C}z}{1 + \mathscr{D}z}, (z \in \mathfrak{D}) \right\}, \tag{9}$$

and

$$\begin{aligned} \lim_{q \to 1^-} \mathcal{S}_q^*(1, -1, \mathscr{C}, \mathscr{D}) &= \mathcal{S}^*(\mathscr{C}, \mathscr{D}) \\ &= \left\{ f \in \mathscr{A} : \frac{2zf'(z)}{f(z) - f(-z)} \prec \frac{1 + \mathscr{C}z}{1 + \mathscr{D}z}, (z \in \mathfrak{D}) \right\}. \end{aligned} \tag{10}$$

Equivalently, a function $f \in \mathscr{A}$ is in the $\mathcal{S}_{p,q}^*(l, m, \mathscr{C}, \mathscr{D})$ if and only if

$$\left| \frac{(l-m)z\partial_{p,q} f(z)/f(lz) - f(mz) - 1}{\mathscr{D}((l-m)z\partial_{p,q} f(z)/f(lz) - f(mz)) - \mathscr{C}} \right| < 1, (z \in \mathfrak{D}). \tag{11}$$

In our main results, in the next section, we evaluate the criteria for functions belonging to this newly defined class. After that, the convex combination property for this class will be discussed. Then utilizing these results, the weighted mean and arithmetic mean properties will be investigated. Further, convolution type results will be discussed in the form of two theorems. At the end of this article, a conclusion and future work will be presented.

2. Main Results

Theorem 2. *Let $f \in \mathscr{A}$ be of the form (1). Then the function $f \in \mathcal{S}_{p,q}^*(l, m, \mathscr{C}, \mathscr{D})$, if and only if the following inequality holds*

$$\sum_{n=2}^{\infty} \left\{ [n]_{p,q}(1 + \mathscr{D}) - (\mathscr{C} + 1) \frac{l^n - m^n}{l - m} \right\} |a_n| < (\mathscr{D} - \mathscr{C}). \tag{12}$$

Proof. Let us suppose that the first inequality (12) holds. Then to show that $f \in \mathcal{S}_{p,q}^*(l, m, \mathscr{C}, \mathscr{D})$, we only need to prove the inequality (11). For this consider

$$\begin{aligned} &\left| \frac{(l-m)z\partial_{p,q} f(z)/f(lz) - f(mz) - 1}{D((l-m)z\partial_{p,q} f(z)/f(lz) - f(mz)) - C} \right| \\ &= \left| \frac{\sum_{n=2}^{\infty} \left[[n]_{p,q} - l^n - m^n/l - m \right] \alpha_n z^n}{(D-C)z - \sum_{n=2}^{\infty} \left[D[n]_{p,q} - C(l^n - m^n/l - m) \alpha_n z^n \right]} \right| \\ &\leq \frac{\sum_{n=2}^{\infty} \left[[n]_{p,q} - l^n - m^n/l - m \right] |\alpha_n|}{(D-C) - \sum_{n=2}^{\infty} \left[D[n]_{p,q} - C(l^n - m^n/l - m) \right] |\alpha_n|} < 1, \end{aligned} \tag{13}$$

where we used and this completes the direct part. Conversely, let $f \in \mathcal{S}_{p,q}^*(l, m, \mathscr{C}, \mathscr{D})$ be of from (1). Then from (11), we have for $z \in \mathfrak{D}$,

$$\left|\frac{(l-m)z\partial_{p,q}f(z)/f(lz)-f(mz)-1}{D\left((l-m)z\partial_{p,q}f(z)/f(lz)-f(mz)\right)-C}\right| = \left|\frac{\sum_{n=2}^{\infty}\left[[n]_{p,q}-l^n-m^n/l-m\right]|\alpha_n z^n|}{(D-C)z-\sum_{n=2}^{\infty}\left[D[n]_{p,q}-C(l^n-m^n/l-m)\right]|\alpha_n z^n|}\right|<1 \tag{14}$$

Since $|\mathrm{Re}z|<|z|<1$, we have

$$\mathrm{Re}\left\{\frac{\sum_{n=2}^{\infty}\left[[n]_{p,q}-l^n-m^n/l-m\right]|\alpha_n z^n|}{(D-C)z-\sum_{n=2}^{\infty}\left[D[n]_{p,q}-C(l^n-m^n/l-m)\right]|\alpha_n z^n|}\right\}<1 \tag{15}$$

Now we choose values of z on the real axis such that $(l-m)z\partial_{p,q}f(z)/f(lz)-f(mz)$ is real. Upon clearing the denominator in (15) and letting $z\longrightarrow 1^-$ through real values, we obtain the required inequality (12).

Theorem 3. *Let $f_i\in\mathcal{S}^*_{p,q}(l,m,\mathscr{C},\mathscr{D})$ and having power series representations*

$$f_i(z)=z+\sum_{k=1}^{\infty}a_{k,i}z^k, \textit{for } i=1,2,3,\cdots,t. \tag{16}$$

Then $\Phi\in\mathcal{S}^*_{p,q}(l,m,\mathscr{C},\mathscr{D})$, where

$$\Phi(z)=\sum_{i=1}^{t}\omega_i f_i(z) \textit{ with } \sum_{i=1}^{t}\omega_i=1. \tag{17}$$

Proof. By Theorem 2, one can write

$$\sum_{n=2}^{\infty}\left\{\frac{\left[[n]_{p,q}(1+\mathscr{D})-(\mathscr{C}+1)(l^n-m^n/l-m)\right]}{(\mathscr{D}-\mathscr{C})}\right\}|a_{n,i}|<1. \tag{18}$$

Therefore

$$\begin{aligned}\Phi(z)&=\sum_{i=1}^{t}\omega_i\left(z+\sum_{n=2}^{\infty}a_{n,i}z^n\right)\\&=z+\sum_{i=1}^{t}\sum_{n=2}^{\infty}\omega_i a_{n,i}z^n\\&=z+\sum_{n=2}^{\infty}\left(\sum_{i=1}^{t}\omega_i a_{n,i}\right)z^n;\end{aligned} \tag{19}$$

however,

$$\begin{aligned}&\sum_{n=2}^{\infty}\frac{\left[[n]_{p,q}(1+\mathscr{D})-(\mathscr{C}+1)(l^n-m^n/l-m)\right]}{(\mathscr{D}-\mathscr{C})}\left(\left|\sum_{i=1}^{t}\omega_i a_{n,i}\right|\right)\\&=\sum_{i=1}^{t}\omega_i\left[\sum_{n=2}^{\infty},\frac{\left[[n]_{p,q}(1+\mathscr{D})-(\mathscr{C}+1)(l^n-m^n/l-m)\right]}{(\mathscr{D}-\mathscr{C})},|a_{n,i}|\right]\leq 1,\end{aligned} \tag{20}$$

then $\Phi\in\mathcal{S}^*_{p,q}(l,m,\mathscr{C},\mathscr{D})$. Hence, the proof is completed.

Theorem 4. *If $f_1,f_2\in\mathcal{S}^*_{p,q}(l,m,\mathscr{C},\mathscr{D})$, then their weighted mean ψ_k is also in $\mathcal{S}^*_{p,q}(l,m,\mathscr{C},\mathscr{D})$, where ψ_k is defined by*

$$\psi_k(z)=\left\{\frac{(1-k)f_1(z)+(1+k)f_2(z)}{2}\right\}. \tag{21}$$

Proof. From (21), one can easily write

$$\psi_k(z)=z+\sum_{n=2}^{\infty}\left\{\frac{(1-k)a_n+(1+k)b_n}{2}\right\}z^n. \tag{22}$$

To prove that $\psi_k\in\mathcal{S}^*_{p,q}(l,m,\mathscr{C},\mathscr{D})$, it is enough to show that

$$\begin{aligned}&\sum_{n=2}^{\infty}\left\{\frac{\left[[n]_{p,q}(1+\mathscr{D})-(\mathscr{C}+1)(l^n-m^n/l-m)\right]}{(\mathscr{D}-\mathscr{C})}\right\}\\&\cdot\left\{\frac{(1-k)a_n+(1+k)b_n}{2}\right\}<1.\end{aligned} \tag{23}$$

For this, consider

$$\begin{aligned}&\sum_{n=2}^{\infty}\left\{\frac{\left[[n]_{p,q}(1+\mathscr{D})-(\mathscr{C}+1)(l^n-m^n/l-m)\right]}{(\mathscr{D}-\mathscr{C})}\right\}\\&\cdot\left\{\frac{(1-k)a_n+(1+k)b_n}{2}\right\}\\&=\frac{(1-j)}{2}\cdot\sum_{n=2}^{\infty}\left\{\frac{\left[[n]_{p,q}(1+\mathscr{D})-(\mathscr{C}+1)(l^n-m^n/l-m)\right]}{(\mathscr{D}-\mathscr{C})}\right\}|a_n|\\&\quad+\frac{(1+j)}{2}\cdot\sum_{n=2}^{\infty}\left\{\frac{\left[[n]_{p,q}(1+\mathscr{D})-(\mathscr{C}+1)(l^n-m^n/l-m)\right]}{(\mathscr{D}-\mathscr{C})}\right\}|b_n|\\&<\frac{(1-k)}{2}+\frac{(1+k)}{2}=1,\end{aligned} \tag{24}$$

where we have used inequality (12). Which completes the proof.

Theorem 5. *Let $f_i \in \mathcal{S}^*_{p,q}(l, m, \mathscr{C}, \mathscr{D})$, with $i = 1, 2, \cdots, j$. Then, their arithmetic mean φ of f_i*

$$\varphi(z) = \frac{1}{j}\sum_{i=1}^{j} f_i(z), \tag{25}$$

is also in the class $\mathcal{S}^*_{p,q}(l, m, \mathscr{C}, \mathscr{D})$.

Proof. From (25), we can write

$$\varphi(z) = \frac{1}{j}\sum_{i=1}^{j}\left(z + \sum_{n=2}^{\infty} a_{n,i}z^n\right) = z + \sum_{n=2}^{\infty}\left(\frac{1}{j}\sum_{i=1}^{j} a_{n,i}\right)z^n. \tag{26}$$

Since $f_i \in \mathcal{S}^*_{p,q}(l, m, \mathscr{C}, \mathscr{D})$ for every $i = 1, 2, \cdots, j$, using (12), we have

$$\begin{aligned}
&\sum_{n=2}^{\infty}\left\{\frac{\left[[n]_{p,q}(1+\mathscr{D}) - (\mathscr{C}+1)(l^n - m^n/l - m)\right]}{(\mathscr{D}-\mathscr{C})}\right\} \cdot \left|\frac{1}{j}\sum_{i=1}^{j} a_{n,i}\right| \\
&= \frac{1}{j}\sum_{i=1}^{j}\left(\sum_{n=2}^{\infty}\left\{\frac{\left[[n]_{p,q}(1+\mathscr{D}) - (\mathscr{C}+1)(l^n - m^n/l - m)\right]}{(\mathscr{D}-\mathscr{C})}\right\} \cdot |a_{n,i}|\right) \\
&\le \frac{1}{j}\sum_{i=1}^{j}(1) = 1,
\end{aligned} \tag{27}$$

which complete the proof.

Theorem 6. *Let $f \in \mathcal{S}^*_{p,q}(l, m, \mathscr{C}, \mathscr{D})$. Then for $|z| = r, 0 < r < 1$,*

$$r - \delta_{p,q}(l, m, \mathscr{C}, \mathscr{D})r^2 < |f(z)| < r + \delta_{p,q}(l, m, \mathscr{C}, \mathscr{D})r^2, \tag{28}$$

where

$$\delta_{p,q}(l, m, \mathscr{C}, \mathscr{D}) = \frac{(\mathscr{C}-\mathscr{D})}{[2]_{p,q}(1-\mathscr{D}) + (\mathscr{C}+1)(l+m)}. \tag{29}$$

$$r - \gamma_{p,q}(l, m, \mathscr{C}, \mathscr{D})r^2 < \left|\partial_{p,q}f(z)\right| < r + \gamma_{p,q}(l, m, \mathscr{C}, \mathscr{D})r^2, \tag{30}$$

where

$$\gamma_{p,q}(l, m, \mathscr{C}, \mathscr{D}) = \frac{(\mathscr{C}-\mathscr{D})}{(1-\mathscr{D}) + (\mathscr{C}+1)(l+m)}. \tag{31}$$

Proof. To prove (28), consider

$$|f(z)| \le r + \sum_{n=2}^{\infty} |a_n||r|^n, \tag{32}$$

as $0 < r < 1$ so $r^n < r^2$ hence

$$|f(z)| < r + r^2\sum_{n=2}^{\infty} |a_n| \le r + \frac{(\mathscr{C}-\mathscr{D})}{[2]_{p,q}(1-\mathscr{D}) + (\mathscr{C}+1)(l+m)}r^2. \tag{33}$$

Similarly,

$$\begin{aligned}
|f(z)| &\ge r - \sum_{n=2}^{\infty} |a_n||r|^n > r - r^2\sum_{n=2}^{\infty} |a_n| \\
&\ge r - \frac{(\mathscr{C}-\mathscr{D})}{[2]_{p,q}(1-\mathscr{D}) + (\mathscr{C}+1)(l+m)}r^2.
\end{aligned} \tag{34}$$

Hence complete the proof of (28). Similarly, we can prove (30).

Theorem 7. *Let $f_i \in \mathcal{S}^*_{p,q}(l, m, \mathscr{C}, \mathscr{D})$, such that*

$$f_i(z) = z + \sum_{n=2}^{\infty} a_{n,i}z^n, i = 1, 2, \tag{35}$$

with condition $|a_{n,2}| \le 1$, then $f_1 * f_2 \in \mathcal{S}^*_{p,q}(l, m, \mathscr{C}, \mathscr{D})$.

Proof. Since form (35), we have

$$f_i(z) = z + \sum_{n=2}^{\infty} a_{n,i}z^n, i = 1, 2. \tag{36}$$

Then convolution is defined as

$$(f_1 * f_2)(z) = z + \sum_{n=2}^{\infty} a_{n,1}a_{n,2}z^n. \tag{37}$$

Since $f_2 \in \mathcal{S}^*_{p,q}(l, m, \mathscr{C}, \mathscr{D})$, with limitation that $|a_{n,2}| \le 1$. Therefore

$$\begin{aligned}
&\sum_{n=2}^{\infty}\left\{\frac{\left[[n]_{p,q}(1+\mathscr{D}) - (\mathscr{C}+1)(l^n - m^n/l - m)\right]}{(\mathscr{D}-\mathscr{C})}\right\}|a_{n,1}||a_{n,2}| \\
&\quad \le \sum_{n=2}^{\infty}\left\{\frac{\left[[n]_{p,q}(1+\mathscr{D}) - (\mathscr{C}+1)(l^n - m^n/l - m)\right]}{(\mathscr{D}-\mathscr{C})}\right\}|a_{n,1}| < 1.
\end{aligned} \tag{38}$$

Hence $f_1 * f_2 \in \mathcal{S}^*_{p,q}(l, m, \mathscr{C}, \mathscr{D})$.

Theorem 8. *Let $f(z) \in \mathcal{S}^*_{p,q}(l, m, \mathscr{C}, \mathscr{D})$. Then*

$$\frac{1}{z}\left[f(z) * \left(\frac{\left(1+\mathscr{D}e^{i\theta}\right)z}{(1-pz)(1-qz)} - \frac{\left(1+\mathscr{C}e^{i\theta}\right)z}{(1-lz)(1-mz)}\right)\right] \ne 0. \tag{39}$$

Proof. Let $f(z) \in \mathcal{S}^*_{p,q}(l, m, \mathscr{C}, \mathscr{D})$. Then by definition of subordination, there exists a Schwarz function $w(z)$, such that $w(0) = 0$ and $|w(z)| < 1$,

$$\frac{(l-m)z\partial_{p,q}f(z)}{f(lz)-f(mz)} = \frac{1+\mathscr{C}w(z)}{1+\mathscr{D}w(z)}, \tag{40}$$

equivalently,

$$\frac{(l-m)z\partial_{p,q}f(z)}{f(lz)-f(mz)} \neq \frac{1+\mathscr{C}e^{i\theta}}{1+\mathscr{D}e^{i\theta}}, \tag{41}$$

$$z\partial_{p,q}f(z)\left(1+\mathscr{D}e^{i\theta}\right) - \frac{f(lz)-f(mz)}{l-m}\left(1+\mathscr{C}e^{i\theta}\right) \neq 0, \tag{42}$$

using the relations

$$\begin{aligned} z\partial_{p,q}f(z) &= f(z) * \frac{z}{(1-pz)(1-qz)}, \\ \frac{f(lz)-f(mz)}{l-m} &= f(z) * \left[\frac{z}{(1-lz)(1-mz)}\right], \end{aligned} \tag{43}$$

now (42), becomes

$$\frac{1}{z}\left[f(z) * \left(\frac{\left(1+\mathscr{D}e^{i\theta}\right)z}{(1-pz)(1-qz)} - \frac{\left(1+\mathscr{C}e^{i\theta}\right)z}{(1-lz)(1-mz)}\right)\right] \neq 0, \tag{44}$$

which completes the proof.

3. Conclusions

Utilizing the concepts of postquantum calculus, we defined a new subclass of analytic functions associated with symmetric circular domain. For this class, we investigated some useful results such as necessary and sufficient problem, convex combination, weight mean, arithmetic mean, distortion bounds, and convolution property. There are some problems open for researchers such as radii problems, extreme point theorem, analytic criteria, and integral mean of inequality. Moreover, this concept is new and can be extended to meromorphic functions and harmonic functions.

Data Availability

Data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors jointly worked on the results and they read and approved the final manuscript.

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