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Research Article

Global Well Posedness for the Thermally Radiative Magnetohydrodynamic Equations in 3D

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In this paper, we study the thermally radiative magnetohydrodynamic equations in 3D, which describe the dynamical behaviors of magnetized fluids that have nonignorable energy and momentum exchange with radiation under the nonlocal thermal equilibrium case. By using exquisite energy estimate, global existence and uniqueness of classical solutions to Cauchy problem in $\mathbb{R}^3$ or $\mathbb{T}^3$ are established when initial data is a small perturbation of some given equilibrium. We can further prove that the rates of convergence of solution toward the equilibrium state are algebraic in $\mathbb{R}^3$ and exponential in $\mathbb{T}^3$ under some additional conditions on initial data. The proof is based on the Fourier multiplier technique.

1. Introduction

In the study of plasma physics, due to the high temperature and high pressure environment, the motion of charged particles flow is usually regarded as compressible fluids, and their dynamics is very often shaped and controlled by magnetic fields and high temperature radiation effects. Meanwhile, it is known that the radiation energy is carried by photons. When the distribution of photon is almost isotropic, based on the standard hydrodynamics, such dynamics can be described by the following 3D thermally radiative magnetohydrodynamic equations (cf. [1, 2]):

$$\begin{aligned}
&\rho_t + \operatorname{div}(\rho u) = 0,\\
&\rho(u_t + u\cdot\nabla u) + \nabla P = \mu\Delta u + (\lambda+\mu)\nabla \operatorname{div} u + H\cdot\nabla H - \frac{1}{2}\nabla\left(|H|^2\right),\\
&H_t - \nu\Delta H + u\cdot\nabla H - H\cdot\nabla u + H \operatorname{div} u = 0, \qquad \operatorname{div} H = 0,\\
&c_\nu\rho(\theta_t + u\cdot\nabla\theta) = \kappa\Delta\theta - P \operatorname{div} u + \lambda(\operatorname{div} u)^2 + 2\mu D\cdot D\\
&\qquad + \nu|\nabla\times H|^2 - \theta^4 + n,\\
&n_t - \Delta n = \theta^4 - n,
\end{aligned} \tag{1}$$

where $\rho=\rho(x,t)>0$, $u=u(x,t)\in\mathbb{R}^3$, $\theta=\theta(x,t)>0$, $H=H(x,t)\in\mathbb{R}^3$, $n=n(x,t)\geq 0$ for $x\in\Omega, t\geq 0$ denote the mass density, velocity field of the fluid, mass temperature, magnetic field, and radiation field, respectively, and $P=R\rho\theta$ denotes the material pressure. The spatial domain $\Omega=\mathbb{R}^3$ or $\mathbb{T}^3$. The parameter $R>0$ is the perfect gas constant, $c_\nu>0$ is the specific heat at constant volume, and κ is the heat conductivity coefficient; λ and μ are the viscosity coefficients of the flow satisfying $\mu>0$ and $3\lambda+2\mu\geq 0$ and $\nu>0$ is the magnetic diffusion coefficient. Throughout this paper, we assume that λ,μ,ν,κ are all positive constants. $D=D(u)$ is the deformation tensor

$$\begin{aligned}
D_{ij} &:= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),\\
D\cdot D &:= \sum_{i,j=1}^{3} D_{ij}^2.
\end{aligned} \tag{2}$$

When the magnetic is ignored (i.e., $H=0$ in (1)), system (1) can be reduced to the nonequilibrium diffusion approximation model in radiation hydrodynamics. This model describes the energy flow due to radiative process in a semiquantitative sense and is particularly accurate if the specific intensity of radiation is almost isotropic (cf. [3–5]). There are some mathematical results on this model. For the global existence of smooth solution for one-dimensional case, see [6]; for the global well-posedness and large time behavior of classical solutions for multidimensional case, see [7]. For the inviscid case, in [8], the authors considered a 1D model and showed the existence of shock profiles for inviscid nonequilibrium gases provided that the initial strength is suitably small. For the local existence of smooth solutions for multidimensional system, see [9]. System ((1)) is the compressible MHD equations coupled with the radiative transport equation with nonlocal terms and is very difficult to solve both numerically and analytically. Ducomet and Feireisl consider the thermally radiative MHD system first and show the existence of global weak solutions for the multidimensional case in [10] (also see, for instance, Li and Guo [11]). For the one-dimensional case, this model has been studied by many authors under the various growth constraints on the heat conductivity κ [12–14].

In this paper, we are focused on the asymptotic and global existence of classical solutions of system (1) with the initial data:

$$(\rho, u, H, \theta, n)|_{t=0} = (\rho_0, u_0, H_0, \theta_0, n_0)(x), \quad x \in \Omega. \tag{3}$$

It is easy to check that $(\rho, u, H, \theta, n) \equiv (1, 0, 0, 1, 1)$ is an equilibrium state of (1). Therefore, it is natural to introduce the transforms

$$\rho = 1 + \varrho, \quad \theta = 1 + \Theta, \quad n = 1 + \eta. \tag{4}$$

Without loss of generality, we assume the positive constants $R = c_v = \kappa = \mu = \lambda = \nu \equiv 1$. Then we can rewrite the system (1) as

$$\varrho_t + (1+\varrho) divu + \nabla\varrho \cdot u = 0, \tag{5}$$

$$u_t + u \cdot \nabla u + \frac{1+\Theta}{1+\varrho}\nabla\rho + \nabla\Theta = \frac{\Delta u}{1+\varrho} + \frac{2\nabla divu}{1+\varrho} + \frac{H \cdot \nabla H - (1/2)\nabla\left(|H|^2\right)}{1+\varrho}, \tag{6}$$

$$H_t + u \cdot \nabla H - H \cdot \nabla u + H divu = \Delta H, \quad divH = 0, \tag{7}$$

$$\Theta_t + u \cdot \nabla\Theta = \frac{\Delta\Theta}{1+\varrho} - (1+\Theta) divu + \frac{(divu)^2}{1+\varrho} + \frac{2D \cdot D}{1+\varrho} - \frac{(1+\Theta)^4}{1+\varrho} + \frac{1+\eta}{1+\varrho} + \frac{|\nabla \times H|^2}{1+\varrho}, \tag{8}$$

$$\eta_t - \Delta\eta = (1+\Theta)^4 - (1+\eta). \tag{9}$$

with initial data

$$(\varrho, u, H, \Theta, \eta)|_{t=0} = (\varrho_0, u_0, H_0, \Theta_0, \eta_0)(x) = (\rho_0 - 1, u_0, H_0, \theta_0 - 1, n_0 - 1)(x). \tag{10}$$

Then, the main results in this paper read as follows:

Theorem 1. *Let $\Omega = \mathbb{R}^3$. Suppose that $\|(\varrho, u_0, H_0, \Theta_0, \eta_0)\|_{H^4}$ is small enough. Then, the Cauchy problem (5)–(10) admits a unique global classical solution $(\varrho, u, H, \Theta, \eta)$ satisfying*

$$\begin{gathered} \varrho, u, H, \Theta, \eta \in C\left([0,\infty]; H^4\right), \\ \sup_{t\geq 0}\|(\varrho, u, H, \Theta, \eta)\|_{H^4} \leq C\|(\varrho_0, u_0, H_0, \Theta_0, \eta_0)\|_{H^4}. \end{gathered} \tag{11}$$

Theorem 2. *Under the conditions of Theorem 1, if we further assume that $\|(\rho_0, u_0, H_0, \Theta_0, \eta_0)\|_{L^1}$ is sufficiently small, then*

$$\|(\varrho, u, H, \Theta, \eta)\|_{L^2} \leq C(1+t)^{-(3/4)}, \tag{12}$$

$$\|\nabla(\varrho, u, H, \Theta, \eta)\|_{H^3} \leq C(1+t)^{-(5/4)}, \tag{13}$$

for all $t \geq 0$.

Theorem 3. *Let $\Omega = \mathbb{T}^3$. Suppose that $\|(\varrho_0, u_0, H_0, \Theta_0, \eta_0)\|_{H^4}$ is small enough, and*

$$\begin{gathered} \int_{\mathbb{T}^3} \varrho_0 dx = 0, \quad \int_{\mathbb{T}^3} (1+\varrho_0) u_0 dx = 0, \quad \int_{\mathbb{T}^3} H_0 dx = 0, \\ \int_{\mathbb{T}^3} \left(\frac{1}{2}(1+\varrho_0)|u_0|^2 + \frac{1}{2}|H_0|^2 + \varrho_0 + \Theta_0 + \rho_0\Theta_0 + \eta_0\right) dx = 0. \end{gathered} \tag{14}$$

Then, the problem (5)–(10) admits a unique global solution $(\varrho, u, H, \Theta, \eta)$ satisfying

$$\begin{gathered} \varrho, u, H, \Theta, \eta \in C\left([0,\infty]; H^4\right), \\ \sup_{t\geq 0} e^{\gamma t}\|(\rho, u, H, \Theta, \eta)\|_{H^4} \leq C\|(\rho_0, u_0, H_0, \Theta_0, \eta_0)\|_{H^4}, \end{gathered} \tag{15}$$

where $\gamma > 0$ is a constant.

The proof of the global existence of classical solution to (5)–(10) relies on the global a priori estimates together with the local existence of classical solutions and continuum argument. The main difficulty in establishing prior estimates in high-order Sobolev spaces is how to control the linear term in (5)–(9), such as $4\Theta/(1+\varrho)$, $\eta/(1+\varrho)$ in (8) and $4\Theta, \eta$ in (1.6). We develop the method in [15] and use the structure of system (5)–(9) itself to construct novel dissipation term

$4\Theta-\eta$ to overcome this difficulty. To prove Theorem 2, we first use a Fourier multiplier technique to establish the $L^p - L^q$ time-decay property of linearized system (47)–(52). Then, the time decay rate can be given by combining the global a priori estimate obtained in Theorem 1 and the above $L^p - L^q$ property and applying the energy estimate technique to the nonlinear problem (5)–(10), whose solutions can be represented by the solution-semigroup operator for the linearized system (47)–(52) by using the Duhamel principle. Here, some nonlinear terms of magnetic field H involved in (6) and (8) may lead to difficulties to gain the desired rate of convergence of solutions. Thus, we will construct some novel functionals such as (79) and (80) and adopt with modification some techniques motivated by [16–18] combined to some vector analysis formula to obtain expected decay rates.

The remainder of this paper is organized as follows. In Section 2, we derive the uniform-in-time a priori estimates and then establish the existence of a global classical solution. In Section 3, we investigate the decay rates of solutions. In Section 4, we adapt our proof to the periodic domain case. Throughout this paper C denotes a positive (generally large) constant and γ a positive (generally small) constant, where both C and γ may take different values in different places. The symbol $A\sim B$ means $CA\le B\le (1/C)A$ for a generic constant $C>0$. For simplicity, we shall use $\|\cdot\|$ to denote norm $L^2(\mathbb{R}^3)$.

2. Global Existence

In what follows, our analysis is based on the Cauchy problem (5)–(10). To obtain the global existence, the most important point is to establish the uniform-in-time a priori estimates.

2.1. A Priori Estimates. Now, we begin to establish the global a priori estimates in the case of the whole space $\mathbb{R}^3$ under the assumption

$$\sup_{t\ge 0}\|(\varrho,u,H,\Theta,\eta)\|_{H^4}\le\delta, \tag{16}$$

where $0<\delta<1$ is a generic constant small enough and (ρ, u, H,Θ,η) is the smooth solution to the Cauchy problem (5)–(10) on $0\le t<T$ for $T>0$. Firstly, we list two important lemmas in Sobolev space.

Lemma 4 (see [16, 19]). *There exist a positive constant C, such that for any $f,g\in H^4(\mathbb{R}^3)$ and any multi-index α with $1\le|\alpha|\le 4$,*

$$\begin{gathered}\|f\|_{L^\infty(\mathbb{R}^3)}\le C\|\nabla f\|^{1/2}_{L^2(\mathbb{R}^3)}\|\nabla^2 f\|^{1/2}_{L^2(\mathbb{R}^3)},\\ \|f\|_{L^p(\mathbb{R}^3)}\le C\|\nabla f\|_{H^1(\mathbb{R}^3)},\quad \text{where } 2\le p\le 6,\\ \|fg\|_{H^3(\mathbb{R}^3)}\le C\|f\|_{H^3(\mathbb{R}^3)}\|\nabla g\|_{H^3(\mathbb{R}^3)},\\ \|\partial^\alpha(fg)\|_{L^2(\mathbb{R}^3)}\le C\|\nabla f\|_{H^3(\mathbb{R}^3)}\|\nabla g\|_{H^3(\mathbb{R}^3)}.\end{gathered}\tag{17}$$

Lemma 5 (Moser-type calculus inequalities) (see [20]). *Let $s\ge 1$ be an integer. Suppose $f\in H^s(\mathbb{R}^3)$, $\nabla f\in L^\infty(\mathbb{R}^3)$ and $g\in H^{s-1}(\mathbb{R}^3)\bigcap L^\infty(\mathbb{R}^3)$. Then, for all multi-index α with $|\alpha|\le s$, we have*

$$\begin{gathered}\|\partial^\alpha(fg)-f\partial^\alpha g\|\le C_s\left(\|\nabla f\|_{L^\infty}\|D^{s-1}g\|+\|D^s f\|\|g\|_{L^\infty}\right),\\ \|D^s f\|=\sum_{|\alpha|=s}\|\partial^\alpha f\|.\end{gathered}\tag{18}$$

Then, we begin to give the priori estimate of ϱ,u,H,Θ,η.

Lemma 6. *Suppose that $(\varrho,u,H,\Theta,\eta)$ be a smooth solution to (5)–(10). Then, for all $0\le t\le T$ with any fixed $T>0$, it holds*

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|(2\rho,2u,H,2\Theta,\eta)\|^2\right)+\gamma\left(\|\nabla(u,H,\Theta,\eta)\|^2\right.\\&\quad\left.+\|\operatorname{div}u\|^2+\|4\Theta-\eta\|^2\right)\\&\le C\left(\|(\varrho,u,H,\Theta,\eta)\|_{H^2}\right.\\&\quad\left.+\|(\varrho,u,H,\Theta,\eta)\|^2_{H^2}\right)\left(\|\nabla(\varrho,u,H,\Theta,\eta)\|^2\right.\\&\quad\left.+\|divu\|^2+\|4\Theta-\eta\|^2\right).\end{aligned}\tag{19}$$

Proof. Multiplying (5)–(9) by 4ρ, $4u$, H, 4Θ, and η and then taking integration and summation, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|2\varrho\|^2+\|2u\|^2+\|H\|^2+\|2\Theta\|^2+\|\eta\|^2\right)\\&\quad+\int\frac{4|\nabla u|^2}{1+\rho}dx+\int\frac{8(divu)^2}{1+\varrho}dx+\int\frac{4|\nabla\Theta|^2}{1+\varrho}dx\\&\quad+\|\nabla H\|^2+\|\nabla\eta\|^2+\|4\Theta-\eta\|^2\\&=-\int 2\rho^2 divudx+\int\frac{4(\rho-\Theta)}{1+\varrho}\nabla\rho\cdot udx\\&\quad-\int 4(u\cdot\nabla u)\cdot udx-\int 4\left(\nabla\frac{1}{1+\varrho}\cdot\nabla u\right)\cdot udx\\&\quad-\int 8\nabla\frac{1}{1+\varrho}\cdot udivudx-\int 4\Theta^2 divudx\\&\quad-\int 4\Theta\nabla\Theta\cdot udx+\int\frac{4\Theta(divu)^2}{1+\varrho}dx+\int\frac{8\Theta D\cdot D}{1+\varrho}dx\\&\quad-\int 4\Theta\nabla\frac{1}{1+\varrho}\cdot\nabla\Theta dx+\int\frac{4\Theta|\nabla\times H|^2}{1+\varrho}dx\\&\quad+\int\frac{4\rho\Theta(4\Theta-\eta)}{1+\varrho}dx-\int\frac{(4\Theta-\eta)\left(6\Theta^2+4\Theta^3+\Theta^4\right)}{1+\varrho}dx\\&\quad+\int\frac{\rho\eta\left(6\Theta^2+4\Theta^3+\Theta^4\right)}{1+\varrho}dx\\&\quad+\int\left(H\cdot\nabla H-\frac{1}{2}\nabla\left(|H|^2\right)\right)\frac{4u}{1+\varrho}dx\\&\quad+\int\left(-u\cdot\nabla H+H\cdot\nabla H-Hdivu\right)\cdot Hdx\\&\equiv\sum_{j=1}^{16}I_j.\end{aligned}\tag{20}$$

For I_1 to I_{12}, using Hölder's and Sobolev's inequalities, we have

$$\begin{aligned}
I_1 &\le C\|\varrho\|_{L^3}\|divu\|\|\varrho\|_{L^6} \le C\|\rho\|_{H^1}\|\nabla\varrho\|\|divu\|,\\
I_2 &\le C(\|\varrho\|_{L^3}+\|\Theta\|_{L^3})\|\nabla\varrho\|\|u\|_{L^6}\\
&\le C(\|\varrho\|_{H^1}+\|\Theta\|_{H^1})\|\nabla\varrho\|\|\nabla u\|,\\
I_3 &\le C\|u\|_{L^3}\|\nabla u\|\|u\|_{L^6} \le C\|u\|_{H^1}\|\nabla u\|^2,\\
I_4+I_5 &\le C\|u\|_{L^6}(\|\nabla u\|_{L^3}+\|divu\|_{L^3})\left\|\nabla\frac{1}{1+\rho}\right\|\\
&\le C\|u\|_{H^2}\|\nabla\varrho\|\|\nabla u\|,\\
I_6 &\le C\|\Theta\|_{L^3}\|divu\|\|\Theta\|_{L^6} \le C\|\Theta\|_{H^1}\|divu\|\|\nabla\Theta\|,\\
I_7 &\le C\|u\|_{L^3}\|\nabla\Theta\|\|\Theta\|_{L^6} \le C\|u\|_{H^1}\|\nabla\Theta\|^2,\\
I_8 &\le C\|\Theta\|_{L^\infty}\|divu\|^2 \le C\|\Theta\|_{H^2}\|divu\|^2,\\
I_9 &= \sum_{i,j=1}^{3}\int\frac{2\Theta\left(u^i_{x_j}+u^j_{x_i}\right)^2}{1+\varrho}dx \le C\sum_{i,j=1}^{3}\|\Theta\|_{L^\infty}\|u^i_{x_j}\|\|u^j_{x_i}\|\\
&\le C\|\Theta\|_{H^2}\|\nabla u\|^2,\\
I_{10} &\le C\left\|\nabla\frac{1}{1+\varrho}\right\|\|\nabla\Theta\|_{L^3}\|\Theta\|_{L^6} \le C\|\Theta\|_{H^2}\|\nabla\varrho\|\|\nabla\Theta\|,\\
I_{11} &\le C\|\Theta\|_{L^\infty}\|\nabla\times H\|^2 \le C\|\Theta\|_{H^2}\|\nabla H\|^2,\\
I_{12} &\le C\|\varrho\|_{L^3}\|4\Theta-\eta\|\|\Theta\|_{L^6} \le C\|\varrho\|_{H^1}\|4\Theta-\eta\|\|\nabla\Theta\|.
\end{aligned} \tag{21}$$

For I_{13}, I_{14}, under the assumption (16), one also has

$$\begin{aligned}
I_{13} &\le C\|4\Theta-\eta\|\left(\|\Theta\|_{L^3}\|\Theta\|_{L^6}+\|\Theta^2\|_{L^3}\|\Theta\|_{L^6}+\|\Theta\|_{L^\infty}\|\Theta^2\|_{L^3}\|\Theta\|_{L^6}\right)\\
&\le C\|4\Theta-\eta\|\left(\|\Theta\|_{H^1}\|\nabla\Theta\|+\|\nabla\Theta\|^3+\|\Theta\|_{H^2}\|\nabla\Theta\|^3\right)\\
&\le C\left(\|\Theta\|_{H^2}+\|\Theta\|^2_{H^2}\right)\left(\|4\Theta-\eta\|^2+\|\nabla\Theta\|^2\right),\\
I_{14} &\le C\|\varrho\|_{H^1}\|\eta\|_{H^1}\left(\|\nabla\Theta\|^2+\|\nabla\Theta\|^3+\|\nabla\Theta\|^4\right)\\
&\le C\left(\|\varrho\|^2_{H^1}+\|\Theta\|^2_{H^1}+\|\eta\|^2_{H^1}\right)\|\nabla\Theta\|^2.
\end{aligned} \tag{22}$$

At last, for I_{15} and I_{16}, we have

$$\begin{aligned}
I_{15} &\le C\|u\|_{L^3}\|\nabla H\|\|H\|_{L^6}+C\|H\|_{L^3}\|divu\|\|H\|_{L^6}\\
&\le C\|u\|_{H^1}\|\nabla H\|^2+C\|H\|_{H^1}\|\nabla H\|\|\nabla u\|,\\
I_{16} &\le C\|u\|_{L^3}\|\nabla H\|\|H\|_{L^6}+C\|H\|_{L^3}\|\nabla u\|\|H\|_{L^6}\\
&\quad+C\|H\|_{L^3}\|divu\|\|H\|_{L^6}\\
&\le C\|u\|_{H^1}\|\nabla H\|^2+C\|H\|_{H^1}\|\nabla H\|\|\nabla u\|.
\end{aligned} \tag{23}$$

Plugging all the above estimates into (20), we obtain (19).

Lemma 7. *Suppose that $(\varrho,u,H,\Theta,\eta)$ be a smooth solution to (5)–(10). Then, for all $0\le t\le T$ with any fixed $T>0$, it holds*

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{1\le|\alpha|\le4}\left(\|2\partial^\alpha\varrho\|^2+\|2\partial^\alpha u\|^2+\|\partial^\alpha H\|^2+\|2\partial^\alpha\Theta\|^2+\|\partial^\alpha\eta\|^2\right)\\
&\quad+\gamma\sum_{1\le|\alpha|\le4}\left(\|\nabla\partial^\alpha u\|^2+\|\operatorname{div}\partial^\alpha u\|^2+\|\nabla\partial^\alpha H\|^2+\|\nabla\partial^\alpha\Theta\|^2\right.\\
&\quad\left.+\|\nabla\partial^\alpha\eta\|^2+\|\partial^\alpha(4\Theta-\eta)\|^2\right)\\
&\le C\left(\|(\varrho,u,H,\Theta,\eta)\|_{H^4}+\|(\varrho,u,H,\Theta,\eta)\|^2_{H^4}\right)\\
&\quad\cdot\left(\|\nabla(\varrho,u,H,\Theta,\eta)\|^2_{H^3}+\|\operatorname{div}u\|^2_{H^3}\right).
\end{aligned} \tag{24}$$

Proof. Applying ∂^α with $1\le|\alpha|\le4$ to (5)–(9) and multiplying by $4\partial^\alpha\varrho$, $4\partial^\alpha u$, $\partial^\alpha H$, $4\partial^\alpha\Theta$, and $\partial^\alpha\eta$, respectively, then taking integration and summation, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|2\partial^\alpha\rho\|^2+\|2\partial^\alpha u\|^2+\|\partial^\alpha H\|^2+\|2\partial^\alpha\Theta\|^2+\|\partial^\alpha\eta\|^2\right)\\
&\quad+\int\frac{4|\nabla\partial^\alpha u|^2}{1+\varrho}dx+\int\frac{8(\operatorname{div}\partial^\alpha u)^2}{1+\varrho}dx+\int\frac{4|\nabla\partial^\alpha\Theta|^2}{1+\varrho}dx\\
&\quad+\|\nabla\partial^\alpha H\|^2+\|\nabla\partial^\alpha\eta\|^2+\|\partial^\alpha(4\Theta-\eta)\|^2\\
&=\int 4[-\partial^\alpha,\varrho div]u\partial^\alpha\varrho dx+\int 4[-\partial^\alpha,u\cdot\nabla]\varrho\partial^\alpha\rho dx\\
&\quad+\int 2|\partial^\alpha\varrho|^2 divudx-\int 4\varrho\partial^\alpha\rho div\partial^\alpha udx\\
&\quad+\int 4[-\partial^\alpha,u\cdot\nabla]u\partial^\alpha udx+\int 4\left[-\partial^\alpha,\frac{1+\Theta}{1+\varrho}\nabla\right]\varrho\partial^\alpha udx\\
&\quad+\int 2|\partial^\alpha u|^2 divudx+\int\frac{4(\Theta-\rho)}{1+\varrho}\partial^\alpha\varrho div\partial^\alpha udx\\
&\quad+\int 4\partial^\alpha\varrho\nabla\frac{1+\Theta}{1+\varrho}\cdot\partial^\alpha udx-\int 4\nabla\left(\frac{1}{1+\varrho}\right)\\
&\quad\cdot\nabla\partial^\alpha u\cdot\partial^\alpha udx+\sum_{0\le\beta<\alpha}C_{\alpha,\beta}\int 4\partial^{\alpha-\beta}\left(\frac{1}{1+\varrho}\right)\partial^\beta\Delta u\\
&\quad\cdot\partial^\alpha udx-\int 8\nabla\left(\frac{1}{1+\varrho}\right)\cdot\partial^\alpha udiv\partial^\alpha udx\\
&\quad+\sum_{0\le\beta<\alpha}C_{\alpha,\beta}\int 8\partial^{\alpha-\beta}\left(\frac{1}{1+\varrho}\right)\partial^\beta\nabla divu\cdot\partial^\alpha udx\\
&\quad-\int\partial^\alpha(u\cdot\nabla H-H\cdot\nabla u+Hdivu)\partial^\alpha Hdx\\
&\quad+\int 4\partial^\alpha\left(\frac{1}{1+\varrho}\left(H\cdot\nabla H-\frac{1}{2}\nabla(|H|)^2\right)\right)\partial^\alpha udx\\
&\quad+\int 4[-\partial^\alpha,u\cdot\nabla]\Theta\partial^\alpha\Theta dx+\int 2|\partial^\alpha\Theta|^2 divudx\\
&\quad+\int 4[-\partial^\alpha,\Theta div]u\partial^\alpha\Theta dx-\int 4\nabla\left(\frac{1}{1+\varrho}\right)\\
&\quad\cdot\nabla\partial^\alpha\Theta\partial^\alpha\Theta dx-\int 4\Theta div\partial^\alpha u\partial^\alpha\Theta dx\\
&\quad+\sum_{0\le\beta<\alpha}C_{\alpha,\beta}\int 4\partial^{\alpha-\beta}\left(\frac{1}{1+\varrho}\right)\partial^\beta\Delta\Theta\partial^\alpha\Theta dx
\end{aligned}$$

$$+\int \frac{16\rho|\partial^{\alpha}\Theta|^{2}}{1+\varrho}dx-\int \frac{4\rho\partial^{\alpha}\eta\partial^{\alpha}\Theta}{1+\varrho}dx$$
$$+\sum_{0\le\beta\le\alpha} C_{\alpha,\beta}\int 4\partial^{\alpha-\beta}\left(\frac{1}{1+\varrho}\right)\partial^{\beta}\big((divu)^{2}$$
$$+2D\cdot D+|\nabla\times H|^{2}\big)\partial^{\alpha}\Theta dx$$
$$-\int 4\partial^{\alpha}\left(\frac{6\Theta^{2}+4\Theta^{3}+\Theta^{4}}{1+\varrho}\right)\partial^{\alpha}\Theta dx$$
$$+\sum_{0\le\beta<\alpha} C_{\alpha,\beta}\int 4\partial^{\alpha-\beta}\left(\frac{1}{1+\varrho}\right)\partial^{\beta}\eta\partial^{\alpha}\Theta dx$$
$$-\sum_{0\le\beta<\alpha} C_{\alpha,\beta}\int 16\partial^{\alpha-\beta}\left(\frac{1}{1+\varrho}\right)\partial^{\beta}\Theta\partial^{\alpha}\Theta dx$$
$$+\int \partial^{\alpha}\left(6\Theta^{2}+4\Theta^{3}+\Theta^{4}\right)\partial^{\alpha}\eta dx\equiv\sum_{j=1}^{28} I_{j}, \tag{25}$$

where $[A,B]$ denotes the commutator $AB-BA$ for two operators A and B; $C_{\alpha,\beta}$ is constant depending only on α and β. We now bound each term on the right-hand side of (25). Utilizing Lemma 5, we get

$$\begin{aligned}
&I_{1}\le C\|\varrho\|_{H^{4}}\|\nabla\varrho\|_{H^{3}}\|divu\|_{H^{3}},\\
&I_{2}\le C\|\varrho\|_{H^{4}}\|\nabla\varrho\|_{H^{3}}\|\nabla u\|_{H^{3}},\\
&I_{5}\le C\|u\|_{H^{4}}\|\nabla u\|_{H^{3}}^{2},\\
&I_{6}\le C\|u\|_{H^{4}}\left(\|\nabla\varrho\|_{H^{3}}\|\nabla\Theta\|_{H^{3}}+\|\nabla\varrho\|_{H^{3}}^{2}\right)\\
&\qquad+C\|\varrho\|_{H^{4}}\|u\|_{H^{4}}\|\nabla\varrho\|_{H^{3}}\|\nabla\Theta\|_{H^{3}},\\
&I_{16}\le C\|\Theta\|_{H^{4}}\|\nabla u\|_{H^{3}}\|\nabla\Theta\|_{H^{3}},\\
&I_{18}\le C\|\Theta\|_{H^{4}}\|divu\|_{H^{3}}\|\nabla\Theta\|_{H^{3}}.
\end{aligned} \tag{26}$$

For I_{11}, we have

$$I_{11}\le C\|u\|_{H^{4}}\|\nabla\varrho\|_{H^{3}}\|\nabla\partial^{\alpha}u\|\le\varepsilon\|\nabla\partial^{\alpha}u\|^{2}+C_{\varepsilon}\|u\|_{H^{4}}^{2}\|\nabla\varrho\|_{H^{3}}^{2}, \tag{27}$$

with $\varepsilon>0$ a small constant, where the first inequality follows that for $\beta<\alpha$,

$$\int \partial^{\alpha-\beta}\left(\frac{1}{1+\rho}\right)\partial^{\beta}\Delta u\partial^{\alpha}u
\le\begin{cases}
\left\|\partial^{\alpha}\left(\frac{1}{1+\varrho}\right)\right\|\|\Delta u\|_{L^{\infty}}\|\partial^{\alpha}u\| & (|\beta|=0),\\
\left\|\partial^{\alpha-\beta}\left(\frac{1}{1+\varrho}\right)\right\|_{L^{3}}\|\partial^{\beta}\Delta u\|_{L^{6}}\|\partial^{\alpha}u\| & (|\beta|=1),\\
\left\|\partial^{\alpha-\beta}\left(\frac{1}{1+\varrho}\right)\right\|_{L^{\infty}}\|\partial^{\beta}\Delta u\|\|\partial^{\alpha}u\| & (|\beta|\ge 2),
\end{cases} \tag{28}$$

and Sobolev's and Young's inequalities were further used. Similarly, we have

$$\begin{aligned}
&I_{13}\le\varepsilon\|\nabla\partial^{\alpha}u\|^{2}+C_{\varepsilon}\|u\|_{H^{4}}^{2}\|\nabla\varrho\|_{H^{3}}^{2},\\
&I_{21}\le\varepsilon\|\nabla\partial^{\alpha}\Theta\|^{2}+C_{\varepsilon}\|\Theta\|_{H^{4}}^{2}\|\nabla\varrho\|_{H^{3}}^{2},\\
&I_{24}\le\|u\|_{H^{4}}\|\Theta\|_{H^{4}}\|\nabla\varrho\|_{H^{3}}\left(\|\nabla u\|_{H^{3}}+\|\operatorname{div}u\|_{H^{3}}+\|\nabla H\|_{H^{3}}\right)\\
&\qquad+\varepsilon\left(\|\nabla\partial^{\alpha}u\|^{2}+\|\operatorname{div}\partial^{\alpha}u\|^{2}+\|\nabla\partial^{\alpha}H\|^{2}\right)\\
&\qquad+C_{\varepsilon}\|\Theta\|_{H^{4}}^{2}\left(\|\nabla u\|_{H^{2}}^{2}+\|\operatorname{div}u\|_{H^{2}}^{2}+\|\nabla H\|_{H^{2}}^{2}\right),\\
&I_{26}+I_{27}\le C\|\Theta\|_{H^{4}}\|\nabla\rho\|_{H^{3}}\left(\|\nabla\eta\|_{H^{2}}+\|\nabla\Theta\|_{H^{2}}\right).
\end{aligned} \tag{29}$$

For I_{15}, we have

$$\begin{aligned}
I_{15}&=\int \frac{4}{1+\varrho}\partial^{\alpha}(H\cdot\nabla H)\partial^{\alpha}udx\\
&\quad+\sum_{0\le\beta<\alpha} C_{\alpha,\beta}\int 4\partial^{\alpha-\beta}\left(\frac{1}{1+\varrho}\right)\partial^{\beta}(H\cdot\nabla H)\partial^{\alpha}udx\\
&\quad-\int \frac{2}{1+\varrho}\partial^{\alpha}\nabla\left(|H|^{2}\right)\partial^{\alpha}udx\\
&\quad-\sum_{0\le\beta<\alpha} C_{\alpha,\beta}\int 2\partial^{\alpha-\beta}\left(\frac{1}{1+\varrho}\right)\partial^{\beta}\nabla\left(|H|^{2}\right)\partial^{\alpha}udx\\
&\equiv\sum_{j=1}^{4} I_{15}^{j}.
\end{aligned} \tag{30}$$

By Lemma 4, we obtain

$$\begin{aligned}
&I_{15}^{1}\le C\|\nabla H\|_{H^{3}}\|\nabla\partial^{\alpha}H\|\|\partial^{\alpha}u\|\le\varepsilon\|\nabla\partial^{\alpha}H\|^{2}+C_{\varepsilon}\|u\|_{H^{4}}^{2}\|\nabla H\|_{H^{3}}^{2},\\
&I_{15}^{2}\le C\|\varrho\|_{H^{4}}\|u\|_{H^{4}}\|\nabla H\|_{H^{3}}^{2},
\end{aligned} \tag{31}$$

since

$$\int \partial^{\alpha-\beta}\left(\frac{1}{1+\varrho}\right)\partial^{\beta}(H\cdot\nabla H)\partial^{\alpha}u
\le\begin{cases}
\left\|\partial^{\alpha}\left(\frac{1}{1+\varrho}\right)\right\|\|H\cdot\nabla H\|_{L^{\infty}}\|\partial^{\alpha}u\| & (|\beta|=0),\\
\left\|\partial^{\alpha-\beta}\left(\frac{1}{1+\varrho}\right)\right\|_{L^{3}}\|\partial^{\beta}(H\cdot\nabla H)\|_{L^{6}}\|\partial^{\alpha}u\| & (|\beta|=1),\\
\left\|\partial^{\alpha-\beta}\left(\frac{1}{1+\varrho}\right)\right\|_{L^{\infty}}\|\partial^{\beta}(H\cdot\nabla H)\|\|\partial^{\alpha}u\| & (|\beta|\ge 2).
\end{cases} \tag{32}$$

Similarly, we can deduce that

$$\begin{aligned}
&I_{15}^{3}\le\varepsilon\|\nabla\partial^{\alpha}H\|^{2}+C_{\varepsilon}\|u\|_{H^{4}}^{2}\|\nabla H\|_{H^{3}}^{2},\\
&I_{15}^{4}\le C\|\rho\|_{H^{4}}\|u\|_{H^{4}}\|\nabla H\|_{H^{3}}^{2}.
\end{aligned} \tag{33}$$

Therefore,

$$I_{15} \leq \varepsilon \|\nabla \partial^\alpha H\|^2 + C(1+\|\varrho\|_{H^4})\|u\|_{H^4}\|\nabla H\|_{H^3}^2. \quad (34)$$

Using Hölder's, Sobolev's, and Young's inequalities and Lemma 4, we can get following bounds:

$$\begin{aligned}
I_3 + I_7 + I_{17} &\leq C\|\text{div } u\|_{L^\infty}\left(\|\partial^\alpha \varrho\|^2+\|\partial^\alpha u\|^2+\|\partial^\alpha \Theta\|^2\right) \\
&\leq C\|u\|_{H^3}\left(\|\nabla \varrho\|_{H^3}^2+\|\nabla u\|_{H^3}^2+\|\nabla \Theta\|_{H^3}^2\right), \\
I_4 &\leq \varepsilon\|\text{div } \partial^\alpha u\|^2 + C_\varepsilon\|\varrho\|_{H^4}^2\|\nabla \rho\|_{H^1}^2, \\
I_8 &\leq \varepsilon\|\text{div } \partial^\alpha u\|^2 + C_\varepsilon\|\varrho\|_{H^4}^2\left(\|\nabla \varrho\|_{H^1}^2+\|\nabla \Theta\|_{H^1}^2\right), \\
I_9 &\leq C\|\varrho\|_{H^4}\|u\|_{H^4}\|\nabla \varrho\|_{H^2}\|\nabla \Theta\|_{H^1} \\
&\quad + C\|\varrho\|_{H^4}\|\nabla u\|_{H^3}(\|\nabla \varrho\|_{H^2}+\|\nabla u\|_{H^2}), \\
I_{10} &\leq \varepsilon\|\nabla \partial^\alpha u\|^2 + C_\varepsilon\|u\|_{H^4}^2\|\nabla \varrho\|_{H^2}^2, \\
I_{12} &\leq \varepsilon\|\text{div } \partial^\alpha u\|^2 + C_\varepsilon\|u\|_{H^4}^2\|\nabla \varrho\|_{H^2}^2, \\
I_{14} &\leq \varepsilon\left(\|\nabla \partial^\alpha u\|^2+\|\text{ div } \partial^\alpha u\|^2+\|\nabla \partial^\alpha H\|^2\right) \\
&\quad + C_\varepsilon\|H\|_{H^4}^2\left(\|\nabla u\|_{H^3}^2+\|\nabla H\|_{H^3}^2\right), \\
I_{19} &\leq \varepsilon\|\nabla \partial^\alpha \Theta\|^2 + C_\varepsilon\|\Theta\|_{H^4}^2\|\nabla \varrho\|_{H^2}^2, \\
I_{20} &\leq \varepsilon\|\text{div } \partial^\alpha u\|^2 + C_\varepsilon\|\Theta\|_{H^4}^2\|\nabla \Theta\|_{H^1}^2, \\
I_{22} + I_{23} &\leq C\|\varrho\|_{H^2}\|\nabla \Theta\|_{H^3}(\|\nabla \Theta\|_{H^3}+\|\nabla \eta\|_{H^3}).
\end{aligned} \quad (35)$$

For the remaining terms, under the assumption (16), one also has

$$\begin{aligned}
I_{25} &\leq C\|\Theta\|_{H^4}\|\nabla \varrho\|_{H^3}\left(\|\nabla \Theta\|_{H^3}+\|\nabla \Theta\|_{H^3}^2+\|\nabla \Theta\|_{H^3}^2\right) \\
&\leq C\|\Theta\|_{H^4}\|\nabla \varrho\|_{H^3}\|\nabla \Theta\|_{H^3}, \\
I_{28} &\leq C\|\eta\|_{H^4}\left(\|\Theta\|_{H^3}^2+\|\Theta\|_{H^3}^3+\|\Theta\|_{H^3}^4\right) \\
&\leq C\|\eta\|_{H^4}\|\nabla \Theta\|_{H^3}^2.
\end{aligned} \quad (36)$$

Putting all the above estimates into (25) and taking the sum over $1 \leq |\alpha| \leq 4$, then (24) follows, and thus, Lemma 7 is proven.

Next, we will give the dissipation rate of ρ.

Lemma 8. *Suppose that $(\varrho, u, H, \Theta, \eta)$ be a smooth solution to (5)–(10). Then, for all $0 \leq t \leq T$ with any fixed $T > 0$, it holds*

$$\begin{aligned}
&\frac{d}{dt}\sum_{|\alpha|\leq 3} \int \nabla \partial^\alpha \varrho \cdot \partial^\alpha u dx + \gamma\|\nabla \varrho\|_{H^3}^2 \\
&\leq C\left(\|\nabla u\|_{H^4}^2+\|divu\|_{H^3}^2+\|\nabla \Theta\|_{H^3}^2\right) \\
&\quad + C\left(\|(\varrho, u, H, \Theta)\|_{H^4}+\|(\varrho, u, H, \Theta)\|_{H^4}^2\right)\|\nabla(\varrho, u, H, \Theta)\|_{H^3}^2.
\end{aligned} \quad (37)$$

Proof. Taking differentiation $\partial^\alpha(|\alpha|\leq 3)$ to (8) and multiplying by $\nabla \partial^\alpha \varrho$, then taking integration, one can get

$$\begin{aligned}
\int |\nabla \partial^\alpha \varrho|^2 dx &= -\int \nabla \partial^\alpha \varrho \cdot \partial^\alpha u_t dx - \int \nabla \partial^\alpha \varrho \partial^\alpha (u \cdot \nabla u) dx \\
&\quad - \int \nabla \partial^\alpha \varrho \cdot \partial^\alpha \left(\left(\frac{\Theta - \varrho}{1 + \varrho}\right)\nabla \varrho\right) dx \\
&\quad - \int \nabla \partial^\alpha \varrho \cdot \nabla \partial^\alpha \Theta dx + \int \nabla \partial^\alpha \varrho \cdot \partial^\alpha \left(\frac{\Delta u}{1 + \varrho}\right) dx \\
&\quad + \int 2\nabla \partial^\alpha \varrho \cdot \partial^\alpha \left(\frac{\nabla divu}{1 + \varrho}\right) dx \\
&\quad + \int \nabla \partial^\alpha \varrho \cdot \partial^\alpha \left(\frac{1}{1 + \varrho}\left(H \cdot \nabla H - \frac{1}{2}\nabla(|H|^2)\right)\right) dx \\
&\equiv \sum_{j=1}^{7} I_j.
\end{aligned} \quad (38)$$

For I_1, applying (5), we have

$$\begin{aligned}
I_1 &= -\frac{d}{dt}\int \nabla \partial^\alpha \varrho \cdot \partial^\alpha u dx + \int \partial^\alpha divu \partial^\alpha((1 + \varrho)divu + \nabla \varrho \cdot u) dx \\
&\leq -\frac{d}{dt}\int \nabla \partial^\alpha \varrho \cdot \partial^\alpha u dx + C\|divu\|_{H^3}^2 + C\|\varrho\|_{H^3}\|\nabla u\|_{H^3}^2.
\end{aligned} \quad (39)$$

Using Hölder's, Sobolev's, and Young's inequalities, we obtain

$$\begin{aligned}
I_2 &\leq C\|\varrho\|_{H^4}\|\nabla u\|_{H^3}^2, \\
I_3 &\leq C\|\varrho\|_{H^4}^2\|\nabla \varrho\|_{H^3}(\|\nabla \varrho\|_{H^3}+\|\nabla \Theta\|_{H^3}), \\
I_4 &\leq \frac{1}{4}\|\nabla \partial^\alpha \varrho\|^2 + C\|\nabla \Theta\|_{H^3}^2, \\
I_5 + I_6 &\leq C\|\varrho\|_{H^4}\|\nabla \varrho\|_{H^3}\|\nabla u\|_{H^3} + C\|\nabla \partial^\alpha \varrho\|\|\nabla u\|_{H^4} \\
&\leq \frac{1}{4}\|\nabla \partial^\alpha \varrho\|^2 + C\|\varrho\|_{H^4}\|\nabla \varrho\|_{H^3}\|\nabla u\|_{H^3} + C\|\nabla u\|_{H^4}^2, \\
I_7 &\leq C\|\varrho\|_{H^4}\|\nabla H\|_{H^3}^2 + C\|\varrho\|_{H^4}^2\|\nabla H\|_{H^3}^2.
\end{aligned} \quad (40)$$

Putting these estimates into (38) and taking the sum over $|\alpha| \leq 3$ gives (37), and Lemma 8 is proven.

2.2. Proof of Global Existence. In this section, we will show there exists a unique global in time solution to the problem (5)–(10). Firstly, combining estimates obtained in Lemmas 6–8, one can finish the proof of uniform-in-time a priori estimates as follows. Define a total temporal energy functional

$\mathscr{E}(t)$ and the corresponding dissipation rate functional $\mathscr{D}(t)$ by

$$\mathscr{E}(t) = \|2\varrho\|^2+\|2u\|^2+\|H\|^2+\|2\Theta\|^2+\|\eta\|^2 + \sum_{1\le|\alpha|\le 4} \left(\|2\partial^\alpha\varrho\|^2+\|2\partial^\alpha u\|^2+\|2\partial^\alpha\Theta\|^2+\|\partial^\alpha H\|^2+\|\partial^\alpha\eta\|^2\right) + \tau_1\sum_{|\alpha|\le 3}\int \nabla\partial^\alpha\varrho\cdot\partial^\alpha u dx, \tag{41}$$

$$\mathscr{D}(t) = \|\nabla\rho\|_{H^3}^2+\|\nabla(u,H,\Theta,\eta)\|_{H^4}^2+\|divu\|_{H^4}^2+\|4\Theta-\eta\|_{H^4}^2, \tag{42}$$

where $0<\tau_1\ll 1$ is a small constant. Under the assumption (16), then

$$\mathscr{E}(t) \sim \|(\varrho,u,H,\Theta,\eta)(t)\|_{H^4}^2, \tag{43}$$

holds true uniformly for all $0\le t<T$. Furthermore, summing ((19)) and ((24)) and $\tau_1\times$ ((37)) and noticing that$\tau_1$is sufficiently small, we have

$$\frac{d}{dt}\mathscr{E}(t)+\gamma\mathscr{D}(t) \le C\left[\mathscr{E}^{1/2}(t)+\mathscr{E}(t)\right]\mathscr{D}(t), \tag{44}$$

for all $0\le t<T$. With the help of (16), one has $\mathscr{E}^{1/2}(t)+\mathscr{E}(t)\le C(\delta+\delta^2)$ with $0<\delta<1$ being small enough. Then, the time integration of (44) yields

$$\mathscr{E}(t)+\gamma\int_0^t \mathscr{D}(s)ds \le \mathscr{E}(0), \tag{45}$$

for all $0\le t<T$. Besides, (16) can be justified by choosing

$$\mathscr{E}(0) \sim \|(\varrho_0,u_0,H_0,\Theta_0,\eta_0)\|_{H^4}^2, \tag{46}$$

which is sufficiently small. For brevity, the proof for local existence of smooth solutions is omitted. Then, the global existence and uniqueness of solutions follow from (45) together with the local existence as well as application of the continuity argument.

3. Convergence Rates

In this section, we consider the convergence rate of solutions obtained in Theorem 1. In order to obtain the desired decay rate estimates in Theorem 2, we firstly consider the linearized system corresponding to (5)–(9):

$$\varrho_t + divu = 0, \tag{47}$$

$$u_t+\nabla\rho+\nabla\Theta-\Delta u-2\nabla divu = 0, \tag{48}$$

$$H_t-\Delta H = 0, \quad divH = 0, \tag{49}$$

$$\Theta_t-\Delta\Theta+divu+4\Theta-\eta = 0, \tag{50}$$

$$\eta_t-\Delta\eta+\eta-4\Theta = 0, \tag{51}$$

with initial data

$$(\varrho,u,H,\Theta,\eta)|_{t=0} = (\varrho_0,u_0,H_0,\Theta_0,\eta_0)(x), \quad x\in\mathbb{R}^3. \tag{52}$$

Denote by $U(t)=(\varrho,u,H,\Theta,\eta)(t)$ to be the solution of the Cauchy problem (47)–(52), then $U(t)$ can be presented as

$$U(t) = \mathbb{A}(t)U_0, \tag{53}$$

where $\mathbb{A}(t)$ is named as the solution operator of (47)–(52) and $U_0=U|_{t=0}$. Then, we utilize the energy method to Cauchy problem (47)–(52) in the Fourier space to present that there is a time-frequency Lyapunov functional which is equivalent to $|U\wedge(t,k)|^2$. This estimate can help us to establish the L^p-L^q time decay property of $U(t)$ as follows.

Theorem 9. *Let $1\le q\le 2$. For any α,α' with $\alpha'\le\alpha$ and $m=|\alpha-\alpha'|$,*

$$\|\partial^\alpha\mathbb{A}(t)U_0\|_{L^2} \le C(1+t)^{-(3/2)((1/q)-(1/2))-(m/2)} \cdot\left(\|\partial^{\alpha'}U_0\|_{L^q}+\|\partial^\alpha U_0\|_{L^2}\right), \tag{54}$$

hold for all $t\ge 0$.

Proof. By taking Fourier transforming in x for (47)–(51), one has

$$\widehat{\varrho}_t + ik\cdot\widehat{u} = 0, \tag{55}$$

$$\widehat{u}_t+ik\widehat{\rho}+ik\widehat{\Theta}+|k|^2\widehat{u}+2k(k\cdot\widehat{u}) = 0, \tag{56}$$

$$\widehat{H}_t+|k|^2\widehat{H} = 0, \tag{57}$$

$$\widehat{\Theta}_t+|k|^2\widehat{\Theta}+ik\cdot\widehat{u}+4\widehat{\Theta}-\widehat{\eta} = 0, \tag{58}$$

$$\widehat{\eta}_t+|k|^2\widehat{\eta}+\widehat{\eta}-4\widehat{\Theta} = 0, \tag{59}$$

where $k\in\mathbb{R}^3$, $i=\sqrt{-1}\in\mathbb{C}$ is the imaginary unit.

Multiplying (55)–(59) by $4\bar{\widehat{\rho}}$, $4\bar{\widehat{u}}$, $\bar{\widehat{H}}$, $4\bar{\widehat{\Theta}}$, $\bar{\widehat{\eta}}$, respectively, its real part gives

$$\partial_t\left|\left(\sqrt{2}\varrho\wedge,\sqrt{2}u\wedge,H\wedge,\sqrt{2}\Theta\wedge,\eta\wedge\right)\right|^2+4|k|^2|u\wedge|^2+|k|^2|H\wedge|^2 + 8|k\cdot u\wedge|^2+4|k|^2|\Theta\wedge|^2+|k|^2|\eta\wedge|^2+|4\Theta\wedge-\eta\wedge|^2 = 0. \tag{60}$$

Multiplying (56) by $ik\bar{\widehat{\rho}}$, utilizing integration by parts in t, and replacing $\partial_t\widehat{\rho}$ by (55), one has

$$\partial_t(\widehat{u}\mid ik\widehat{\varrho})+|k|^2|\varrho\wedge|^2 = |k\cdot u\wedge|^2+3|k|^2 ik\cdot\widehat{u}\bar{\widehat{\varrho}}-|k|^2\widehat{\Theta}\bar{\widehat{\varrho}}, \tag{61}$$

here $(\cdot|\cdot)$ means the complex inner product. For the real part of (61) and with the help of Cauchy-Schwarz inequality, one has

$$\begin{aligned}\partial_t \operatorname{Re}(\widehat{u} \mid ik\widehat{\varrho}) + |k|^2|\varrho\wedge|^2 \le |k\cdot u\wedge|^2 + \varepsilon|k|^2|\varrho\wedge|^2 \\ + C_\varepsilon|k|^2|k\cdot u\wedge|^2 + \varepsilon|k|^2|\varrho\wedge|^2 \\ + C_\varepsilon|k|^2|\Theta\wedge|^2,\end{aligned} \tag{62}$$

with $\varepsilon > 0$ being a small constant. Multiplying it by $1/(1+|k|^2)$, we conclude that there exists $\gamma > 0$ such that

$$\partial_t \frac{\operatorname{Re}(\widehat{u} \mid ik\widehat{\varrho})}{1+|k|^2} + \frac{\gamma|k|^2|\varrho\wedge|^2}{1+|k|^2} \le C|u\wedge|^2 + \frac{C|k|^2|\Theta\wedge|^2}{1+|k|^2}. \tag{63}$$

Now, we define the time-frequency Lyapunov functional as

$$\begin{aligned}\mathscr{E}\left(\widehat{U}(t,k)\right) = \left|\left(\sqrt{2}\varrho\wedge, \sqrt{2}u\wedge, H\wedge, \sqrt{2}\Theta\wedge, \eta\wedge\right)\right|^2 \\ + \tau_2 \frac{\operatorname{Re}(\widehat{u} \mid ik\widehat{\varrho})}{1+|k|^2},\end{aligned} \tag{64}$$

where $0 < \tau_2 \ll 1$ is sufficiently small. It also holds that $\mathscr{E}(\widehat{U}) \sim |U\wedge|^2$. Moreover, by suitably choosing constants τ_1, the sum of equations (19), (24), $\tau_1\times$ (37) gives the linear combination $(60) + \tau_1 \times$ (63) which gives (60)

$$\partial_t \mathscr{E}\left(\widehat{U}(t,k)\right) + \frac{\gamma|k|^2}{1+|k|^2}\mathscr{E}\left(\widehat{U}(t,k)\right) \le 0. \tag{65}$$

As in [15, 21], the desired time decay estimates (54) and directly follows from the above estimate, and the detailed proof is omitted for brevity.

Now, we prove the rate of convergence (12). We quote a technical lemma in [19] for later proofs.

Lemma 10. *Given any $0 < \beta_1 \neq 1$ and $\beta_2 > 1$,*

$$\int_0^t (1+t-s)^{-\beta_1}(1+s)^{-\beta_2} ds \le C(1+t)^{-\min\{\beta_1,\beta_2\}} \tag{66}$$

for all $t \ge 0$.

By the Duhamel principle, the solution of nonlinear Cauchy problem (5)–(10) can be formally written as

$$U(t) = \mathbb{A}(t)U_0 + \int_0^t \mathbb{A}((t-s)(Y_1, Y_2, Y_3, Y_4, Y_5))ds, \tag{67}$$

with

$$\begin{aligned}
Y_1 &= -\varrho divu - \nabla\varrho\cdot u,\\
Y_2 &= -u\cdot\nabla u - \frac{\Theta-\varrho}{1+\varrho}\nabla\varrho - \frac{\varrho}{1+\varrho}\Delta u - \frac{2\varrho}{1+\varrho}\nabla divu \\
&\quad + \frac{1}{1+\varrho}\left(H\cdot\nabla H - \frac{1}{2}\nabla\left(|H|^2\right)\right),\\
Y_3 &= u\cdot\nabla H - H\cdot\nabla u + H divu,\\
Y_4 &= -u\cdot\nabla\Theta - \frac{\varrho}{1+\varrho}\Delta\Theta - \Theta divu + \frac{(divu)^2}{1+\varrho} + \frac{2D\cdot D}{1+\varrho} \\
&\quad - \frac{\varrho\eta}{1+\varrho} - \frac{4\rho\Theta}{1+\varrho} - \frac{6\Theta^2+4\Theta^3+\Theta^4}{1+\varrho} + \frac{|\nabla\times H|^2}{1+\varrho},\\
Y_5 &= 6\Theta^2 + 4\Theta^3 + \Theta^4.
\end{aligned} \tag{68}$$

By the definitions of $\mathscr{E}(t)$ and $\mathscr{D}(t)$ in (41) and (42), respectively, we have

$$\mathscr{E}(t) \le C\left(\mathscr{D}(t) + \|U\|^2\right). \tag{69}$$

From (44), we have

$$\frac{d}{dt}\mathscr{E}(t) + \frac{\gamma}{C}\left(\mathscr{E}(t) - \|U\|^2\right) \le \frac{d}{dt}\mathscr{E}(t) + \gamma\mathscr{D}(t) \le 0, \tag{70}$$

which implies

$$\frac{d}{dt}\mathscr{E}(t) + \gamma\mathscr{E}(t) \le C\|U\|^2. \tag{71}$$

Gronwall's inequality gives

$$\mathscr{E}(t) \le e^{-\gamma t}\mathscr{E}(0) + C\int_0^t e^{-\lambda(t-s)}\|U\|^2 ds. \tag{72}$$

Next, we give estimate of $\|U(t)\|$. Firstly, we further rewrite (67) as

$$\begin{aligned}U(t) &= \mathbb{A}(t)U_0 + \int_0^t \mathbb{A}(t-s)(Y_1, Y_2, Y_3, 0, 0)ds \\ &\quad + \int_0^t \mathbb{A}(t-s)(0,0,0,Y_4,Y_5)ds \\ &\equiv \sum_{i=1}^3 J_i(t).\end{aligned} \tag{73}$$

Define

$$\mathscr{E}_\infty(t) = \sup_{0\le s\le t}(1+s)^{3/2}\mathscr{E}(s). \tag{74}$$

One has that from (54) and Lemma 10,

$$\begin{aligned}
\|J_1(t)\| &\le C(1+t)^{-(3/4)}\|U_0\|_{L^1\cap L^2},\\
\|J_2(t)\| &\le C\int_0^t (1+t-s)^{-(3/4)}\|(Y_1,Y_2,Y_3)\|_{L^1\cap L^2}ds\\
&\le C\int_0^t (1+t-s)^{-(3/4)}\mathcal{E}(s)ds\\
&\le C\int_0^t (1+t-s)^{-(3/4)}(1+s)^{-(3/2)}ds\mathcal{E}_\infty(t)\\
&\le C(1+t)^{-(3/4)}\mathcal{E}_\infty(t),\\
\|J_3(t)\| &\le C\int_0^t (1+t-s)^{-(3/4)}\|(Y_4,Y_5)\|_{L^1\cap L^2}dsds\\
&\le C\int_0^t (1+t-s)^{-(3/4)}\left(\mathcal{E}(s)+\mathcal{E}^2(s)\right)ds\\
&\le C(1+t)^{-(3/4)}\left(\mathcal{E}_\infty(t)+\mathcal{E}_\infty^2(t)\right).
\end{aligned} \tag{75}$$

Therefore, it follows that

$$\|U\|^2 \le C(1+t)^{-(3/2)}\left\{\|U_0\|^2_{L^1\cap L^2}+\mathcal{E}^2_\infty(t)+\mathcal{E}^4_\infty(t)\right\}. \tag{76}$$

Substituting (76) into (72), we get

$$\mathcal{E}_\infty(t)\le C\left\{\|U_0\|^2_{H^4\bigcap L^1}+\mathcal{E}^2_\infty(t)+\mathcal{E}^4_\infty(t)\right\}, \tag{77}$$

which implies that $\mathcal{E}_\infty(t)\le C\|U_0\|^2_{H^4\bigcap L^1}$ for all $t\ge 0$, provided that $\|U_0\|_{H^4\bigcap L^1}$ is sufficiently small. Thus,

$$\mathcal{E}(t)\le C(1+t)^{-(3/2)}\|U_0\|^2_{H^4\bigcap L^1}. \tag{78}$$

This can deduce (12).

We continue to prove the rate of convergence (13). Firstly, we define the new energy functional and dissipation rate functional by

$$\begin{aligned}
\mathcal{M}(t) = &\sum_{1\le|\alpha|\le 4}\left(\|2\partial^\alpha \varrho\|^2+\|2\partial^\alpha u\|^2+\|\partial^\alpha H\|^2+\|2\partial^\alpha\Theta\|^2+\|\partial^\alpha\eta\|^2\right)\\
&+\tau_1\sum_{|\alpha|\le 3}\int\nabla\partial^\alpha\varrho\cdot\partial^\alpha u dx,
\end{aligned} \tag{79}$$

$$\begin{aligned}
\mathcal{N}(t) = &\sum_{1\le|\alpha|\le 3}\left(\|\partial^\alpha\nabla\varrho\|^2+\|\partial^\alpha\nabla(4\Theta-\eta)\|^2\right)\\
&+\sum_{1\le|\alpha|\le 4}\left(\|\partial^\alpha\nabla(u,H,\Theta,\eta)\|^2+\|\partial^\alpha divu\|^2\right).
\end{aligned} \tag{80}$$

By using Lemma 4 and similar arguments to those in the proof of Lemmas 6–8, we obtained

$$\frac{d}{dt}\mathcal{M}(t)+\gamma\mathcal{N}(t)\le C\left(\mathcal{M}^{1/2}(t)+\mathcal{M}(t)\right)\mathcal{N}(t). \tag{81}$$

Adding the term $\|\nabla U\|^2$ to both sides of (81) gives

$$\frac{d}{dt}\mathcal{M}(t)+\gamma\mathcal{M}(t)\le C\|\nabla U\|^2, \tag{82}$$

if $\mathcal{M}(t)$ is small enough. Being similar to the proof of $\|U\|$ and defining $\mathcal{M}_\infty(t)=\sup_{0\le s\le t}(1+s)^{5/2}\mathcal{H}(s)$ and by (73), we can deduce that

$$\|\nabla U\|^2\le C(1+t)^{-(5/2)}\left\{\|U_0\|^2_{L^1}+\|\nabla U_0\|^2+\mathcal{M}^2_\infty(t)+\mathcal{M}^4_\infty(t)\right\}. \tag{83}$$

From (81), (83), and Gronwall's inequality, we have

$$\begin{aligned}
\mathcal{M}(t)\le\; &e^{-\gamma t}\mathcal{M}(0)+C(1+t)^{-(5/2)}\left\{\|\nabla U_0\|^2+\|U_0\|^2_{L_1}+\mathcal{M}^2_\infty(t)\right.\\
&\left.+\mathcal{M}^4_\infty(t)\right\},
\end{aligned} \tag{84}$$

and hence

$$\mathcal{M}_\infty(t)\le C\left\{\|\nabla U_0\|^2_{H^3}+\|U_0\|^2_{L_1}+\mathcal{M}^2_\infty(t)+\mathcal{M}^4_\infty(t)\right\}. \tag{85}$$

Thus, since $\|\nabla U_0\|^2_{H^3}+\|U_0\|^2_{L_1}$ can be small enough, this implies

$$\mathcal{M}_\infty(t)\le C\left(\|\nabla U_0\|^2_{H^3}+\|U_0\|^2_{L_1}\right), \tag{86}$$

for all $t\ge 0$, that is,

$$\mathcal{M}(t)\le C(1+t)^{-(5/2)}\left(\|\nabla U_0\|^2_{H^3}+\|U_0\|^2_{L_1}\right), \tag{87}$$

which means

$$\|\nabla(\rho,u,H,\Theta,\eta)\|_{H^3}\le C(1+t)^{-(5/4)}, \tag{88}$$

for all $t\ge 0$, this completes the proof of Theorem 2.

4. The Periodic Case

In this section, we deal with the spatial domain $\Omega=\mathbb{T}^3$. For smooth solution of the system (1), it is not hard to get the following conservation laws in the case of torus,

$$\begin{gathered}
\frac{d}{dt}\int_{\mathbb{T}^3}\rho dx=0,\\
\frac{d}{dt}\int_{\mathbb{T}^3}\rho u dx=0,\quad \frac{d}{dt}\int_{\mathbb{T}^3}H dx=0,\\
\frac{d}{dt}\int_{\mathbb{T}^3}\left(\frac{1}{2}\rho|u|^2+\frac{1}{2}|H|^2+\rho\theta+n\right)dx=0,
\end{gathered} \tag{89}$$

and by the assumption (14), it follows that

$$\int_{\mathbb{T}^3} \rho dx = 0, \int_{\mathbb{T}^3} (1+\rho) u dx = 0, \int_{\mathbb{T}^3} H dx = 0,$$
$$\int_{\mathbb{T}^3} \left(\frac{1}{2}(1+\rho)|u|^2 + \frac{1}{2}|H|^2 + \rho + \Theta + \rho\Theta + \eta \right) dx = 0, \tag{90}$$

for all $t \geq 0$.

Proof of Theorem 11. We only give the proof of the global a priori estimates. Firstly, let the temporal energy functional $\mathscr{E}(t)$ and the corresponding dissipation rate functional $\mathscr{D}(t)$ be defined in the same way as in (41) and (42), respectively, for the case of the whole space $\Omega = \mathbb{R}^3$. Similarly, we conclude that

$$\frac{d}{dt}\mathscr{E}(t) + \gamma \mathscr{D}(t) \leq C\left(\|(\rho, u, H, \Theta, \eta)\|_{H^4} + \|(\rho, u, H, \Theta, \eta)\|_{H^4}^2 \right) \cdot \left(\|\nabla(\rho, u, H, \Theta, \eta)\|_{H^3}^2 + \|divu\|_{H^3}^2 \right). \tag{91}$$

Thanks to Poincaré's inequality and the conservation laws (90), we have

$$\|\rho\| \leq C\|\nabla\rho\|, \quad \|H\| \leq C\|\nabla H\|, \tag{92}$$

$$\begin{aligned} \|u\| &\leq \|u + \rho u\| + \|\rho u\| \leq C\|\nabla u\| + C\|\nabla(\rho u)\| + \|\rho\|\|u\|_{L^\infty} \\ &\leq C\|\nabla u\| + C\|u\|_{H^2}\|\nabla\rho\| + C\|\rho\|_{H^2}\|\nabla u\|, \end{aligned} \tag{93}$$

$$\begin{aligned} \|\Theta + \eta\| &= \left\| \frac{1}{2}(1+\rho)|u|^2 + \frac{1}{2}|H|^2 + \rho + \Theta + \rho\Theta + \eta \right\| \\ &\quad + \left\| \frac{1}{2}(1+\rho)|u|^2 + \frac{1}{2}|H|^2 + \rho + \rho\Theta \right\| \\ &\leq C\|u\|_{H^2}\|\nabla u\| + C\|\rho\|_{H^2}\|u\|_{H^2}\|\nabla u\| + C\|u\|_{H^2}^2\|\nabla\rho\| \\ &\quad + C\|H\|_{H^2}\|\nabla H\| + C\|\Theta\|_{H^2}\|\nabla\rho\| + C\|\rho\|_{H^2}\|\nabla\Theta\| \\ &\quad + C(\|\nabla\rho\| + \|\nabla\Theta\| + \|\nabla\eta\|). \end{aligned} \tag{94}$$

Then, define

$$\mathscr{D}_{\mathbb{T}}(t) = \mathscr{D}(t) + \tau_3\left(\|\rho\|^2 + \|u\|^2 + \|H\|^2 + \|\Theta + \eta\|^2 \right), \tag{95}$$

where $0 < \tau_3 \ll 1$ is sufficiently small. Notice

$$\mathscr{D}_{\mathbb{T}}(t) \sim \|\rho\|_{H^4}^2 + \|(u, H, \Theta, \eta)\|_{H^5}^2, \tag{96}$$

uniformly for all $t \geq 0$. Combining (91)–(94) together, we have

$$\frac{d}{dt}\mathscr{E}(t) + \gamma \mathscr{D}_{\mathbb{T}}(t) \leq C\left[\mathscr{E}^{1/2}(t) + \mathscr{E}(t) + \mathscr{E}^2(t) \right] \mathscr{D}_{\mathbb{T}}(t). \tag{97}$$

Using the fact that $\mathscr{E}(t)$ is small enough and uniform in time, and $\mathscr{E}(t) \leq C\mathscr{D}_{\mathbb{T}}(t)$, we then obtain

$$\frac{d}{dt}\mathscr{E}(t) + \gamma \mathscr{E}(t) \leq 0, \tag{98}$$

for all $t \geq 0$. Applying Gronwall's inequality to (98), one has

$$\mathscr{E}(t) \leq e^{-\gamma t}\mathscr{E}(0). \tag{99}$$

This gives the desired exponential decay of $\mathscr{E}(t) \sim \|(\rho, u, H, \Theta, \eta)\|_{H^4}^2$, and hence completes the proof of Theorem 11.

Data Availability

Not applicable.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed equally and significantly in writing this article. All the authors read and approved the final manuscript.

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