Research Article

Hybrid Fixed Point Theorem with Applications to Forced Damped Oscillations and Infinite Systems of Fractional Order Differential Equations

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In this manuscript, hybrid common fixed point results in the setting of a b -metric space are established. Our results generalized the results of Fisher, Khan, and Piri et al. for set-valued mapping in b -metric spaces. Applications to forced damped oscillations, infinite systems of fractional order differential equations, and system of functional equations are also studied. We construct an example to support our main result.

1. Introduction and Preliminaries

The idea of a metric space was generalized by Czerwik [1] and Bakhtin [2]. They presented metric spaces called b -metric spaces. Several researchers took the idea of Czerwik and illustrated interesting results. For details, see [3–7]. For recent generalizations to b -metric spaces by employing control functions in the triangle inequality to replace the constant of the b -metric triangle inequality, we refer to [8–12] and the references therein.

In 1973, the contraction was introduced by Geraghty [13] in which the contraction constant was quite changed by mapping owing to its interesting properties. After that, several papers for rational Geraghty contractive mappings have appeared (for details, see [14–18]). Khan [19] introduced one of the best works in this line, and Fisher [20] modified it. By rational expressions, Khan [19] and Fisher [20] results were lately extended by Piri et al. [21] by introducing a new general contractive condition. Fixed point

results via F -Khan contractions were studied by Piri et al. [22] on complete metric spaces, and they discuss their application to integral equations. In [23], Ullah et al. established fixed point results and discuss the application to an infinite system of fractional order differential equations.

Nadler [24] elaborated and extended the Banach contraction principle [25] to set-valued mapping by using the Hausdorff metric. After different generalizations of the Nadler contraction principle, Wardowski [26] introduced a contraction called F -contraction. In this way, Wardowski generalized the Banach contraction principle (BCP) in a different manner from the known results of literature. Following this direction, Sgroi and Vetro [27] studied set-valued F -contractions and discussed their application on certain functional and integral equations.

Cosentino and Vetro [25] extended F -contraction in the setting of b -metric spaces and proved some fixed point results. Ali et al. [28] studied the fixed point, generalize the result of Cosentino et al. [25] for a new class of F -contractions in the set-

ting of b -metric spaces, and apply the result to obtain existence results for Volterra-type integral inclusion in b -metric spaces. Several authors generalized F -contraction by combining it with some existing contractive conditions (see [27, 29–32]).

In the current work, we derive a hybrid (single and multi-valued) common fixed point result for the F -Khan-type contraction in the b -metric space. Also, we shall provide an example and applications for the validity of the established result. Throughout this paper, $CB(\Lambda)$ indicates the family of nonempty subsets of Λ , which is bounded and closed. $\mathbb{R}^+$, $\mathbb{N}_0$, and $\mathbb{N}$ signify the set of any nonnegative real numbers, the set of nonnegative integers, and the set of positive integers. Now, we recall a few basic results and definitions.

Definition 1 [1]. Consider a nonempty set Λ and let $s \geq 1$. Assume $d : \Lambda \times \Lambda \longrightarrow \mathbb{R}^+$ is a function satisfying the conditions:

- (1) $d(\xi_{11}, \xi_{22}) = 0 \Leftrightarrow \xi_{11} = \xi_{22}$ for all $\xi_{11}, \xi_{22} \in \Lambda$
- (2) $d(\xi_{11}, \xi_{22}) > 0$, $\xi_{11} \neq \xi_{22}$, for all $\xi_{11}, \xi_{22} \in \Lambda$
- (3) $d(\xi_{11}, \xi_{22}) = d(\xi_{22}, \xi_{11})$, where $\xi_{11}, \xi_{22} \in \Lambda$
- (4) $d(\xi_{11}, \xi_{22}) \leq s(d(\xi_{11}, \xi_{33}) + d(\xi_{33}, \xi_{22}))$ for all $\xi_{11}, \xi_{22}, \xi_{33} \in \Lambda$

Then, the triple (Λ, d, s) is called the b -metric space.

Definition 2 [1]. Assume (Λ, d, s) is a b -metric space, where $s \geq 1$. Let σ_n be a sequence in Λ . Then, $\sigma \in \Lambda$ is said to be the limit of the sequence σ_n if

$$\lim_{n \rightarrow \infty} d(\sigma_n, \sigma) = 0, \quad (1)$$

and the sequence σ_n is said to be convergent in Λ .

Definition 3 [1]. If for each $\epsilon > 0$, there is a positive integer $\mathbb{N}$ such that $d(\sigma_n, \sigma_m) < \epsilon$ for all $n, m > \mathbb{N}$, then a sequence σ_n is said to be a b -Cauchy sequence.

Definition 4 [1]. A b -metric space (Λ, d, s) is said to be complete (or a b -complete metric space) if every Cauchy sequence in (Λ, d, s) is convergent in Λ .

Definition 5 [33]. Assume $s \geq 1$ is a real number and $\mathbb{F}$ represents the family of functions $F : \mathbb{R}^+ \longrightarrow \mathbb{R}$, with the below conditions:

- (F₁) For each $\{\gamma_n\} \subset \mathbb{R}^+$ which is a positive term sequence, F is strictly increasing.
- (F₂) $\lim_{n \rightarrow \infty} \gamma_n = 0 \Leftrightarrow \lim_{n \rightarrow \infty} F(\gamma_n) = \infty$.
- (F₃) For each positive number sequence, i.e., $\{\gamma_n\} \subset \mathbb{R}^+$, $\lim_{n \rightarrow \infty} \gamma_n = 0$, there exists $k \in (0, 1)$ such that $\lim_{n \rightarrow \infty} (\gamma_n)^k F(\gamma_n) = 0$.
- (F₄) For $\{\gamma_n\} \subset \mathbb{R}^+$, such that $\lambda + F(s\gamma_n) \leq F(\gamma_{n-1}) \forall n \in \mathbb{N}$ and some $\lambda \in \mathbb{R}^+$, then $\lambda + F(s^n \gamma_n) \leq F(s^{n-1} \gamma_{n-1})$.

Now, we give some basics defined in a b -metric space (Λ, d, s) for set-valued mappings. Define the mapping $\mathbb{H} : CB(\Lambda) \times CB(\Lambda) \longrightarrow \mathbb{R}^+$ for $\Lambda_1, \Lambda_2 \in CB(\Lambda)$ by

$$\mathbb{H}(\Lambda_1, \Lambda_2) = \max \{\delta(\Lambda_1, \Lambda_2), \delta(\Lambda_2, \Lambda_1)\}, \quad (2)$$

where

$$\begin{aligned} \delta(\Lambda_1, \Lambda_2) &= \sup \{d(a, \Lambda_2), a \in \Lambda_1\}, \\ \delta(\Lambda_2, \Lambda_1) &= \sup \{d(b, \Lambda_1), b \in \Lambda_2\}, \\ d(a, \Lambda) &= \inf \{d(a, \sigma_1), \sigma_1 \in \Lambda\}. \end{aligned} \quad (3)$$

Lemma 6 [16, 34]. Assume a b -metric space (Λ, d, s) . Suppose $\Lambda_1, \Lambda_2, \Lambda_3 \in CB(\Lambda)$ and $\sigma_1, \sigma_2 \in \Lambda$; then, the following holds:

- (1) $d(\sigma_1, \Lambda_2) \leq d(\sigma_1, b)$ for any $b \in \Lambda_2$
- (2) $\delta(\Lambda_1, \Lambda_2) \leq \mathbb{H}(\Lambda_1, \Lambda_2)$
- (3) $d(\sigma_1, \Lambda_2) \leq \mathbb{H}(\Lambda_1, \Lambda_2)$ for any $\sigma_1 \in \Lambda_1$

$$\mathbb{H}(\Lambda_1, \Lambda_1) = 0, \quad (4)$$

$$(4) \mathbb{H}(\Lambda_1, \Lambda_2) = \mathbb{H}(\Lambda_2, \Lambda_1)$$

$$\begin{aligned} \mathbb{H}(\Lambda_1, \Lambda_3) &\leq s[\mathbb{H}(\Lambda_1, \Lambda_2) + \mathbb{H}(\Lambda_2, \Lambda_3)], \\ \mathbb{H}(\sigma_1, \Lambda_1) &\leq s[d(\sigma_1, \sigma_2) + d(\sigma_2, \Lambda_1)]. \end{aligned} \quad (5)$$

Lemma 7 [20, 24]. Let (Λ, d) be a metric space. Further, $\Lambda_1, \Lambda_2 \in CB(\Lambda)$; then, for $\delta \geq 1$ and for each $\zeta_1 \in \Lambda_1$, there exists $\zeta_2(\zeta_1) \in \Lambda_2$ such that $d(\zeta_1, \zeta_2) \leq \delta \mathbb{H}(\Lambda_1, \Lambda_2)$.

Lemma 8 [16, 34]. Assume a b -metric space (Λ, d, s) and $\Lambda_1, \Lambda_2 \in CB(\Lambda)$. Then, for $\delta > 1, \zeta_1 \in \Lambda_1$, there exists $\zeta_2(\zeta_1) \in \Lambda_2$ such that $d(\zeta_1, \zeta_2) \leq \delta \mathbb{H}(\Lambda_1, \Lambda_2)$.

Theorem 9 [24]. Assume $\lambda : \Lambda \longrightarrow CB(\Lambda)$ is a multivalued mapping on a complete metric space (Λ, d) such that

$$\mathbb{H}(\lambda\sigma_1, \lambda\sigma_2) \leq \mu(d(\sigma_1, \sigma_2)), \quad \forall \sigma_1, \sigma_2 \in \Lambda, \quad (6)$$

where $\mu \in (0, 1)$. Then, λ has a fixed point.

Theorem 10 [25]. Let (Λ, d, s) be a b -metric space, where $s \geq 1$. For $F \in \mathbb{F}$, a multivalued mapping $\lambda : \Lambda \longrightarrow CB(\Lambda)$ is called a Nadler-type F -contraction such that for $\tau \in \mathbb{R}^+$,

$$2\tau + F(s\mathbb{H}(\lambda\xi_1, \lambda\xi_2)) \leq F(d(\xi_1, \xi_2)). \quad (7)$$

$\forall \xi_1, \xi_2 \in \Lambda$ $\lambda\xi_1 \neq \lambda\xi_2$; then, λ has a fixed point in Λ .

2. Main Results

Definition 11. Let (Λ, d, s) be a b -metric space. Let $\theta : \Lambda \longrightarrow \Lambda$ and $\Theta : \Lambda \longrightarrow CB(\Lambda)$ be two mappings. Then, Θ is said to be F -Khan contraction w.r.t θ if there exists $0 < \tau < \infty$ and $F \in \mathbb{F}$ such that for all $\xi, \zeta \in \Lambda$, $\max \{d(\xi, \theta\zeta), d(\theta\xi, \zeta)\} \neq 0$; then, $\theta\xi \neq \theta\zeta$ and

$$\tau + F(s\mathbb{H}(\theta\xi, \theta\zeta)) \leq \begin{cases} F\left(\frac{d(\xi, \theta\xi)d(\xi, \theta\zeta) + d(\zeta, \theta\zeta)d(\zeta, \theta\xi)}{\xi\{d(\xi, \theta\zeta), d(\theta\xi, \zeta)\}}\right), & \text{if } \max\{d(\xi, \theta\zeta), d(\theta\xi, \zeta)\} \neq 0, \\ 0, & \text{if } \max\{d(\xi, \theta\zeta), d(\theta\xi, \zeta)\} = 0. \end{cases} \quad (8)$$

Theorem 12. Assume (Λ, d, s) is a complete b -metric space. Let (θ, Θ) be the F -Khan contraction. Further, θ is continuous. Then, $C(\theta, \Theta) \neq \emptyset$.

- (i) θ and Θ have a common fixed point if $\theta\theta\alpha = \theta\alpha$, and θ is occasionally Θ -weakly commuting at α . Then, θ and Θ have a common fixed point

Proof. Let $\zeta_0 \in \Lambda$ and $\zeta_1 \in \Theta\zeta_0$. Let $\zeta_2 := \theta\zeta_1$. By Lemma 8, there exists $\zeta_3 \in \Theta\zeta_2$ such that $d(\zeta_3, \zeta_2) \leq \mathbb{H}(\Theta\zeta_2, \{\theta\zeta_1\})$. Inductively, we let $\zeta_{2n} := \theta\zeta_{2n-1}$, and by Lemma 8, we choose $\zeta_{2n+1} \in \Theta\zeta_{2n}$ such that

$$d(\zeta_{2n-1}, \zeta_{2n}) \leq \mathbb{H}(\Theta\zeta_{2n}, \{\theta\zeta_{2n-1}\}). \quad (9)$$

Using equation (8), we have

$$\begin{aligned} & \tau + F(s\mathbb{H}(\Theta\zeta_{2n}, \Theta\zeta_{2n+1})) \\ & \leq F\left(\frac{d(\zeta_{2n}, \theta\zeta_{2n})d(\zeta_{2n}, \theta\zeta_{2n+1}) + d(\zeta_{2n+1}, \theta\zeta_{2n+1})d(\zeta_{2n+1}, \theta\zeta_{2n})}{\max\{d(\theta\zeta_{2n}, \zeta_{2n+1}), d(\zeta_{2n}, \theta\zeta_{2n+1})\}}\right) \\ & \leq F(d(\zeta_{2n}, \theta\zeta_{2n})), \end{aligned} \quad (10)$$

which implies

$$\tau + F(s\mathbb{H}(\Theta\zeta_{2n}, \Theta\zeta_{2n+1})) \leq F(d(\zeta_{2n}, \theta\zeta_{2n})). \quad (11)$$

We deduce that

$$\tau + F(s d(\zeta_{2n+1}, \zeta_{2n+2})) \leq F(d(\zeta_{2n}, \zeta_{2n+1})). \quad (12)$$

Let $Q_n = d(\zeta_{2n+1}, \zeta_{2n+2}) > 0$, $\forall n \in \mathbb{N}$. It follows from (12) and axiom F_4 that

$$\tau + F(s^n d(\zeta_{2n+1}, \zeta_{2n+2})) \leq F(s^{n-1} d(\zeta_{2n}, \zeta_{2n+1})), \quad \forall n \in \mathbb{N}. \quad (13)$$

Thus, by equation (13),

$$\begin{aligned} F(s^n Q_n) & \leq F(s^{n-1} Q_{n-1}) - \tau, \\ F(s^{n-1} Q_{n-1}) & \leq F(s^{n-2} Q_{n-2}) - 2\tau, \\ & \vdots \\ F(s^n Q_n) & \leq F(s^0 Q_0) - n\tau, \end{aligned} \quad (14)$$

which implies that

$$\lim_{n \rightarrow \infty} F(s^n Q_n) = -\infty. \quad (15)$$

By using F_2 , we have

$$\lim_{n \rightarrow \infty} s^n Q_n = 0. \quad (16)$$

By the F_3 property, there exists $0 < k < 1$ such that

$$\lim_{n \rightarrow \infty} (s^n Q_n)^k F(s^n Q_n) = 0. \quad (17)$$

Equation (14) implies that

$$F(s^n p_n) \leq F(p_0) - n\tau. \quad (18)$$

Multiplying (18) by $(s^n p_n)^k$, we have

$$(s^n p_n)^k F(s^n p_n) \leq (s^n p_n)^k F(p_0) - n\tau (s^n p_n)^k, \quad (19)$$

which implies that

$$(s^n p_n)^k F(s^n p_n) - (s^n p_n)^k F(p_0) \leq -n\tau (s^n p_n)^k \leq 0. \quad (20)$$

Applying limit $n \rightarrow \infty$, we have

$$\lim_{n \rightarrow \infty} n(s^n p_n)^k = 0. \quad (21)$$

From (21), there exists $n_1 \in \mathbb{N}$ such that $n(s^n Q_n)^k < 1$ such that

$$s^n Q_n \leq \frac{1}{nn^{1/k}}, \quad \forall n \geq n_1. \quad (22)$$

To show that $\{\zeta_n\}$ is a b -Cauchy sequence, consider $m, n \in \mathbb{N}$ such that $m > n > n_1$, using triangular inequality, and using (18), we have

$$\begin{aligned}
d(\zeta_{2n}, \zeta_{2m}) &\leq sd(\zeta_{2n}, \zeta_{2n+1}) + s^2 d(\zeta_{2n+1}, \zeta_{2n+2}) + \dots \\
&\quad + s^{m-n} d(\zeta_{2m-1}, \zeta_{2m}), = sQ_{n-1} + s^2 Q_n + \dots \\
&\quad + s^{m-n} Q_{m-2}, = \sum_{j=n-1}^{m-2} s^{j-n+2} Q_j \\
&\leq \sum_{j=n-1}^{\infty} s^{j-n+2} Q_j \leq \sum_{j=n-1}^{\infty} s^{2-n} \frac{1}{j^{1/k}}.
\end{aligned} \tag{23}$$

By taking limit, we get $d(\zeta_n, \zeta_m) \longrightarrow 0$. Hence, $\{\zeta_n\}$ is a b -Cauchy sequence, but a b -metric space (Λ, d, s) is a complete space so there exists $\zeta \in \Lambda$ such that $\zeta_n \longrightarrow \zeta$ as $n \longrightarrow \infty$.

The next step is to show that ζ is a common fixed point of the mapping Θ and θ . We have

$$d(\zeta_{2n+2}, \Theta\zeta) \leq H(\Theta\zeta_{2n+1}, \Theta\zeta) \leq sH(\Theta\zeta_{2n+1}, \Theta\zeta), \tag{24}$$

which implies that

$$d(\zeta_{2n+2}, \Theta\zeta) \leq sH(\Theta\zeta_{2n+1}, \Theta\zeta). \tag{25}$$

Since F is strictly increasing, therefore

$$F(d(\zeta_{2n+2}, \Theta\zeta)) \leq F(sH(\Theta\zeta_{2n+1}, \Theta\zeta)). \tag{26}$$

Adding τ to both sides and using equation (7), we have

$$\begin{aligned}
&\tau + F(d(\zeta_{2n+2}, \Theta\zeta)) \\
&\leq \tau + F(sH(\Theta\zeta_{2n+1}, \Theta\zeta)) \\
&\leq F\left(\left\{\frac{d(\zeta_{2n}, \Theta\zeta_{2n})d(\zeta_{2n}, \Theta\zeta) + d(\zeta, \Theta\zeta)d(\zeta, \Theta\zeta_{2n})}{\max\{d(\zeta_{2n+1}, \Theta\zeta), d(\Theta\zeta_{2n+1}, \zeta)\}}\right\}\right).
\end{aligned} \tag{27}$$

Since $\tau \in \mathbb{R}^+$, we have

$$\begin{aligned}
&F(d(\zeta_{2n+2}, \Theta\zeta)) \\
&\leq F\left(\left\{\frac{d(\zeta_{2n}, \Theta\zeta_{2n})d(\zeta_{2n}, \Theta\zeta) + d(\zeta, \Theta\zeta)d(\zeta, \Theta\zeta_{2n})}{\max\{d(\zeta_{2n+1}, \Theta\zeta), d(\Theta\zeta_{2n+1}, \zeta)\}}\right\}\right).
\end{aligned} \tag{28}$$

Since F is strictly increasing, therefore

$$d(\zeta_{2n+2}, \Theta\zeta) \leq \left\{\frac{d(\zeta_{2n}, \Theta\zeta_{2n})d(\zeta_{2n}, \Theta\zeta) + d(\zeta, \Theta\zeta)d(\zeta, \Theta\zeta_{2n})}{\max\{d(\zeta_{2n+1}, \Theta\zeta), d(\Theta\zeta_{2n+1}, \zeta)\}}\right\}. \tag{29}$$

Applying limit $n \longrightarrow \infty$, we get

$$d(\zeta, \Theta\zeta) \leq 0, \tag{30}$$

which implies $\zeta \in \Theta\zeta$.

But

$$d(\zeta_{2n+2}, \Theta\zeta) = d(\Theta\zeta_{2n+1}, \Theta\zeta). \tag{31}$$

By taking limit and using continuity of θ , we have

$$d(\zeta, \Theta\zeta) = \lim_{n \rightarrow \infty} d(\Theta\zeta_{2n+1}, \Theta\zeta) = d(\theta\zeta, \Theta\zeta), \tag{32}$$

which implies $\theta\zeta \in \Theta\zeta$. Since $\theta\theta\zeta = \theta\zeta$ and $\theta\zeta \in \Theta\theta\zeta$, therefore $\gamma = \theta\gamma \in \Theta\gamma$.

By putting $\theta = \Theta$ in Theorem 12, we get the following.

Corollary 13. Assume (Λ, d, s) is a complete b -metric space. Let Θ be an F -Khan contraction. Then, Θ has a fixed point.

Consider the following class of function.

Definition 14. Assume $s \geq 1$ is a real number and $\mathbb{F}_1$ represents the family of functions $F : \mathbb{R}^+ \longrightarrow \mathbb{R}$, which is strictly increasing.

Definition 15. Let (Λ, d, s) be a b -metric space. Let $\theta : \Lambda \longrightarrow \Lambda$ and $\Theta : \Lambda \longrightarrow CB(\Lambda)$ be two mapping. Then, Θ is said to be the F -Khan contraction w.r.t θ if there exists $0 < \tau < \infty$ and $F \in \mathbb{F}_1$ such that for all $\xi, \zeta \in \Lambda$, $\max\{d(\xi, \Theta\zeta), d(\theta\xi, \zeta)\} \neq 0$; then, $\theta\xi \neq \theta\zeta$ and

$$\tau + F(sH(\Theta\xi, \Theta\zeta)) \leq \begin{cases} F\left(\frac{d(\xi, \theta\xi)d(\xi, \theta\zeta) + d(\zeta, \theta\zeta)d(\zeta, \theta\xi)}{\xi\{d(\xi, \Theta\zeta), d(\theta\xi, \zeta)\}}\right), & \text{if } \max\{d(\theta\xi, \zeta), d(\theta\xi, \zeta)\} \neq 0, \\ 0, & \text{if } \max\{d(\xi, \theta\zeta), d(\theta\xi, \zeta)\} = 0. \end{cases} \tag{33}$$

Theorem 16. Assume (Λ, d, s) is a complete b -metric space. Let (θ, Θ) be an F -Khan contraction. Further, if θ is continuous, then $C(\theta, \Theta) \neq \emptyset$.

- (i) θ and Θ have a common fixed point if $\theta\theta\alpha = \theta\alpha$, and θ is occasionally Θ -weakly commuting at α . Then, θ and Θ have a common fixed point

Remarks. Our result extended the results of

- (i) Fisher [20] for set-valued mapping in the setting of b -metric spaces
- (ii) Khan [19] for set-valued mapping in b -metric spaces
- (iii) Piri et al. [21, 22] for set-valued mapping in b -metric spaces

Example 17. Consider the sequence $\{\{S_q\}: q \in \{1, 2 \dots 100\}\}$ as follows:

$$S_1 = 1.2, S_2 = 2.3, \dots, S_q = q(q+1). \quad (34)$$

Let $\Lambda = \{S_q : q \in \{1, 2, \dots, 100\}\}$ and $d : \Lambda \times \Lambda \longrightarrow [0, \infty)$ be defined by

$$d(\xi, \zeta) = \max \{\xi, \zeta\}^2, \quad \text{if } \xi \neq \zeta \quad d(\xi, \zeta) = 0, \text{ if } \xi = \zeta. \quad (35)$$

Then, (Λ, d, s) is a complete b -metric space. Define the mapping $\Theta : \Lambda \longrightarrow CB(\Lambda)$ by

$$\Theta(S_1) = \{S_1\}, \Theta(S_q) = \{S_{q-1}\}, \quad (36)$$

FIGURE 1: Forced oscillations.

and $\theta : \Lambda \longrightarrow \Lambda$ by

$$\theta(S_1) = S_1, \theta(S_q) = S_{q-1}. \quad (37)$$

Let us consider the following calculation. First, observe that $\max \{d(x_q, \theta x_p), d(\theta x_q, x_p)\} \neq 0 \Leftrightarrow [(p > 2 \wedge q = 1) \vee (p > q > 1)]$.

For each $p \in \mathbb{N}$, $p > 2$, we have

$$\begin{aligned} & \frac{H(\Theta S_p, \Theta S_1)}{d(S_p, \Theta S_p)d(S_p, \Theta S_1) + d(S_1, \Theta S_1)d(S_1, \Theta S_p) / \max \{d(S_p, \Theta S_1)d(S_1, \Theta S_p)\}} \\ &= \frac{S_{p-1}^2}{S_p^2 \cdot S_1^2 + S_1^2 \cdot S_{p-1}^2 / S_p^2} \leq \frac{S_{p-1}^2}{S_p^2} = \frac{(p-1)^2}{(p+1)^2} < e^{-0.02}. \end{aligned} \quad (38)$$

For each $p, q \in \mathbb{N}$, $p > q > 1$, we have

$$\begin{aligned} & \frac{H(\Theta S_p, \Theta S_q)}{d(S_p, \Theta S_p)d(S_p, \Theta S_q) + d(S_q, \Theta S_q)d(S_q, \Theta S_p) / \max \{d(S_p, \Theta S_q)d(S_q, \Theta S_p)\}} \\ &= \frac{S_p^2 S_{p-1}^2}{S_p^2 S_{p-1}^2 + S_q^2 S_p^2} \leq \frac{S_p^2}{S_q^2 + S_p^2} = \frac{(p(p-1))^2}{(q(q+1))^2 + (p(p+1))^2} < e^{-0.02}. \end{aligned} \quad (39)$$

Multiplying 1.01 on both sides and taking log to the base e on both sides, we get inequality (7) and also find that $\tau = 0.004365 \dots$. Therefore, θ and Θ have a fixed point.

3. Application to Force Damping Oscillation

Throughout in this section, $\max \{d(h, \theta k), d(\theta h, k)\} \neq 0$.

Assume that an object of mass m moves to and fro on the x -axis around an equilibrium position $x = 0$ (see Figure 1). The object has position $x(t)$ at time t . It undergoes a force due to a spring:

$$F_s = -kx, \quad (40)$$

Furthermore, a damping force that resists the movement of the object is shown:

$$F_d = b \frac{dx}{dt}. \quad (41)$$

Now, by the second law of motion,

$$F_{\text{net}} = m \frac{d^2 x}{dt^2}, \quad (42)$$

where m , k , and b are all positive constants. Up to that feature, the system is simply the damped harmonic oscillator.

Now, suppose the additional time-dependent force $f(t)$ is applied to the object. Then, by Newton's second law,

$$\begin{aligned} -kx - b \frac{dx}{dt} + f(t) &= m \frac{d^2 x}{dt^2}, m \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + kx \\ &= f(t), x(0) = 0, x'(0) = 0. \end{aligned} \quad (43)$$

The problem (43) can be written in the form of the Fredholm integral equation:

$$v(t_1) = \int_0^1 G((\beta, s)) \Theta_1(t_1, v(s_1)) ds_1, t_1 \in [0, 1]. \quad (44)$$

Here, Green's function for critically damping oscillation is defined as

$$\mathcal{G}(\beta, s_1) = \begin{cases} -s_1 e^{\tau(\beta-t)}, & 0 \leq s_1 \leq \beta \leq 1, \\ -t e^{\tau(t-\beta)}, & 0 \leq \beta \leq s_1 \leq 1, \end{cases} \quad (45)$$

where τ can be found in terms of m , b , and k .

Let $\Lambda = C[0, 1]$ be the set of all continuous functions defined on $[0, 1]$. For $u \in C([0, 1])$, define the supremum norm as

$$\|v\| = \left(\sup_{t_1 \in [0, t]} |v(t_1)| \right) e^{-\tau}. \quad (46)$$

Let $C([0, 1], \mathbb{R})$ be endowed with the b -metric:

$$d(v, \zeta) = \sup_{t_1 \in [0, 1]} \|v(t_1) - \zeta(t_1)\| e^{-\tau}. \quad (47)$$

With these contexts, $C([0, 1], \mathbb{R}, \|\cdot\|)$ becomes the Banach space. We give the following theorem.

Theorem 18. Assume the assumptions given below hold.

(A₁) There exists a continuous function $\Theta_1 : [0, 1] \times [0, 1] \longrightarrow [0, \infty)$ and for $t_1, s_1 \in [0, t]$ such that

$$|\Theta_1(t_1, s_1, u) - \Theta_1(t_1, s_1, v)| \leq \sqrt{\frac{e^{-\tau}}{s}} M_s(h, k),$$

$$M_s(h, k) = \sqrt{\frac{e^{-\tau}}{s} \left\{ \frac{d(h, \theta h)d(h, \theta k) + d(k, \theta k)d(k, \theta h)}{\max \{d(h, \theta h), d(\theta h, k)\}} \right\}}, \quad (48)$$

where $\theta : C([0, 1]) \longrightarrow C([0, 1])$ is any function.

(A₂) $\int_0^1 G(\beta, s_1) ds_1 \leq 1$, for some $\tau > 0$.

Then, the problem (43) has a solution in $C([0, 1])$.

Proof. Define $\Theta : C([0, 1]) \longrightarrow C([0, 1])$ by

$$\Theta v(t_1) = \int_0^1 G((\beta, s_1)) \Theta_1(t_1, v(s_1)) ds_1, \quad t_1 \in [0, t]. \quad (49)$$

We have

$$\begin{aligned} |\Theta v - \Theta \zeta| &= \int_0^1 G((\beta, s_1)) |\Theta_1(t_1, s_1, v(s_1)) - \Theta_1(t_1, s_1, \zeta(s_1))| ds_1 \\ &\leq \frac{1}{\sqrt{s}} \int_0^1 G(\beta, s_1) \sqrt{e^{-\tau} M_s(h, k)} ds_1 \\ &\leq \frac{1}{\sqrt{s}} G(\beta, s_1) \sqrt{e^{-\tau} M_s(h, k)} ds_1 \\ &\leq \frac{1}{e^{-2\tau} \sqrt{s}} \sqrt{e^{-\tau} M_s(h, k)} \int_0^1 (\beta, s_1) ds_1 \\ &\leq \frac{1}{\sqrt{s}} \sqrt{e^{-\tau} M_s(h, k)}. \end{aligned} \quad (50)$$

Squaring both sides, we get

$$|\Theta v - \Theta \zeta|_2 \leq \frac{1}{2} e^{-\tau} \left\{ \frac{d(h, \theta h)d(h, \theta k) + d(k, \theta k)d(k, \theta h)}{\max \{d(h, \theta k), d(\theta h, k)\}} \right\}. \quad (51)$$

Multiplying on both sides $e^{-2\tau}$ and taking sup, we get

$$sd(\Theta v, \Theta \zeta) \leq e^{-2\tau} \left\{ \frac{d(h, \theta h)d(h, \theta k) + d(k, \theta k)d(k, \theta h)}{\max \{d(h, \theta k), d(\theta h, k)\}} \right\}. \quad (52)$$

By taking $F(v) = \ln(v)$ from Theorem 12, the integral equation (44) has a common solution.

Similarly, we can apply our theorem for the existence of underdamping oscillation and overdamping oscillation.

4. Application to Infinite Systems of Fractional Order Differential Equations

Now, we have to derive sufficient conditions for the solutions in space $\mathbf{c}$ to the following nonlinear infinite systems of fractional order differential equations:

$$\begin{cases} \vartheta_{\alpha_i} = \frac{e^{-\tau}}{b(t)} b_i(t) \vartheta_i + \theta_i(\vartheta, \eta_1, \eta_2, \dots), \alpha_i \in (0, 1), \\ \vartheta_{\beta_j} = \frac{e^{-\tau}}{b(t)} b_j(t) \vartheta_j + \theta_j(\vartheta, \vartheta_1, \vartheta_2, \dots), \beta_j \in (0, 1), \end{cases} \quad (53)$$

with the initial condition $\vartheta_0 = \vartheta_i^0$, where $t \in J$, $i, j = 1, 2, \dots$, and τ is the positive real number. J is any fixed interval on the real line. Let $\Lambda = \mathbf{c}$ be the space of all real sequences whose limit is finite.

$$\|\eta\| = (\sup |\eta_i|) e^{-\tau}. \quad (54)$$

Let $(\mathbf{c}, \mathbb{R})$ be endowed with the b -metric:

$$d(\eta, \vartheta) = \sup \|\eta_i - \vartheta_i\| e^{-\tau}. \quad (55)$$

Theorem 19. Assume the assumptions given below hold.

- (A₁) $\vartheta_0 = \vartheta_i^0 \in \mathbf{c}$, $\theta = (\theta_1, \theta_2, \dots) : J \times \mathbf{c} \longrightarrow \mathbf{c}$.
 (A₂) $b_i(t)$ represents the continuous function on J .
 (A₃)

$$\begin{aligned} & |\theta_i(t, \eta) - \theta_i(t, \vartheta)| \\ & \leq \frac{e^{-\tau}}{2s} \left\{ \frac{(d(\eta, \theta_j \eta) d(\eta, \theta_j \vartheta) + d(\vartheta, \theta_j \vartheta) d(\vartheta, \theta_j \eta))}{\eta \{d(\eta, \theta_j \vartheta), d(\theta_j \eta, \vartheta)\}} \right\}^{1/2}, \end{aligned} \quad (56)$$

where $\eta = (\eta_1, \eta_2, \eta_3, \dots)$ and $\vartheta = (\vartheta_1, \vartheta_2, \vartheta_3, \dots)$.
 (A₄)

$$\begin{aligned} & |b_i(t)\eta_i - b_j(t)\vartheta_i| \\ & \leq \frac{b(t)}{2s} \left\{ \frac{(d(\eta, \theta_j \eta) d(\eta, \theta_j \vartheta) + d(\vartheta, \theta_j \vartheta) d(\vartheta, \theta_j \eta))}{\eta \{d(\eta, \theta_j \vartheta), d(\theta_j \eta, \vartheta)\}} \right\}^{1/2}, \end{aligned} \quad (57)$$

then, the infinite systems of fractional order differential (53) have a common solution in $\mathbf{c}$.

Proof. Define $h_i(t, \eta), h_j(t, \eta) : J \times \mathbf{c} \longrightarrow \mathbf{c}$ by

$$\begin{cases} h_i(t, \eta) = \frac{e^{-\tau}}{b(t)} b_i(t)\eta_i + \theta_i(\eta, \eta_1, \eta_2, \dots), \\ h_j(t, \vartheta) = \frac{e^{-\tau}}{b(t)} b_j(t)\eta_j + \theta_j(\vartheta, \vartheta_1, \vartheta_2, \dots). \end{cases} \quad (58)$$

We have

$$\begin{aligned} & |h_i(t, \eta) - h_i(t, \vartheta)| \\ & = \left| \frac{e^{-\tau}}{b(t)} b_i(t)\eta_i + \theta_i(\eta, \eta_1, \eta_2, \dots) \right. \\ & \quad \left. - \frac{e^{-\tau}}{b(t)} b_j(t)\vartheta_i - \theta_i(\vartheta, \vartheta_1, \vartheta_2, \dots) \right|, \\ & \leq \left| \frac{e^{-\tau}}{b(t)} b_i(t)\eta_i - \frac{e^{-\tau}}{b(t)} b_j(t)\vartheta_j \right| \\ & \quad + |\theta_i(\eta, \eta_1, \eta_2, \dots) - \theta_j(\vartheta, \vartheta_1, \vartheta_2, \dots)|, \\ & \leq \frac{e^{-\tau}}{2s} \left\{ \frac{d(\eta, \theta_j \eta) d(\eta, \theta_j \vartheta) + d(\vartheta, \theta_j \vartheta) d(\vartheta, \theta_j \eta)}{\max \{d(\eta, \theta_j \vartheta), d(\theta_j \eta, \vartheta)\}} \right\}^{1/2} \\ & \quad + \frac{e^{-\tau}}{2s} \left\{ \frac{d(\eta, \theta_j \eta) d(\eta, \theta_j \vartheta) + d(\vartheta, \theta_j \vartheta) d(\vartheta, \theta_j \eta)}{\eta \{d(\eta, \theta_j \vartheta), d(\theta_j \eta, \vartheta)\}} \right\}^{1/2}. \end{aligned} \quad (59)$$

Squaring both sides, we get

$$\begin{aligned} & |h_i(t, \eta) - h_i(t, \vartheta)|^2 e^{-\tau} \\ & \leq \frac{e^{-2\tau}}{s^2} \left\{ \frac{d(\eta, \theta_j \eta) d(\eta, \theta_j \vartheta) + d(\vartheta, \theta_j \vartheta) d(\vartheta, \theta_j \eta)}{\max \{d(\eta, \theta_j \vartheta), d(\theta_j \eta, \vartheta)\}} \right\}. \end{aligned} \quad (60)$$

Taking the supremum, we get

$$\begin{aligned} & s^2 d(h_i(t, \eta) - h_i(t, \vartheta)) \\ & \leq e^{-\tau} \left\{ \frac{d(\eta, \theta_j \eta) d(\eta, \theta_j \vartheta) + d(\vartheta, \theta_j \vartheta) d(\vartheta, \theta_j \eta)}{\max \{d(\eta, \theta_j \vartheta), d(\theta_j \eta, \vartheta)\}} \right\}. \end{aligned} \quad (61)$$

By taking $F(v) = \ln(v)$ from Theorem 12, the infinite systems of fractional order differential equation (53) have a common solution.

5. Application to Functional Equations

In this section, we study the solvability of functional equations using the established fixed point theorem.

Let Λ_1 and Λ_2 be Banach spaces, $\Theta_1 \subset \Lambda_1$, $\Theta_2 \subset \Lambda_2$, and $\mathbb{R}$ be the field of real numbers. Suppose $\Lambda = B(\Theta_1)$ represents the set of all functions defined on Θ_1 which is bounded and real valued. Define $d : \Lambda \times \Lambda \longrightarrow \mathbb{R}^+$ by

$$d(v, \zeta) = |v - \zeta|^2. \quad (62)$$

Then, (Λ, d, s) is called the b -metric space. Consider the system of functional equations given as

$$\begin{cases} p(v) = \sup_{\zeta \in \Theta_2} \{g(v, \zeta) + \Phi_1(v, \zeta, Q(\tau(v, \zeta)))\}, & v \in \Theta_1, \\ q(v) = \sup_{\zeta \in \Theta_2} \{g(v, \zeta) + \Phi_2(v, \zeta, Q(\tau(v, \zeta)))\}, & v \in \Theta_1, \end{cases} \quad (63)$$

Here, $g : \Theta_1 \times \Theta_2 \longrightarrow \mathbb{R}$ and $\Phi_1, \Phi_2 : \Theta_1 \times \Theta_2 \times \mathbb{R} \longrightarrow \mathbb{R}$ are functions which are bounded. Θ_1 and Θ_2 are the state and decision spaces, respectively. $\tau : \Theta_1 \times \Theta_2 \longrightarrow \Theta_1$ shows the transformation of the process, and $p(v)$ and $q(v)$ signify sup return functions with initial state Λ . Let $\Theta, \theta : B(\Theta_1) \longrightarrow B(\Theta_1)$ be defined by

$$\begin{cases} \Theta(h_1(v)) = \sup_{\zeta \in \Theta_2} \{g(v, \zeta) + \Phi_1(v, \zeta, P(\tau(v, \zeta)))\}, & v \in \Theta_1, \\ \theta(h_1(v)) = \sup_{\zeta \in \Theta_2} \{g(v, \zeta) + \Phi_2(v, \zeta, P(\tau(v, \zeta)))\}, & v \in \Theta_1, \end{cases} \quad (64)$$

Theorem 20. Let $\Phi_1, \Phi_2 : \Theta_1 \times \Theta_2 \times \mathbb{R} \longrightarrow \mathbb{R}$ and $g : \Theta_1 \times \Theta_2 \longrightarrow \mathbb{R}$ be continuous and bounded and satisfy the following assumption.

$$|\Phi_1(v, \zeta, h(v)) - \Phi_1(v, \zeta, k(v))| \leq \frac{e^{-\tau}}{\sqrt{s}} \left\{ \sqrt{\frac{d(h, \theta h)d(h, \theta k) + d(k, \theta k)d(k, \theta h)}{\max\{d(h, \theta k), d(\theta h, k)\}}} \right\}, \quad (65)$$

for all $v \in \Theta_1$ and $\zeta \in \Theta_2$. Then, the system of functional equation (63) has a common solution.

Proof. Let δ be an arbitrary positive number, $v \in \Theta_1$, and $h_1, h_2 \in B(\Lambda)$; then, there exist $\zeta_1, \zeta_2 \in \Theta_2$ such that

$$\Theta(h_1(v)) < g(v, \zeta_1) + \Phi_1(v, \zeta_1, h_1(\tau_1)) + \delta, \quad (66)$$

$$\Theta(h_2(v)) < g(v, \zeta_2) + \Phi_1(v, \zeta_2, h_2(\tau_2)) + \delta, \quad (67)$$

$$\Theta(h_1(v)) \geq g(v, \zeta_2) + \Phi_1(v, \zeta_2, h_1(\tau_2)), \quad (68)$$

$$\Theta(h_2(v)) \geq g(v, \zeta_1) + \Phi_1(v, \zeta_1, h_2(\tau_1)). \quad (69)$$

Using equations (66) and (69),

$$\begin{aligned} \Theta(h_1(v)) - \Theta(h_2(v)) &\leq \Phi_1(v, \zeta_1, h_1(\tau(v, \zeta_1))) - \Phi_1(v, \zeta_1, h_2(\tau(v, \zeta_1))) + \delta \\ &\leq |\Phi_1(v, \zeta_1, h_2(\tau(v, \zeta_1))) - \Phi_1(v, \zeta_1, h_1(\tau(v, \zeta_1)))| + \delta \\ &\leq \frac{e^{-\tau}}{\sqrt{s}} \left\{ \sqrt{\frac{d(h, \theta h)d(h, \theta k) + d(k, \theta k)d(k, \theta h)}{\max\{d(h, \theta k), d(\theta h, k)\}}} \right\} + \delta. \end{aligned} \quad (70)$$

It follows that

$$\begin{aligned} \Theta(h_1(v)) - \Theta(h_2(v)) &\leq \frac{e^{-\tau}}{\sqrt{s}} \left\{ \sqrt{\frac{d(h, \theta h)d(h, \theta k) + d(k, \theta k)d(k, \theta h)}{\max\{d(h, \theta k), d(\theta h, k)\}}} \right\} + \delta. \end{aligned} \quad (71)$$

Similarly, using equations (67) and (68), we have

$$\begin{aligned} \Theta(h_2(v)) - \Theta(h_1(v)) &\leq \frac{e^{-\tau}}{\sqrt{s}} \left\{ \sqrt{\frac{d(h, \theta h)d(h, \theta k) + d(k, \theta k)d(k, \theta h)}{\max\{d(h, \theta k), d(\theta h, k)\}}} \right\} + \delta. \end{aligned} \quad (72)$$

Using equations (71) and (72), we get

$$|\Theta(h_1(v)) - \Theta(h_2(v))| \leq \frac{e^{-\tau}}{\sqrt{s}} \left\{ \sqrt{\frac{d(h, \theta h)d(h, \theta k) + d(k, \theta k)d(k, \theta h)}{\max\{d(h, \theta k), d(\theta h, k)\}}} \right\} + \delta, \quad (73)$$

for all $v \in \Theta_1$, and $\delta > 0$ is arbitrary; therefore,

$$\begin{aligned} s(|\Theta(h_1(v)) - \Theta(h_2(v))|^2) &\leq e^{-2\tau} \left(\left\{ \sqrt{\frac{d(h, \theta h)d(h, \theta k) + d(k, \theta k)d(k, \theta h)}{\max\{d(h, \theta k), d(\theta h, k)\}}} \right\} \right)^2. \end{aligned} \quad (74)$$

Taking logarithms, we have

$$\begin{aligned} \ln(s|\Theta(h_1(v)) - \Theta(h_2(v))|^2) &\leq \ln \left(e^{-2\tau} \left\{ \sqrt{\frac{d(h, \theta h)d(h, \theta k) + d(k, \theta k)d(k, \theta h)}{\max\{d(h, \theta k), d(\theta h, k)\}}} \right\} \right)^2. \end{aligned} \quad (75)$$

After simple calculation, we get

$$\begin{aligned} 2\tau + \ln(s|\Theta(h_1(v)) - \Theta(h_2(v))|^2) &\leq \ln \left(\frac{d(h, \theta h)d(h, \theta k) + d(k, \theta k)d(k, \theta h)}{\max\{d(h, \theta k), d(\theta h, k)\}} \right). \end{aligned} \quad (76)$$

By taking $F(v) = \ln(v)$ from Theorem 12, the system of functional equation (63) has a common solution.

Data Availability

No data were used to support the study.

Conflicts of Interest

We have no competing interests.

Authors' Contributions

All the authors contributed equally, and they read and approved the final manuscript for publication.

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