Research Article

Existence of Solutions for Fractional Differential Equations with p -Laplacian Operator and Integral Boundary Conditions

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In this paper, we investigate a class of integral boundary value problems of fractional differential equations with a p -Laplacian operator. Existence of solutions is obtained by using the fixed point theorem, and an example is given to show the applicability of our main result.

1. Introduction

In this paper, we consider the nonlinear fractional differential equations with a p -Laplacian operator and integral boundary conditions

$$\begin{cases} {}^c D_{0+}^\beta \left(\phi_p({}^c D_{0+}^\alpha u(t)) \right) + f(t, u(t)) = 0, & t \in [0, 1], \\ u(1) = \lambda \int_0^1 u(s) ds, & u'(1) = 0, \\ {}^c D_{0+}^\alpha u(1) = b {}^c D_{0+}^\alpha u(\xi), \end{cases} \quad (1)$$

where $1 < \alpha \leq 2$, $0 < \beta \leq 1$, $0 < \xi$, $b, \lambda < 1$, ${}^c D_{0+}^\alpha$ and ${}^c D_{0+}^\beta$ are the Caputo fractional derivative.

$\phi_p(s) = |s|^{p-2}s$ is the p -Laplacian operator such that $(1/p) + (1/q) = 1$, $p > 1$, and $\phi_p^{-1}(s) = \phi_q(s)$, and $f(t, u): [0, 1] \times [0, \infty) \rightarrow [0, \infty)$ is a given continuous function.

In recent years, boundary value problems of fractional differential equations have significantly been discussed by some researchers because fractional calculus theory and methods have been widely used in various fields of natural sciences and social sciences. In the field of physical mechanics, fractional calculus not only provides suitable mathematical tools for the study of soft matter but also provides new research ideas and plays an irreplaceable role in the modeling of soft matter [1–3]. Some nonlinear analysis tools such as

coincidence degree theory [4, 5], upper and lower solution method [6–8], fixed point theorems [9–11], and variational methods [12–14] have been widely used to discuss existence of solutions for boundary value problems of fractional differential equations.

On the other hand, it is well known that differential equation models with p -Laplacian operators are often used to simulate practical problems such as tides caused by celestial gravity and elastic deformation of beams and rich results of fractional differential equations with a p -Laplacian operator have been obtained [15–18]. In particular, in [15], by using the fixed point theorem, Yan et al. studied the existence of solutions for boundary value problems of fractional differential equations with a p -Laplacian operator:

$$\begin{cases} -D_{0+}^\alpha \left(\phi_p \left(D_{0+}^\beta u(t) \right) \right) = f(t, u(t)), \\ u(0) = u(1) = u'(0) = u'(1), \\ D_{0+}^\beta u(0) = 0, \quad D_{0+}^\beta u(1) = b D_{0+}^\beta u(\eta), \end{cases} \quad (2)$$

where $1 < \alpha \leq 2$, $3 < \beta \leq 4$, $0 < \eta < 1$, $0 < b < \eta^{(1-\alpha)/(p-1)}$, $(1/p) + (1/q) = 1$, $1 < p$, $\phi_p^{-1}(s) = \phi_q(s)$, D_{0+}^α , D_{0+}^β is the standard Riemann-Liouville derivative, and $f(t, u): (0, 1) \times (0, \infty) \rightarrow [0, \infty)$ is a given continuous function.

Moreover, during the last decade, the integral boundary value problem of fractional differential equations is also a

hot issue for scholars and some good results have been achieved [19–23]. In [24], by using the method of the upper and lower solutions and Schauder's and Banach's fixed points theorem, Abdo et al. obtained the existence and uniqueness of a positive solution of the fractional differential equations with integral boundary equations:

$$\begin{cases} {}^c D_{0+}^\alpha u(t) = f(t, u(t)), & t \in (0, 1], \\ u(0) = \lambda \int_0^1 u(s) ds + d, \end{cases} \quad (3)$$

where $0 < \alpha \leq 1$, $\lambda \geq 0$, $d > 0$, ${}^c D_{0+}^\alpha$ is the standard Caputo derivative, and $f : [0, 1] \times [0, \infty) \rightarrow [0, \infty)$ is a given continuous function.

In [25], Bai and Qiu discuss the existence of positive solutions for boundary value problems of fractional differential equations:

$$\begin{cases} {}^c D_{0+}^\alpha u(t) + f(t, u(t)) = 0, & t \in (0, 1], \\ u(0) = u'(1) = u''(0) = 0, \end{cases} \quad (4)$$

where $2 < \alpha \leq 3$, $f \in C([0, 1] \times [0, +\infty), [0, +\infty))$, and ${}^c D_{0+}^\alpha$ is the standard Caputo derivative.

Motivated by the works mentioned above, we concentrate on the solutions for the nonlinear fractional differential equation (1). We obtain the existence result of the fractional differential equations with integral boundary equations by using the Schauder fixed point theorem and other mathematical analysis techniques.

The rest of this paper is organized as follows. In Section 2, we give some notations and lemmas. Section 3 is devoted to study existence of solutions for boundary value problems of fractional differential equations. Finally, we provide an example to illustrate our results.

2. Preliminaries

In the section, we present some definitions and lemmas, which are required for building our theorems.

Definition 1 (see [1]). The fractional integral of order $\alpha (\alpha > 0)$ of function $f : [0, \infty) \rightarrow \mathbb{R}$ is given by

$$I_{0+}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad (5)$$

where $\Gamma(\alpha)$ is the Gamma function, provided the right side is pointwise defined on $(0, +\infty)$.

Definition 2 (see [2]). The Caputo fractional derivative of order $\alpha (\alpha > 0)$ of function $f : [0, \infty) \rightarrow \mathbb{R}$ is given by

$${}^c D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds, \quad (6)$$

where $t > 0$, $n = [\alpha] + 1$, $\Gamma(\alpha)$ is the Gamma function.

Lemma 3 (see [26]). For $\alpha > 0$, the solution of fractional differential equation ${}^c D_{0+}^\alpha u(t) = 0$ is given by $u(t) = c_0 + c_1 t + \dots + c_{n-1} t^{n-1}$, $c_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, n-1$, $n = [\alpha] + 1$, and $[\alpha]$ denotes the integer part of the real number α .

Lemma 4 (see [1]). For $\alpha > 0$, then (i) $I_{0+}^\alpha ({}^c D_{0+}^\alpha u(t)) = u(t) + c_0 + c_1 t + \dots + c_{n-1} t^{n-1}$, $c_i \in \mathbb{R}$, $n = [\alpha] + 1$ and (ii) ${}^c D_{0+}^\alpha I_{0+}^\alpha u(t) = u(t)$.

Lemma 5 (see [1]). Let X be a Banach space and $\Omega \subset X$ a convex, closed, and bounded set. If $T : \Omega \rightarrow \Omega$ is a continuous operator such that $T\Omega \subset X$, $T\Omega$ is relatively compact, then T has at least one fixed point in Ω .

Let $\phi_p ({}^c D_{0+}^\alpha u(t)) = v(t)$, then $v(1) = b^{p-1} v(\xi)$. We now consider the following equations:

$$\begin{cases} -{}^c D_{0+}^\beta v(t) = y(t), & t \in [0, 1], \\ v(1) = b^{p-1} v(\xi). \end{cases} \quad (7)$$

Lemma 6. Let $y \in C[0, 1]$, then (7) has a unique solution

$$v(t) = \int_0^1 H(t, \tau) y(\tau) d\tau, \quad (8)$$

where

$$H(t, \tau) = \begin{cases} \frac{(1-\tau)^{\beta-1}}{(1-b^{p-1})\Gamma(\beta)} - \frac{(t-\tau)^{\beta-1}}{\Gamma(\beta)}, & 0 \leq \tau \leq t \leq 1, \quad \xi \leq \tau, \\ \frac{(1-\tau)^{\beta-1}}{(1-b^{p-1})\Gamma(\beta)} - \frac{(t-\tau)^{\beta-1}}{\Gamma(\beta)} - \frac{b^{p-1}(\xi-\tau)^{\beta-1}}{(1-b^{p-1})\Gamma(\beta)}, & 0 \leq \tau \leq t \leq 1, \quad \xi \leq \tau, \\ \frac{(1-\tau)^{\beta-1}}{(1-b^{p-1})\Gamma(\beta)}, & 0 \leq t \leq \tau \leq 1, \quad \xi \leq \tau, \\ \frac{(1-\tau)^{\beta-1}}{(1-b^{p-1})\Gamma(\beta)} - \frac{b^{p-1}(\xi-\tau)^{\beta-1}}{(1-b^{p-1})\Gamma(\beta)}, & 0 \leq t \leq \tau \leq 1, \quad \xi \leq \tau. \end{cases} \quad (9)$$

Proof. Suppose v satisfies boundary value problem (7), by (i) of Lemma 4, we can obtain

$$v(t) = -I_{0+}^{\beta} y(t) - c_0 = -\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} y(s) ds - c_0. \quad (10)$$

Using the boundary condition $v(1) = b^{p-1} v(\xi)$, we can obtain

$$c_0 = \frac{b^{p-1}}{(1-b^{p-1})\Gamma(\beta)} \int_0^{\xi} (\xi-s)^{\beta-1} y(s) ds - \frac{1}{(1-b^{p-1})\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} y(s) ds. \quad (11)$$

Thus,

$$\begin{aligned} v(t) = & -\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} y(s) ds \\ & - \frac{b^{p-1}}{(1-b^{p-1})\Gamma(\beta)} \int_0^{\xi} (\xi-s)^{\beta-1} y(s) ds \\ & + \frac{1}{(1-b^{p-1})\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} y(s) ds = \int_0^1 H(t,s) y(s) ds. \end{aligned} \quad (12)$$

From the above analysis, the equation

$$\begin{cases} {}^c D_{0+}^{\beta} (\phi_p({}^c D_{0+}^{\alpha} u(t))) + y(t) = 0, \\ u(1) = \lambda \int_0^1 u(s) ds, \quad u'(1) = 0, \\ {}^c D_{0+}^{\alpha} u(1) = b^c D_{0+}^{\alpha} u(\xi) \end{cases} \quad (13)$$

is equivalent to

$$\begin{cases} {}^c D_{0+}^{\alpha} u(t) = \phi_q \left(\int_0^1 H(t,\tau) y(\tau) d\tau \right), \quad 0 < t < 1, \\ u'(1) = 0, \quad u(1) = \lambda \int_0^1 u(s) ds. \end{cases} \quad (14)$$

Lemma 7. Let $y \in C[0, 1]$. Then (14) has a unique solution:

$$\begin{aligned} u(t) = & \left\{ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi_q \left(\int_0^1 H(s,\tau) y(\tau) d\tau \right) ds \right. \\ & - \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \phi_q \left(\int_0^1 H(s,\tau) y(\tau) d\tau \right) ds \\ & + \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} \phi_q \left(\int_0^1 H(s,\tau) y(\tau) d\tau \right) ds \\ & \left. - \frac{1}{\Gamma(\alpha-1)} \int_0^1 t(1-s)^{\alpha-2} \phi_q \left(\int_0^1 H(s,\tau) y(\tau) d\tau \right) ds + \lambda A, \right\} \end{aligned} \quad (15)$$

where

$$\begin{aligned} A = & \frac{1}{1-\lambda} \left[\frac{1}{\Gamma(\alpha+1)} \int_0^1 (1-s)^{\alpha} \phi_q \left(\int_0^1 H(s,\tau) y(\tau) d\tau \right) ds \right. \\ & - \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \phi_q \left(\int_0^1 H(s,\tau) y(\tau) d\tau \right) ds \\ & \left. + \frac{1}{2\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} \phi_q \left(\int_0^1 H(s,\tau) y(\tau) d\tau \right) ds \right]. \end{aligned} \quad (16)$$

Proof. By (i) of Lemma (4), we can obtain $u(t) = (1/\Gamma(\alpha)) \int_0^t (t-s)^{\alpha-1} \phi_q \left(\int_0^1 H(s,\tau) y(\tau) d\tau \right) ds - c_1 - c_2 t$. Then, $u'(t) = (1/\Gamma(\alpha-1)) \int_0^t (t-s)^{\alpha-2} \phi_q \left(\int_0^1 H(s,\tau) y(\tau) d\tau \right) ds - c_2$, using the boundary condition $u'(1) = 0$, we can obtain

$$c_2 = \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} \phi_q \left(\int_0^1 H(s,\tau) y(\tau) d\tau \right) ds. \quad (17)$$

Another, because

$$u(1) = \lambda \int_0^1 u(s) ds, \quad (18)$$

$$u(1) = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \phi_q \left(\int_0^1 H(s,\tau) y(\tau) d\tau \right) ds - c_1 - c_2, \quad (19)$$

we have

$$\begin{aligned} c_1 = & \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \phi_q \left(\int_0^1 H(s,\tau) y(\tau) d\tau \right) ds \\ & - \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} \phi_q \left(\int_0^1 H(s,\tau) y(\tau) d\tau \right) ds - \lambda \int_0^1 u(s) ds. \end{aligned} \quad (20)$$

Now, we express $\int_0^1 u(s) ds$, let

$$\begin{aligned} A = & \int_0^1 u(t) dt = \int_0^1 \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q \left(\int_0^1 H(s,\tau) y(\tau) d\tau \right) ds dt \\ & - \int_0^1 \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q \left(\int_0^1 H(s,\tau) y(\tau) d\tau \right) ds dt \\ & + \int_0^1 \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \phi_q \left(\int_0^1 H(s,\tau) y(\tau) d\tau \right) ds dt \\ & - \int_0^1 \int_0^1 \frac{t(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \phi_q \left(\int_0^1 H(s,\tau) y(\tau) d\tau \right) ds dt \\ & + \int_0^1 \lambda A dt = \int_0^1 \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)} \phi_q \left(\int_0^1 H(s,\tau) y(\tau) d\tau \right) ds \\ & - \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q \left(\int_0^1 H(s,\tau) y(\tau) d\tau \right) ds \\ & + \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \phi_q \left(\int_0^1 H(s,\tau) y(\tau) d\tau \right) ds \\ & - \int_0^1 \frac{(1-s)^{\alpha-2}}{2\Gamma(\alpha-1)} \phi_q \left(\int_0^1 H(s,\tau) y(\tau) d\tau \right) ds + \lambda A. \end{aligned} \quad (21)$$

We obtain

$$\begin{aligned} A = & \frac{1}{1-\lambda} \left[\frac{1}{\Gamma(\alpha+1)} \int_0^1 (1-s)^\alpha \phi_q \left(\int_0^1 H(s, \tau) y(\tau) d\tau \right) ds \right. \\ & - \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \phi_q \left(\int_0^1 H(s, \tau) y(\tau) d\tau \right) ds \\ & \left. + \frac{1}{2\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} \phi_q \left(\int_0^1 H(s, \tau) y(\tau) d\tau \right) ds \right]. \end{aligned} \quad (22)$$

Therefore,

$$\begin{aligned} u(t) = & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi_q \left(\int_0^1 H(s, \tau) y(\tau) d\tau \right) ds \\ & - \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \phi_q \left(\int_0^1 H(s, \tau) y(\tau) d\tau \right) ds \\ & + \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} \phi_q \left(\int_0^1 H(s, \tau) y(\tau) d\tau \right) ds \\ & - \frac{1}{\Gamma(\alpha-1)} \int_0^1 t(1-s)^{\alpha-2} \phi_q \left(\int_0^1 H(s, \tau) y(\tau) d\tau \right) ds + \lambda A. \end{aligned} \quad (23)$$

Reverse, if

$$\begin{aligned} u(t) = & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi_q \left(\int_0^1 H(s, \tau) y(\tau) d\tau \right) ds \\ & - \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \phi_q \left(\int_0^1 H(s, \tau) y(\tau) d\tau \right) ds \\ & + \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} \phi_q \left(\int_0^1 H(s, \tau) y(\tau) d\tau \right) ds \\ & - \frac{1}{\Gamma(\alpha-1)} \int_0^1 t(1-s)^{\alpha-2} \phi_q \left(\int_0^1 H(s, \tau) y(\tau) d\tau \right) ds + \lambda A, \end{aligned} \quad (24)$$

by (ii) of Lemma (4), we can obtain that $u(t)$ is a solution of (14).

The proof is completed.

Lemma 8. *The functions H is continuous on $[0, 1] \times [0, 1]$ and has the following properties:*

- (1) $H(t, \tau) \leq H(\tau, \tau)$, $t, \tau \in [0, 1]$
- (2) $\int_0^1 H(\tau, \tau) d\tau \leq (1/(1-b^{p-1}))\Gamma(\beta+1)$

Proof. (1) For any $t, \tau \in [0, 1]$, by (9), it is obvious that $H(t, \tau) \leq H(\tau, \tau)$. (2) For any $t, \tau \in [0, 1]$, by (9), we conclude that

$$H(\tau, \tau) \leq \frac{(1-\tau)^{\beta-1}}{(1-b^{p-1})\Gamma(\beta)}. \quad (25)$$

Therefore,

$$\int_0^1 H(\tau, \tau) d\tau \leq \frac{1}{(1-b^{p-1})\Gamma(\beta+1)}. \quad (26)$$

This completes the proof.

3. Main Results

In this section, we will show the existence results for boundary value problem (1) by the Schauder fixed point theorem.

Let $I = [0, 1]$, $U = \{u(t) \mid u(t) \in C(I)\}$, and define the norm $\|u\| = \max_{t \in [0, 1]} |u(t)|$, $(U, \|\cdot\|)$ is a Banach space.

Theorem 9. *Assume that the following conditions (H_1) and (H_2) are satisfied:*

(H_1) $f(t, u): [0, 1] \times [0, \infty) \rightarrow [0, \infty)$ is continuous

(H_2) There exists a constant $k > 0$, satisfying $f(t, u) \leq L\phi_p(u)$, $t \in [0, 1]$, $\|u\| \leq k$, where

$$0 < L \leq \left[\phi_p \left(\frac{2(1-\lambda)\Gamma(\alpha+2)}{(2-\lambda)(\alpha^2+\alpha)} \right) \left(\frac{1}{(1-b^{p-1})\Gamma(\beta+1)} \right)^{-1} \right], \quad (27)$$

then the problem (1) has at least one solution.

Proof. Let $P = \{u(t) \mid \|u(t)\| \leq k, t \in [0, 1]\}$; thus, $P \subset U$ is convex, bounded, and closed.

Define an operator $T : P \rightarrow U$ by

$$\begin{aligned} Tu(t) = & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi_q \left(\int_0^1 H(s, \tau) f(\tau, u(\tau)) d\tau \right) ds \\ & - \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \phi_q \left(\int_0^1 H(s, \tau) f(\tau, u(\tau)) d\tau \right) ds \\ & + \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} \phi_q \left(\int_0^1 H(s, \tau) f(\tau, u(\tau)) d\tau \right) ds \\ & - \frac{1}{\Gamma(\alpha-1)} \int_0^1 t(1-s)^{\alpha-2} \phi_q \left(\int_0^1 H(s, \tau) f(\tau, u(\tau)) d\tau \right) ds \\ & + \frac{\lambda}{1-\lambda} \left[\frac{1}{\Gamma(\alpha+1)} \int_0^1 (1-s)^\alpha \phi_q \left(\int_0^1 H(s, \tau) f(\tau, u(\tau)) d\tau \right) ds \right. \\ & - \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \phi_q \left(\int_0^1 H(s, \tau) f(\tau, u(\tau)) d\tau \right) ds \\ & \left. + \frac{1}{2\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} \phi_q \left(\int_0^1 H(s, \tau) f(\tau, u(\tau)) d\tau \right) ds \right]. \end{aligned} \quad (28)$$

For any $u \in P$, then by (H_2) , we have

$$f(t, u) \leq L\phi_p(u) \leq L\phi_p(k). \quad (29)$$

By Lemma (8), we conclude that

$$\begin{aligned}
 |Tu(t)| &\leq \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi_q \left(\int_0^1 H(\tau, \tau) f(\tau, u(\tau)) d\tau \right) ds \right. \\
 &\quad - \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \phi_q \left(\int_0^1 H(\tau, \tau) f(\tau, u(\tau)) d\tau \right) ds \\
 &\quad + \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} \phi_q \left(\int_0^1 H(\tau, \tau) f(\tau, u(\tau)) d\tau \right) ds \\
 &\quad - \frac{1}{\Gamma(\alpha-1)} \int_0^1 t(1-s)^{\alpha-2} \phi_q \left(\int_0^1 H(\tau, \tau) f(\tau, u(\tau)) d\tau \right) ds \\
 &\quad + \frac{\lambda}{1-\lambda} \left[\frac{1}{\Gamma(\alpha+1)} \int_0^1 (1-s)^\alpha \phi_q \left(\int_0^1 H(\tau, \tau) f(\tau, u(\tau)) d\tau \right) ds \right. \\
 &\quad - \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \phi_q \left(\int_0^1 H(\tau, \tau) f(\tau, u(\tau)) d\tau \right) ds \\
 &\quad \left. + \frac{1}{2\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} \phi_q \left(\int_0^1 H(\tau, \tau) f(\tau, u(\tau)) d\tau \right) ds \right] \Big| \\
 &= \left| \frac{t^\alpha}{\Gamma(\alpha+1)} \phi_q \left(\int_0^1 H(\tau, \tau) f(\tau, u(\tau)) d\tau \right) \right. \\
 &\quad - \frac{1}{\Gamma(\alpha+1)} \phi_q \left(\int_0^1 H(\tau, \tau) f(\tau, u(\tau)) d\tau \right) \\
 &\quad + \frac{1}{\Gamma(\alpha)} \phi_q \left(\int_0^1 H(\tau, \tau) f(\tau, u(\tau)) d\tau \right) \\
 &\quad - \frac{t}{\Gamma(\alpha)} \phi_q \left(\int_0^1 H(\tau, \tau) f(\tau, u(\tau)) d\tau \right) \\
 &\quad + \frac{\lambda}{1-\lambda} \left[\frac{1}{\Gamma(\alpha+2)} \phi_q \left(\int_0^1 H(\tau, \tau) f(\tau, u(\tau)) d\tau \right) \right. \\
 &\quad - \frac{1}{\Gamma(\alpha+1)} \phi_q \left(\int_0^1 H(\tau, \tau) f(\tau, u(\tau)) d\tau \right) \\
 &\quad \left. + \frac{1}{2\Gamma(\alpha)} \phi_q \left(\int_0^1 H(\tau, \tau) f(\tau, u(\tau)) d\tau \right) \right] \Big| \\
 &\leq \left| \frac{1}{\Gamma(\alpha+1)} \phi_q \left(\int_0^1 H(\tau, \tau) f(\tau, u(\tau)) d\tau \right) \right. \\
 &\quad - \frac{1}{\Gamma(\alpha+1)} \phi_q \left(\int_0^1 H(\tau, \tau) f(\tau, u(\tau)) d\tau \right) \\
 &\quad + \frac{1}{\Gamma(\alpha)} \phi_q \left(\int_0^1 H(\tau, \tau) f(\tau, u(\tau)) d\tau \right) \\
 &\quad + \frac{\lambda}{1-\lambda} \left[\frac{1}{\Gamma(\alpha+2)} \phi_q \left(\int_0^1 H(\tau, \tau) f(\tau, u(\tau)) d\tau \right) \right. \\
 &\quad - \frac{1}{\Gamma(\alpha+1)} \phi_q \left(\int_0^1 H(\tau, \tau) f(\tau, u(\tau)) d\tau \right) \\
 &\quad \left. + \frac{1}{2\Gamma(\alpha)} \phi_q \left(\int_0^1 H(\tau, \tau) f(\tau, u(\tau)) d\tau \right) \right] \Big| \\
 &= \left| \frac{2\alpha^2 + 2\alpha - \lambda\alpha^2 - 3\lambda\alpha}{2(1-\lambda)\Gamma(\alpha+2)} \phi_q \left(\int_0^1 H(\tau, \tau) f(\tau, u(\tau)) d\tau \right) \right| \\
 &\leq \left| \frac{(2-\lambda)(\alpha^2 + \alpha)}{2(1-\lambda)\Gamma(\alpha+2)} \phi_q \left(\int_0^1 H(\tau, \tau) f(\tau, u(\tau)) d\tau \right) \right| \\
 &\leq \left| \frac{(2-\lambda)(\alpha^2 + \alpha)}{2(1-\lambda)\Gamma(\alpha+2)} \phi_q \left(\int_0^1 H(\tau, \tau) L\phi_p(u(\tau)) d\tau \right) \right| \\
 &\leq \left| \frac{(2-\lambda)(\alpha^2 + \alpha)}{2(1-\lambda)\Gamma(\alpha+2)} \phi_q \left(\int_0^1 H(\tau, \tau) L\phi_p(k) d\tau \right) \right| \\
 &\leq \frac{(2-\lambda)(\alpha^2 + \alpha)}{2(1-\lambda)\Gamma(\alpha+2)} \left| \phi_q \left(\int_0^1 H(\tau, \tau) d\tau \right) \right| \phi_q(L)k \\
 &\leq \frac{(2-\lambda)(\alpha^2 + \alpha)}{2(1-\lambda)\Gamma(\alpha+2)} \left| \phi_q \left(\frac{1}{(1-b^{p-1})\Gamma(\beta+1)} \right) \right| \phi_q(L)k \leq k.
 \end{aligned}
 \tag{30}$$

Thus, $T(P) \subseteq P$. By (H_2) , we have

$$\begin{aligned}
 &\left| \phi_q \left(\int_0^1 H(s, \tau) f(\tau, u(\tau)) d\tau \right) ds \right| \\
 &\leq \left| \phi_q \left(\frac{L\phi_p(k)}{(1-b^{p-1})\Gamma(\beta+1)} \right) \right| \leq \left(\frac{L\phi_p(k)}{(1-b^{p-1})\Gamma(\beta+1)} \right)^{q-1}.
 \end{aligned}
 \tag{31}$$

We express

$$M := \left(\frac{L\phi_p(k)}{(1-b^{p-1})\Gamma(\beta+1)} \right)^{q-1}.
 \tag{32}$$

For each $t_1, t_2 \in [0, 1]$, $t_1 < t_2$, we get

$$\begin{aligned}
 |Tu(t_2) - Tu(t_1)| &= \left| \left(\frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2-s)^{\alpha-1} \phi_q \left(\int_0^1 H(s, \tau) f(\tau, u(\tau)) d\tau \right) ds \right. \right. \\
 &\quad - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} \phi_q \left(\int_0^1 H(s, \tau) f(\tau, u(\tau)) d\tau \right) ds \\
 &\quad - \left(\frac{1}{\Gamma(\alpha-1)} \int_0^t t_2(1-s)^{\alpha-2} \phi_q \left(\int_0^1 H(s, \tau) f(\tau, u(\tau)) d\tau \right) ds \right. \\
 &\quad \left. \left. - \frac{1}{\Gamma(\alpha-1)} \int_0^1 t_1(1-s)^{\alpha-2} \phi_q \left(\int_0^1 H(s, \tau) f(\tau, u(\tau)) d\tau \right) ds \right) \right| \\
 &\leq \left(\frac{L\phi_p(k)}{(1-b^{p-1})\Gamma(\beta+1)} \right)^{q-1} \left| \frac{t_2^\alpha - t_1^\alpha}{\Gamma(\alpha+1)} - \frac{t_2 - t_1}{\Gamma(\alpha)} \right| \\
 &\leq M \left| \frac{(t_2^{\alpha-1} + t_2^{\alpha-2}t_1 + t_2^{\alpha-3}t_1^2 + \dots + t_2t_1^{\alpha-2} + t_1^{\alpha-1})}{\Gamma(\alpha+1)} - \frac{1}{\Gamma(\alpha)} \right| |t_2 - t_1|.
 \end{aligned}
 \tag{33}$$

As $t_2 \rightarrow t_1$, the right-hand side of the previous inequality is independent of u and tends to zero; thus, $T(P)$ is equicontinuous. From the Arzela-Ascoli Theorem, T is compact. Applying Schauder's fixed point theorem, T has at least one fixed point $u \in P$. Therefore, the problem (1) has at least one positive solution u in P .

4. Applications

In this section, we will give an example to illustrate our main results.

Example 1. Consider the following equation:

$$\begin{cases} {}^c D_{0^+}^{1/2} \left(\phi_{4/3} \left({}^c D_{0^+}^{3/2} u(t) \right) \right) = \frac{\phi_p |u(t)|}{(t+5)^2}, & t \in (0, 1), \\ u(1) = \frac{1}{2} \int_0^1 u(s) ds, u'(1) = 0, & {}^c D_{0^+}^\alpha u(1) = \frac{1}{2} {}^c D_{0^+}^\alpha u \left(\frac{1}{2} \right), \end{cases}
 \tag{34}$$

where $\beta = 1/2$, $\alpha = 3/2$, $p = 4/3$, $q = 4$, $\lambda = \xi = b = 1/2$, and $f(t, u) = \phi_p |u(t)| / (t+5)^2$. Since f is continuous and

$$|f(t, u)| \leq \frac{1}{25} \phi_p |u(t)|,
 \tag{35}$$

for $(t, u) \in [0, 1] \times [0, \infty)$, we obtain

$$L = \frac{1}{25}. \quad (36)$$

Then,

$$\left[\phi_p \left(\frac{2(1-\lambda)\Gamma(\alpha+2)}{(2-\lambda)(\alpha^2+\alpha)} \right) \left(\frac{1}{(1-b^{p-1})\Gamma(\beta+1)} \right)^{-1} \right] \approx 0.1534. \quad (37)$$

It is obvious that

$$L < \left[\phi_p \left(\frac{2(1-\lambda)\Gamma(\alpha+2)}{(2-\lambda)(\alpha^2+\alpha)} \right) \left(\frac{1}{(1-b^{p-1})\Gamma(\beta+1)} \right)^{-1} \right] \approx 0.1534. \quad (38)$$

By Theorem (9), we conclude that the problem (34) has at least one solution.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

Both authors made an equal contribution.

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