Research Article

V-Proximal Trustworthy Banach Spaces

Messaoud Bounkhel and Mostafa Bachar

Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia

Correspondence should be addressed to Messaoud Bounkhel; bounkhel@ksu.edu.sa

Received 15 December 2019; Accepted 10 April 2020; Published 5 June 2020

Academic Editor: Gestur Ólafsson

Copyright © 2020 Messaoud Bounkhel and Mostafa Bachar. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In a recent work (2016), the first author proved the fuzzy sum rule for the V -proximal subdifferential under some natural assumptions on an equivalent norm of the Banach spaces. In the present paper, we are going to prove that the class of Banach spaces satisfying the fuzzy sum rule is very large and contains all L^p spaces $1 < p < \infty$ as well as the sequence spaces ℓ^p ($1 < p < \infty$), the Sobolev spaces $W^{p,n}$ ($1 < p < \infty$), and the Schatten trace ideals $\mathbf{C}^p$ ($1 < p < \infty$).

1. Preliminaries

Let X be a Banach space with dual X^* , $f : X \rightarrow \mathbf{R} \cup \{\infty\}$ a function, $\bar{x} \in \text{dom } f : \{x \in X : f(x) < \infty\}$. Throughout the paper, $\mathbf{B}$ denotes the closed unit ball in X and $\langle \cdot, \cdot \rangle$ is the dual pairing between X and its dual X^* . We define (see [1, 2]) the analytic (resp., the geometric) V -proximal subdifferential of f at $\bar{x}$ as follows:

$$\partial_A^\pi f(\bar{x}) := \{x^* \in X^* : \exists \delta, \sigma > 0, \langle x^*, x - \bar{x} \rangle \leq f(x) - f(\bar{x}) + \sigma V(J(\bar{x}), x), \forall x \in \bar{x} + \delta \mathbf{B}\} \quad (1)$$

(resp. $\partial_G^\pi f(\bar{x}) := \{x^* \in X^* : (x^*, -1) \in N^\pi(\text{epi } f; (\bar{x}, f(\bar{x})))\}$), where $N^\pi(S; u) := \partial_A^\pi \psi_S(u)$ is the V -proximal normal cone associated with S at $u \in S$. Here, $J : X \rightarrow X^*$ is the normalized duality mapping and $V : X^* \times X \rightarrow \mathbf{R}$ is a functional defined by

$$V(x^*, x) = \|x^*\|^2 - 2\langle x^*, x \rangle + \|x\|^2, \text{ for any } x^* \in X^*, x \in X. \quad (2)$$

We recall, respectively, the well-known concepts of proximal subdifferential and Fréchet subdifferential (see, for instance, [3]):

- (i) $x^* \in \partial^P f(x)$ if and only if there exist $\delta, \sigma > 0$ such that $\langle x^*, x' - x \rangle \leq f(x') - f(x) + \sigma \|x' - x\|^2$, for any $x' \in x + \delta \mathbf{B}$
- (ii) $x^* \in \partial^F f(x)$ if and only if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\langle x^*, x' - x \rangle \leq f(x') - f(x) + \varepsilon \|x' - x\|, \text{ for any } x' \in x + \delta \mathbf{B} \quad (3)$$

The Fréchet and proximal normal cones are defined as $N^F(S; \bar{x}) = \partial^F \psi_S(\bar{x})$ and $N^P(S; \bar{x}) = \partial^P \psi_S(\bar{x})$. Notice that (see [3–5]) Fréchet and proximal subdifferential can be defined geometrically by the formulas

$$\partial^F f(\bar{x}) = \{x^* \in X^* : (x^*, -1) \in N^F(S; \bar{x})\}, \quad (4)$$

$$\partial^P f(\bar{x}) = \{x^* \in X^* : (x^*, -1) \in N^P(S; \bar{x})\}. \quad (5)$$

Using the same terminology used in Ioffe [6], we will say that X is a V -proximal trustworthy space provided that for any $\varepsilon > 0$, any two functions $f_1, f_2 : X \rightarrow \mathbf{R} \cup \{\infty\}$ and any $u \in X$ such that f_1 is lower semicontinuous and f_2 is Lipschitz around u , the following fuzzy sum rule holds:

$$\partial_A^\pi(f_1 + f_2)(u) \subset \bigcup \left\{ \partial_A^\pi f_1(u_1) + \partial_A^\pi f_2(u_2) : u_i \in U_{f_i}(u, \varepsilon), \quad i = 1, 2 \right\} + \varepsilon \mathbf{B}_*. \quad (6)$$

Here, $U_{f_i}(u, \varepsilon) := \{x \in u + \varepsilon \mathbf{B} \text{ such that } |f_i(x) - f_i(u)| < \varepsilon\}$ and $\mathbf{B}_*$ denotes the closed unit ball in X^* . It has been proved in Theorem 2.3 in [1] that if X is a Banach space with an equivalent norm $\|\cdot\|$ such that $\|\cdot\|^s$ (for some $s \geq 2$) is C^2 -differentiable on $X \setminus \{0\}$ and let V be the functional associated to that norm $\|\cdot\|$, then X is a V -proximal trustworthy space. Indeed, we have proved the following result.

Theorem 1. *Let X be a Banach space with an equivalent norm $\|\cdot\|$ such that $\|\cdot\|^s$ (for some $s \geq 2$) is C^2 -differentiable on $X \setminus \{0\}$, and let V be the functional associated to that norm $\|\cdot\|$. For any $\varepsilon > 0$, any two functions $f_1, f_2 : X \rightarrow \mathbf{R} \cup \{\infty\}$ and any $u \in X$ such that f_1 is lower semicontinuous and f_2 is Lipschitz around u , the following fuzzy sum rule holds:*

$$\partial_A^\pi(f_1 + f_2)(u) \subset \bigcup \left\{ \partial_A^\pi f_1(u_1) + \partial_A^\pi f_2(u_2) : u_i \in U_{f_i}(u, \varepsilon), i = 1, 2 \right\} + \varepsilon \mathbf{B}_*. \quad (7)$$

For a positive measure space (Ω, Σ, μ) , we denote by L^p , $p \in [1, \infty)$, the Banach space $L^p(\Omega, \Sigma, \mu)$ with its canonical norm $\|x\| = (\int_\Omega |x(w)|^p d\mu(w))^{1/p}$.

We recall the following result from Theorem 1.1 in Section 5.1 in [7] (see also [8]).

Theorem 2.

- (i) *If p is an even integer, then $\|\cdot\|^p$ is C^∞ -differentiable on $L^p \setminus \{0\}$*
- (ii) *If p is an odd integer, then $\|\cdot\|^p$ is C^{p-1} -smooth on $L^p \setminus \{0\}$*
- (iii) *If p is not integer, then $\|\cdot\|^p$ is $C^{[p]}$ -smooth on $L^p \setminus \{0\}$, where $[p]$ is the integer part of p*

The following corollary follows directly from Theorem 2.

Corollary 3. *For the canonical norm of L^p , we have $\|\cdot\|^p$ which is C^2 -differentiable on $L^p \setminus \{0\}$, for any $p \geq 2$.*

Unfortunately, for the case of $p \in (1, 2)$, the function $\|\cdot\|^p$ is not C^2 -differentiable on $L^p \setminus \{0\}$ and so the fuzzy sum rule cannot be covered by Theorem 1. This our objective in the next few lines, and so we obtain that L^p is V -proximal trustworthy, for any $p \in (1, \infty)$. In the sequel of this section, we assume that $X := L^p$ with $p \in (1, 2)$. It is well-known that X is 2-uniformly convex (see, for instance, [7]), that is, there is a constant $c > 0$ such that

$$\delta_X(\varepsilon) \geq c\varepsilon^2, \quad \text{for all } 0 < \varepsilon \leq 2. \quad (8)$$

The following lemma is taken from [9].

Lemma 4. *If E is a uniformly convex Banach space, then the inequality*

$$V(J(x), y) \geq 8C^2 \delta_E \left(\frac{\|x - y\|}{4C} \right) \quad (9)$$

holds for all x and y in E , where $C = \sqrt{(\|x\|^2 + \|y\|^2)/2}$.

In page 51 in [9], the following parallelogram inequality is presented:

$$2\|x\|^2 + 2\|y\|^2 - \|x - y\|^2 \geq (p - 1)\|x + y\|^2 \quad (1 < p \leq 2) \quad \text{for any } x, y \in X. \quad (10)$$

It follows that

$$\frac{\|x - y\|}{4C} \leq \frac{1}{2}, \quad \text{for any } x, y \in X, \quad (11)$$

and so the previous lemma with the inequality (8) yields the following important result.

Proposition 5. *For some positive constant $c > 0$, we have for any x and y in X ($X = L^p$ ($1 < p \leq 2$)),*

$$V(J(x), y) \geq \frac{c}{2} \|x - y\|^2. \quad (12)$$

Using this proposition, we get obviously, in our setting ($X = L^p$ ($1 < p \leq 2$)), the inclusions

$$\partial^p f(\bar{x}) \subset \partial_A^\pi f(\bar{x}), \quad (13)$$

$$N^p(S; \bar{x}) \subset N^\pi(S; \bar{x}). \quad (14)$$

However, the inclusions $\partial_A^\pi f(\bar{x}) \subset \partial_G^\pi f(\bar{x}) \subset \partial^F f(\bar{x})$ and $N^\pi(S; \bar{x}) \subset N^F(S; \bar{x})$ have been proved, in [1], for Banach spaces with the assumption: For some $\bar{\sigma}, \bar{\delta} > 0$ and $q \in (1, 2]$,

$$V(J(\bar{x}), x) \leq \bar{\sigma} \|x - \bar{x}\|^q, \quad \text{for all } x \in \bar{x} + \bar{\delta} \mathbf{B}. \quad (15)$$

This assumption is valid whenever the space is q -uniformly convex Banach spaces. For reason of completeness, we present their proofs.

Proposition 6. *Assume that there exist constants $\bar{\sigma}, \bar{\delta} > 0$ and $q \in (1, 2]$ such that 1.8 holds. Then,*

$$\begin{aligned} N^\pi(S; \bar{x}) &\subset N^F(S; \bar{x}), \\ \partial_A^\pi f(\bar{x}) &\subset \partial_G^\pi f(\bar{x}) \subset \partial^F f(\bar{x}). \end{aligned} \quad (16)$$

Proof. We prove only the first relation; the second one can be conducted similarly. Let $x^* \in N^\pi(S; \bar{x})$. Then, by Proposition 3.4 in [2], there exists $\sigma > 0$ such that

$$\langle x^*, x - \bar{x} \rangle \leq \sigma V(J(\bar{x}), x), \quad \text{for all } x \in S. \quad (17)$$

Now, let $\varepsilon > 0$ be given and take $\mu = (1/2) \min \{\bar{\delta}, (\varepsilon/\sigma\bar{\sigma})^{a/(q-1)}\}$. Then, by (15), for $x \in S \cap (\bar{x} + \mu\mathbf{B}) \subset \bar{x} + \bar{\delta}\mathbf{B}$, one has

$$\begin{aligned} \langle x^*, x - \bar{x} \rangle &\leq \sigma V(J(\bar{x}), x) \leq \sigma\bar{\sigma} \|x - \bar{x}\|^q \\ &= \sigma\bar{\sigma} \|x - \bar{x}\|^{q-1} \|x - \bar{x}\| < \varepsilon \|x - \bar{x}\|. \end{aligned} \quad (18)$$

Hence, for any $\varepsilon > 0$, there exists $\mu > 0$ such that $\langle x^*, x - \bar{x} \rangle \leq \varepsilon \|x - \bar{x}\|$, for any $x \in S \cap (\bar{x} + \mu\mathbf{B})$, that is $x^* \in N^F(S; \bar{x})$ and so $N^\pi(S; \bar{x}) \subset N^F(S; \bar{x})$.

Using Remark 7.7 in [9] and the fact that $X = L^p(p \in (1, 2])$ is p -uniformly smooth, we obtain

$$V(J(x), y) \leq \beta \rho_X \left(\frac{4\|x - y\|}{R} \right) \leq \beta \frac{4^p \|x - y\|^p}{R^p}, \quad \forall x, y \in R\mathbf{B} \quad (19)$$

and hence, for any $\bar{x} \in X$ and any $\delta > 0$, we can write

$$V(J(\bar{x}), x) \leq \bar{\sigma} \|x - \bar{x}\|^p, \quad \forall x \in \bar{x} + \delta\mathbf{B}, \quad (20)$$

where $\bar{\sigma}$ depends on $\bar{x}, \delta$, and β . Consequently, as a direct corollary of Proposition 6, we have, in our setting of L^p spaces ($p \in (1, 2]$), both inclusions

$$N^\pi(S; \bar{x}) \subset N^F(S; \bar{x}), \quad (21)$$

$$\partial_A^\pi f(\bar{x}) \subset \partial_G^\pi f(\bar{x}) \subset \partial^F f(\bar{x}) \quad (22)$$

that hold for any lower semicontinuous function f and any nonempty closed set S .

Now, we recall from Ioffe [10] and Fabian [11] the following two important results.

Proposition 7. Suppose that X is uniformly convex and that S is a closed subset of X . Let $x^- \in S$ and $x^* \in N^F(S; x^-)$. Then, for any $\varepsilon > 0$, there exist $x_\varepsilon \in S$ and $x_\varepsilon^* \in N^P(S; x_\varepsilon)$ such that $\|x_\varepsilon - x^-\| < \varepsilon$ and $\|x_\varepsilon^* - x^*\| < \varepsilon$.

Theorem 8. Let X be an Asplund space. For any $\varepsilon > 0$, any two functions $f_1, f_2 : X \rightarrow \mathbf{R} \cup \{\infty\}$ and any $u \in X$ such that f_1 is lower semicontinuous and f_2 is Lipschitz around u , the following fuzzy sum rule holds:

$$\partial^F(f_1 + f_2)(u) \subset \cup \left\{ \partial^F f_2(u_2) : u_i \in U_{f_i}(u, \varepsilon), i = 1, 2 \right\} + \varepsilon \mathbf{B}_*. \quad (23)$$

Now we are ready to prove the main result of this section.

Theorem 9. Let $X = L^p(p \in (1, 2])$. For any $\varepsilon > 0$, any two functions $f_1, f_2 : X \rightarrow \mathbf{R} \cup \{\infty\}$ and any $u \in X$ such that f_1 is lower semicontinuous and f_2 is Lipschitz around u , the following fuzzy sum rule holds:

$$\partial_A^\pi(f_1 + f_2)(u) \subset \cup \left\{ \partial_A^\pi f_1(u_1) + \partial_A^\pi f_2(u_2) : u_i \in U_{f_i}(u, \varepsilon), i = 1, 2 \right\} + \varepsilon \mathbf{B}_*. \quad (24)$$

Proof. Fix any $\varepsilon > 0$. Let $x^* \in \partial_A^\pi(f_1 + f_2)(u)$. Then, by (21) we have $x^* \in \partial^F(f_1 + f_2)(u)$. Then, by Theorem 8, there exist $v_i \in U_{f_i}(u, \varepsilon/2)$ and $x_i^* \in \partial^F f_i(v_i)$, $i = 1, 2$ such that $\|x^* - (x_1^* + x_2^*)\| < \varepsilon/2$. Thus, $(x_i^*, -1) \in N^F(\text{epif}_i; (v_i, f_i(v_i)))$, $i = 1, 2$. Let $\delta \in (0, \min \{\varepsilon/2, \varepsilon/(4(1 + \max \{\|x_1\|, \|x_2\|\}))\})$. We use now Proposition 7 to get $i = 1, 2$, $(x_{i,\delta}, r_{i,\delta}) \in \text{epif}_i$ and $(x_{i,\delta}^*, -\alpha_{i,\delta}) \in N^P(\text{epif}_i; (x_{i,\delta}, r_{i,\delta}))$ such that

$$\begin{aligned} \|(x_{i,\delta}, r_{i,\delta}) - (v_i, f_i(v_i))\| &< \frac{\delta}{2}, \\ \|(x_{i,\delta}^*, -\alpha_{i,\delta}) - (x_i^*, -1)\| &< \frac{\delta}{2}. \end{aligned} \quad (25)$$

Clearly, $\alpha_{i,\delta} > 0$ and so $r_{i,\delta} = f_i(x_{i,\delta})$, for $i = 1, 2$. Thus, by the inclusions (5) and (13), we obtain

$$\begin{aligned} \frac{x_{i,\delta}}{\alpha_{i,\delta}} &\in \partial^P f_i(x_{i,\delta}) \subset \partial_A^\pi f_i(x_{i,\delta}) \\ \|(x_{i,\delta}, f_i(x_{i,\delta})) - (v_i, f_i(v_i))\| &< \frac{\delta}{2}, \\ \|(x_{i,\delta}^*, -\alpha_{i,\delta}) - (x_i^*, -1)\| &< \frac{\delta}{2}. \end{aligned} \quad (26)$$

Set $y_i^* := x_{i,\delta}^*/\alpha_{i,\delta}$ and $u_i := x_{i,\delta}$ ($i = 1, 2$). Then, we have

$$\begin{aligned} \|u_i - u\| &\leq \|u_i - v_i\| + \|v_i - u\| \leq \frac{\delta}{2} + \frac{\varepsilon}{2} < \varepsilon, \\ \|f_i(u_i) - f_i(u)\| &\leq \|f_i(u_i) - f_i(v_i)\| + \|f_i(v_i) - f_i(u)\| \\ &< \frac{\delta}{2} + \frac{\varepsilon}{2} < \varepsilon. \end{aligned} \quad (27)$$

Also, we have

$$\begin{aligned} \|y_i^* - x_i^*\| &\leq \|y_i^* - \alpha_{i,\delta}^{-1} x_i^*\| + \|\alpha_{i,\delta}^{-1} x_i^* - x_i^*\| \\ &\leq \alpha_{i,\delta}^{-1} \|x_{i,\delta}^* - x_i^*\| + |\alpha_{i,\delta}^{-1} - 1| \|x_i^*\| \\ &\leq \frac{\delta}{2\alpha_{i,\delta}} + \frac{|1 - \alpha_{i,\delta}|}{\alpha_{i,\delta}} \|x_i^*\| \leq \frac{\delta}{2\alpha_{i,\delta}} (1 + \|x_i^*\|) \\ &< \delta (1 + \|x_i^*\|) \leq \frac{\varepsilon}{4}. \end{aligned} \quad (28)$$

Thus,

$$\begin{aligned} \|x^* - (y_1^* + y_2^*)\| &\leq \|x^* - (x_1^* + x_2^*)\| + \sum_{i=1}^2 \|y_i^* - x_i^*\| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned} \quad (29)$$

Therefore, for any $\varepsilon > 0$, there exist $u_1 \in U_{f_1}(u, \varepsilon)$, $u_2 \in U_{f_2}(u, \varepsilon)$, $y_1^* \in \partial_{A f_1}^\pi(u_1)$, and $y_2^* \in \partial_{A f_2}^\pi(u_2)$ such that

$$x^* \in y_1^* + y_2^* + \varepsilon \mathbf{B}_*. \quad (30)$$

This completes the proof.

Remark 10.

- (i) An inspection of the previous proof shows that Theorem 9 holds for any 2-uniformly convex and p -uniformly smooth. Indeed, all the assumptions of Propositions 6–7 and Theorem 8 are satisfied whenever the space is 2-uniformly convex and p -uniformly smooth. Consequently, all the spaces $\mathbb{P}$, $\mathbf{C}^p$, $W^{p,m}$, L^p for $p \in (1, 2]$ are V -proximal trustworthy. For the proof of the uniform convexity and uniform smoothness of the spaces $\mathbb{P}$, L^p , $W^{p,m}$, we refer, for instance, to Remark 1.6.9 in [9] and for the Schatten trace ideals $\mathbf{C}^p$, we refer to [12]
- (ii) For the case $p \geq 2$, we have the following:
 - (i) All Sobolev spaces $W^{p,m}$ and L^p are V -proximal trustworthy for any $p \geq 2$ by combining Theorem 2 and Theorem 1.
 - (ii) The spaces $\mathbf{C}^p$ for $p \geq 2$ are V -proximal trustworthy by combining Theorem 1. and the following result proved in Theorem 2 in [13].

Theorem 11.

- (i) If p is an even integer, then the norm in $\mathbf{C}^p$ is C^∞ -differentiable away from zero
- (ii) If p is an odd integer, then the norm in $\mathbf{C}^p$ is C^{p-1} -differentiable away from zero and is not C^{p-1} -differentiable
- (iii) If p is not integer, then the norm in $\mathbf{C}^p$ is $C^{[p]}$ -differentiable away from zero and is not $C^{[p]+1}$ -differentiable

2. Approximate Mean Value Theorem

This section is concerned with the approximate mean value theorem for both analytic and geometric V -proximal subdifferentials in V -proximal trustworthy spaces. It is well-known that approximate mean value theorem (AMVT) and its variants are very important in nonsmooth analysis and optimization. It has been proved for all the existing subdifferentials (see, for instance, [14]). Since the analytic V -proximal subdifferential ∂_A^π is the smallest one in some important cases (for instance, L^p , $\mathbb{P}$, $\mathbf{C}^p$, $W^{p,m}$ with $p \geq 2$), we cannot deduce that the AMVT form the ones proved for the other existing subdifferentials ([14]). For this reason, it is needed to prove the AMVT for the analytic V -proximal

subdifferential ∂_A^π in V -proximal trustworthy spaces. For the geometric V -proximal subdifferential ∂_G^π , the AMVT follows directly from the inclusion $\partial_A^\pi f \subset \partial_G^\pi f$.

Theorem 12 (Approximate mean value theorem). *Let X be a smooth Banach space which is V -proximal trustworthy and let $f : X \rightarrow \mathbf{R} \cup \{\infty\}$, be l.s.c. function finite at two distinct points $a \neq b$. Let $r \in \mathbf{R}$ with $r \leq f(b) - f(a)$. Then, there exists $c \in [a, b]$ and a sequence $x_n \rightarrow c$ with $f(x_n) \rightarrow f(c)$ and $x_n^* \in \partial_A^\pi f(x_n)$ such that*

$$\begin{aligned} \liminf_n \langle x_n^*, c - x_n \rangle &\geq 0, \\ \liminf_n \langle x_n^*, b - a \rangle &\geq r \\ f(c) - f(a) &\leq |r|. \end{aligned} \quad (31)$$

Proof. Since X is smooth, we can take $x^* \in X^*$ so that $r := \langle x^*, a - b \rangle$. Set $g(x) := f(x) + \langle x^*, x \rangle + \psi_{[a,b]}(x)$. Clearly, $g(b) \geq g(a)$ and so g attains its minimum at some point $c \in [a, b]$ and so $0 \in \partial_A^\pi [f(\cdot) + \langle x^*, \cdot \rangle + \psi_{[a,b]}(\cdot)](c)$. Using the fuzzy sum rule for ∂_A^π in Theorem 9, we get a sequence $x_n \rightarrow c$ with $f(x_n) \rightarrow f(c)$, $y_n \rightarrow c$, and $x_n^* \in \partial_A^\pi f(x_n)$, $y_n^* \in N^\pi([a, b]; y_n)$ such that $\|x_n^* + y_n^* + x^*\| < 1/n$. Then, $\{y_n\}_n \subset [a, b]$ for n sufficiently large and hence,

$$\langle y_n^*, z - y_n \rangle \leq 0, \quad \forall z \in [a, b], \forall n. \quad (32)$$

Hence, $\langle x_n^*, c - x_n \rangle = \langle x_n^* + x^* + y_n^*, c - x_n \rangle + \langle -y_n^*, c - y_n \rangle + \langle -y_n^*, y_n - x_n \rangle + \langle -x^*, c - x_n \rangle \geq \langle x_n^* + x^* + y_n^*, c - x_n \rangle + \langle -y_n^*, y_n - x_n \rangle + \langle -x^*, c - x_n \rangle$ (by (32)).

Taking the liminf on both sides of the previous inequality yields

$$\begin{aligned} \liminf_n \langle x_n^*, c - x_n \rangle &\geq \liminf_n [\langle x_n^* + x^* + y_n^*, c - x_n \rangle \\ &\quad + \langle -y_n^*, y_n - x_n \rangle + \langle -x^*, c - x_n \rangle] \geq 0, \end{aligned} \quad (33)$$

and hence, (i) is proved. Let us show (ii). Since $y_n \in [a, b]$, we can write $y_n = t_n b + (1 - t_n)a$ for a sequence $\{t_n\}_n \subset [0, 1]$ converging to some $t_0 \in [0, 1]$ satisfying $c = t_0 b + (1 - t_0)a$. Hence, $b - y_n = (1 - t_n)(b - a)$ and so $b - a = (1/(1 - t_n))(b - y_n)$. Hence,

$$\begin{aligned} \liminf_n \langle x_n^*, b - a \rangle &= \langle x^*, a - b \rangle + \liminf_n \langle x_n^* + x^*, b - a \rangle \\ &= r + \liminf_n \langle -y_n^*, b - a \rangle \\ &= r + \frac{1}{1 - t_0} \liminf_n \langle y_n^*, y_n - b \rangle \geq (\text{by 29}). \end{aligned} \quad (34)$$

To prove (iii), we use the fact that c is the minimum of g on $[a, b]$ and hence, $g(c) \leq g(a)$; that is, $f(c - f(a)) \leq \langle x^*, a - c \rangle \leq 1/(1 - t_0) \langle x^*, a - b \rangle = (1/(1 - t_0))r \leq |r|$.

We close this section with the following result which is an application of the AMVT. Many other applications of the previous version of AMVT can be proved.

Theorem 13 (Lipschitz behavior). *Let X be a smooth Banach space which is V -proximal trustworthy and let U be an open convex subset of X . Let $f: X \rightarrow \mathbf{R} \cup \{+\infty\}$ be a l.s.c. function with $U \cap \text{dom} f \neq \emptyset$. The following two assertions are equivalent:*

- (1) *For some positive constant $L > 0$, we have $\sup \{\|x^*\| : x^* \in \partial_A^\pi f(x)\} \leq L$, for all $x \in U$*
- (2) *f is Lipschitz on U with rank L*

Proof. The implication (2) $\Rightarrow$ (1) follows directly from the fact that $\partial_A^\pi f(x)$ is always included in the Clarke subdifferential. So, we have to prove the implication (1) $\Rightarrow$ (2). Let $x, y \in U$ and let $r := f(y) - f(x)$. Applying Theorem 12 to get a point $c \in [x, y] \subset U$ and a sequence $x_n \rightarrow c$ with $f(x_n) \rightarrow f(c)$ and $x_n^* \in \partial_A^\pi f(x_n)$ such that

$$f(y) - f(x) \leq \liminf_n \langle x_n^*, y - x \rangle. \quad (35)$$

Let $\varepsilon > 0$. By the definition of the liminf, there exist $z \in U$ and $z^* \in \partial_A^\pi f(z)$ such that

$$f(y) - f(x) \leq \langle z^*, y - x \rangle + \varepsilon \leq \|z^*\| \|y - x\| + \varepsilon \leq L \|y - x\| + \varepsilon. \quad (36)$$

By taking $\varepsilon \rightarrow 0$, we obtain

$$f(y) - f(x) \leq L \|y - x\|, \quad (37)$$

and by exchanging the roles of x and y we arrive to

$$|f(y) - f(x)| \leq L \|y - x\|, \quad \text{for all } x, y \in U. \quad (38)$$

This completes the proof.

Data Availability

Not applicable.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to this work.

Acknowledgments

The authors extend their appreciation to the Deanship of Scientific Research at King Saud University for funding the work through the research group Project No. RGP-024.

References

- [1] M. Bounkhel, "Calculus rules for π -proximal subdifferentials in smooth Banach spaces," *Journal of Function Spaces*, vol. 2016, Article ID 1917387, 12 pages, 2016.
- [2] M. Bounkhel and R. Al-Yusof, "Proximal analysis in reflexive smooth Banach spaces," *Nonlinear Analysis*, vol. 73, no. 7, pp. 1921–1939, 2010.
- [3] M. Bounkhel, "Regularity concepts in nonsmooth analysis, theory and applications," in *Springer Optimization and Its Applications*, vol. 59, Springer, 2012.
- [4] F. H. Clarke, Y. S. Ledyev, R. J. Stern, and P. R. Wolenski, *Nonsmooth Analysis and Control Theory, Graduate Texts in Mathematics*, vol. 178, Springer-Verlag, 1998.
- [5] B. Mordukhovich and Y. Shao, "Nonsmooth sequential analysis in Asplund spaces," *Transactions of the American Mathematical Society*, vol. 348, no. 4, pp. 1235–1281, 1996.
- [6] A. D. Ioffe, "On subdifferentiability spaces," *Annals of the New York Academy of Sciences*, vol. 410, pp. 107–119, 1983.
- [7] R. Deville, G. Godefroy, and V. Zizler, *Smoothness and renormings in Banach spaces, Pitman Monographs and Surveys in Pure and Applied Mathematics*, Longman Scientific & Technical, Harlow, UK, 1993.
- [8] R. Deville, R. Gonzalo, and J. A. Jaramilo, "Renormings of $L_p(L_q)$," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 126, no. 1, pp. 155–169, 1999.
- [9] Y. I. Alber and I. Ryazantseva, *Nonlinear Ill-Posed Problems of Monotone Type*, Springer, 2006.
- [10] A. D. Ioffe, "Proximal analysis and approximate subdifferentials," *Journal of the London Mathematical Society*, vol. 2, pp. 175–192, 1990.
- [11] M. J. Fabian, "Subdifferentiability and trustworthiness in the light of a new variational principle of Borwein and Preiss," *Acta Universitatis Carolinae Mathematica et Physica*, vol. 30, no. 2, pp. 51–56, 1989.
- [12] K. Ball, E. A. Carlen, and E. H. Lieb, "Sharp uniform convexity and smoothness inequalities for trace norms," *Inventiones Mathematicae*, vol. 115, no. 1, pp. 463–482, 1994.
- [13] N. Tomczak-Jaegermann, "On the differentiability of the norm in trace classes S_p ," *Séminaire d'analyse fonctionnelle (Polytechnique)*, vol. 22, pp. 1–8, 1975.
- [14] D. Aussel, J. N. Corvellec, and M. Lassond, "Mean value property and subdifferential criteria for lower semicontinuous functions," *Transactions of the American Mathematical Society*, vol. 347, no. 10, pp. 4147–4161, 1995.