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Research Article

Bilinear Equation of the Nonlinear Partial Differential Equation and Its Application

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The homogeneous balance of undetermined coefficient method is firstly proposed to derive a more general bilinear equation of the nonlinear partial differential equation (NLPDE). By applying perturbation method, subsidiary ordinary differential equation (sub-ODE) method, and compatible condition to bilinear equation, more exact solutions of NLPDE are obtained. The KdV equation, Burgers equation, Boussinesq equation, and Sawada-Kotera equation are chosen to illustrate the validity of our method. We find that the underlying relation among the (G'/G)-expansion method, Hirota's method, and HB method is a bilinear equation. The proposed method is also a standard and computable method, which can be generalized to deal with other types of NLPDE.

1. Introduction

The nonlinear partial differential equation (NLPDE) is known to describe a wide variety of phenomena not only in physics but also in biology, chemistry, and several other fields [1–3]. The investigation of the exact solutions for NLPDE plays an important role on the study of nonlinear physical phenomena [4–9]. In recent years, many powerful methods are used to obtain the exact solutions of NLPDE, for example, the inverse scattering method [7], Bäcklund and Darboux transformation method [8], homotopy perturbation method [9], first integral method [10–12], variational iteration method [13, 14], sub-ODE method [15–17], Jacobi elliptic function method [18], tanh-sech method [19], (G'/G)-expansion method [20, 21], Hirota's method [22–24], and homogeneous balance (HB) method [25–27].

As the direct methods, the (G'/G)-expansion method, Hirota's method, and HB method are very effective for constructing the exact solutions of NLPDE. Exact traveling wave solutions, N-soliton solutions, and solitary wave solutions of some NLPDE are obtained by using the above three methods [20–30]. Fan improved the HB method to investigate the BT, Lax pairs, symmetries, and exact solutions for some NLPDE [31, 32]. He also showed that there are many links among the HB method, Weiss-Tabor-Carnevale method, and Clarkson-Kruskal method.

However, there is no unified direct method which can be used to deal with all types of NLPDE. And also, no literature is available to illustrate the underlying relations among the three direct methods.

In the present paper, by improving some key steps in the HB method [26], we propose a new method, HB of undetermined coefficient method, which can be used to derive the bilinear equation of NLPDE. Based on the bilinear equation, by applying the perturbation method, sub-ODE method, and compatible condition, more exact solutions of NLPDE are obtained. We illustrate the real meaning of balance numbers. We show the underlying relations among the (G'/G)-expansion method, Hirota's method, and HB method.

This paper is organized as follows: the HB of undetermined coefficient method is described in Section 2. In Sections 3 and 4, the KdV equation and Burgers equation are chosen as examples to illustrate the method, respectively. In Section 5, the bilinear equations of Boussinesq equation and Sawada-Kotera equation are derived, respectively. Some brief conclusions are given in Section 6.

2. Description of the HB of Undetermined Coefficient Method

Let us consider a general NLPDE, say, in two variables

$$P(u, u_t, u_x, u_{xx}, u_{xt}, \cdots) = 0, \quad (1)$$

where P is a polynomial function of its arguments and the subscripts denote the partial derivatives. The HB of undetermined coefficient method consists of three steps.

Step 1. Suppose that the solution of Equation (1) is of the form

$$u = a_{mn}(\ln w)_{m,n} + \sum_{\substack{i,j=0 \\ i+j\neq 0,m+n}}^{i=m,j=n} a_{ij}(\ln w)_{i,j} + a_{00}, \quad (2)$$

where $u = u(x,t)$, $w = w(x,t)$, $(\ln w)_{i,j} = \partial^{i+j}(\ln w(x,t))/\partial x^i \partial t^j$, m, n (balance numbers), and $a_{ij}(i = 0, 1, \cdots, m, j = 0, 1, \cdots, n)$ (balance coefficients) are constants to be determined later. By balancing the highest nonlinear terms and the highest order partial derivative terms, balance numbers are obtained. Substituting Equation (2) into Equation (1) and balancing the terms with $(w_x/w)^i(w_t/w)^j$ yield a set of algebraic equations for $a_{ij}(i = 1, \cdots, m, j = 1, \cdots, n)$.

Step 2. Solving the set of algebraic equations and simplifying Equation (1), we can get the bilinear equation of Equation (1) directly or after integrating some times (Generally, integrating times equal to the orders of the lowest partial derivative of Equation (1).) with respect to x, t.

Step 3. Generally, in order to obtain the exact solutions of Equation (1), there are three methods to deal with the bilinear equation of Equation (1):

(i) Applying the perturbation method to the bilinear equation of Equation (1), N-soliton solution of Equation (1) can be obtained.

(ii) By using traveling wave transformations

$$w(x,t) = w(\xi), \quad \xi = x - Vt, \quad (3)$$

the bilinear equation of Equation (1) satisfies the following ODE:

$$w'' + \lambda w' + \mu w = 0, \quad (4)$$

where the prime denotes the derivation with respect to ξ and λ, μ, and V are constants to be determined later.

Substituting Equations (3) and (4) into Equation (1), it is converted into the following equation:

$$l_1 w^2 + l_2 w w' + l_3 w'^2 = 0, \quad (5)$$

where l_1, l_2, and l_3 are polynomial functions of V, λ, and μ.

Setting $l_1 = l_2 = l_3 = 0$ yields a set of algebraic equations for V, λ, and μ. Solving the set of algebraic equations and using the solutions of Equation (4), w can be determined. Substituting w into Equation (2), the exact traveling wave solutions of Equation (1) are obtained.

(iii) By applying the compatible condition $(w_{xt} = w_{tx})$ to the bilinear equation of Equation (1), it is reduced to an ODE. Solving the ODE, more exact solutions of Equation (1) can be obtained.

Next, we choose the KdV equation and Burgers equation to illustrate our method.

3. Application to the KdV Equation

Let us consider the celebrated KdV equation in the form

$$u_t + u u_x + \delta u_{xxx} = 0, \quad (6)$$

where δ is a constant. Suppose that the solution of Equation (6) is of the form

$$u = a_{mn}(\ln w)_{m,n} + \sum_{\substack{i,j=0 \\ i+j\neq 0,m+n}}^{i=m,j=n} a_{ij}(\ln w)_{i,j} + a_{00}, \quad (7)$$

where m, n, and a_{ij} $(i = 0, 1, \cdots, m, j = 0, 1, \cdots, n)$ are constants to be determined later.

Balancing u_{xxx} and uu_x in Equation (6), it is required that $m + 3 = 2m + 1$ and $n = 2n$. Then, Equation (7) can be written as

$$u = a_{20}(\ln w)_{xx} + a_{10}(\ln w)_x + a_{00}. \quad (8)$$

From Equation (8), one can calculate the following derivatives:

$$u = a_{20}\left(\frac{w_{xx}}{w} - \frac{w_x^2}{w^2}\right) + a_{10}\frac{w_x}{w} + a_{00},$$

$$u_x = a_{20}\left(\frac{w_{xxx}}{w} - \frac{3w_{xx}w_x}{w^2} + \frac{2w_x^3}{w^3}\right) + a_{10}\left(\frac{w_{xx}}{w} - \frac{w_x^2}{w^2}\right),$$

$$u_t = a_{20}\left(\frac{w_{xxt}}{w} - \frac{w_{xx}w_t + 2w_x w_{xt}}{w^2} + 2\frac{w_x^2 w_t}{w^3}\right) + a_{10}\left(\frac{w_{xt}}{w} - \frac{w_x w_t}{w^2}\right),$$

$$u_{xxx}=a_{20}\left(\frac{w_{xxxxx}}{w}-\frac{5w_{xxxx}w_x+10w_{xxx}w_{xx}}{w^2}+\frac{20w_{xxx}w_x^2+30w_{xx}^2w_x}{w^3}-\frac{60w_{xx}w_x^3}{w^4}+\frac{24w_x^5}{w^5}\right)+a_{10}\left(\frac{w_{xxxx}}{w}-\frac{4w_{xxx}w_x+3w_{xx}^2}{w^2}+\frac{12w_{xx}w_x^2}{w^3}-\frac{6w_x^4}{w^4}\right). \tag{9}$$

Equating the coefficients of $(w_x/w)^5$ and $(w_x/w)^4$ on the left-hand side of Equation (6) to zero yields a set of algebraic equations for a_{20} and a_{10} as follows:

$$\begin{aligned}&-2a_{20}^2+24\delta a_{20}=0,.\\&3a_{20}a_{10}-6\delta a_{10}=0.\end{aligned} \tag{10}$$

Solving the above algebraic equations, we get $a_{20}=12\delta$ and $a_{10}=0$. Substituting a_{20} and a_{10} back into Equation (8), we get

$$u=12\delta(\ln w)_{xx}+a_{00}. \tag{11}$$

where a_{00} is an arbitrary constant. Substituting Equation (11) into Equation (6), we get

$$12\delta(K_1+K_2+K_3)=0, \tag{12}$$

where

$$\begin{aligned}K_1&=\frac{w_{xxt}}{w}-\frac{2w_xw_{xt}+w_{xx}w_t}{w^2}+\frac{2w_x^2w_t}{w^3},\\K_2&=a_{00}\left(\frac{w_{xxx}}{w}-\frac{3w_{xx}w_x}{w^2}+\frac{2w_x^3}{w^3}\right),\\K_3&=\delta\left(\frac{w_{xxxxx}}{w}+\frac{2w_{xxx}w_{xx}-5w_{xxxx}w_x}{w^2}+\frac{16w_{xxx}w_x^2-6w_xw_{xx}^2}{w^3}\right).\end{aligned} \tag{13}$$

Simplifying Equation (12) and integrating once with respect to x, we get

$$\frac{\partial}{\partial x}\left(\frac{(w_{xt}w-w_xw_t)+\delta\left(w_{xxxx}w-4w_xw_{xxx}+3w_{xx}^2\right)+a_{00}\left(w_{xx}w-w_x^2\right)}{w^2}\right)=0. \tag{14}$$

Equation (14) is identical to

$$\begin{aligned}&(w_{xt}w-w_xw_t)+\delta\left(w_{xxxx}w-4w_xw_{xxx}+3w_{xx}^2\right)\\&\quad+a_{00}\left(w_{xx}w-w_x^2\right)-C(t)w^2=0,\end{aligned} \tag{15}$$

where $C(t)$ is an arbitrary function of t and a_{00} is an arbitrary constant.

In particular, taking $C(t)$ as zero in Equation (15), we get the bilinear equation of Equation (6) as follows:

$$\begin{aligned}&(w_{xt}w-w_xw_t)+\delta\left(w_{xxxx}w-4w_xw_{xxx}+3w_{xx}^2\right)\\&\quad+a_{00}\left(w_{xx}w-w_x^2\right)=0.\end{aligned} \tag{16}$$

Equation (16) can be written concisely in terms of D-operators as

$$\left(D_xD_t+\delta D_x^4+a_{00}D_x^2\right)w\cdot w=0, \tag{17}$$

where

$$D_x^mD_t^na\cdot b=(\partial_x-\partial_{x'})^m(\partial_t-\partial_{t'})^na(x,t)b\left(x',t'\right)\Big|_{x'=x,t'=t}. \tag{18}$$

Remark 1. Applying Hirota's method [22–24], the bilinear equation of Equation (6) can be written as

$$\left(D_xD_t+\delta D_x^4\right)w\cdot w=0. \tag{19}$$

Equation (19) is obtained by setting $a_{00}=0$ in Equation (17). Obviously, Equation (19) is a special case of Equation (17).

(i) Now, we apply the perturbation method to Equation (17) to derive N-soliton solution of Equation (6). Suppose that w can be expanded as follows:

$$w=1+\varepsilon w_1+\varepsilon^2w_2+\cdots\varepsilon^Nw_N+\cdots, \tag{20}$$

where ε is a parameter and $w_i=w_i(x,t)\ (i=1,2,\cdots)$.

Substituting Equation (20) into Equation (17) and arranging it at each order of ε, we get

$$\begin{aligned}&\varepsilon: D_x\left(D_t+a_{00}D_x+\delta D_x^3\right)(w_1\cdot 1+1\cdot w_1)=0,\\&\varepsilon^2: D_x\left(D_t+a_{00}D_x+\delta D_x^3\right)(w_2\cdot 1+w_1\cdot w_1+1\cdot w_2)=0,\\&\varepsilon^3: D_x\left(D_t+a_{00}D_x+\delta D_x^3\right)(w_3\cdot 1+w_2\cdot w_1+w_1\cdot w_2+1\cdot w_3)=0,\\&\cdots\end{aligned} \tag{21}$$

The order-ε equation can be rewritten as a linear differential equation for w_1 as follows:

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}+a_{00}\frac{\partial}{\partial x}+\delta\frac{\partial^3}{\partial x^3}\right)w_1=0. \tag{22}$$

Solving Equation (22), we get

$$w_1=e^{P_1x-\left(a_{00}P_1+\delta P_1^3\right)t}, \tag{23}$$

where P_1 is an arbitrary constant.

The coefficient of ε^2 can be rearranged as follows:

$$\begin{aligned}&2\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}+a_{00}\frac{\partial}{\partial x}+\delta\frac{\partial^3}{\partial x^3}\right)w_2\\&\quad=-D_x\left(D_t+a_{00}D_x+\delta D_x^3\right)w_1\cdot w_1.\end{aligned} \tag{24}$$

Substituting Equation (23) into Equation (24), the right-hand side of Equation (24) equals zero. Therefore, we can choose

$$w_2=0. \tag{25}$$

Substituting Equations (23) and (25) into Equation (20), we get

$$w=1+e^{P_1x-\left(a_{00}P_1+\delta P_1^3\right)t+\xi_1^0}, \tag{26}$$

where P_1, a_{00}, and ξ_1^0 are arbitrary constants.

Substituting Equation (26) into Equation (11), 1-soliton solution of Equation (6) can be obtained. If we choose $w_1=e^{P_1x-(a_{00}P_1+\delta P_1^3)t}+e^{P_2x-(a_{00}P_2+\delta P_2^3)t}$ in Equation (22), similar to the process of obtaining 1-soliton solution, we can get 2-soliton solution of Equation (6) as follows:

$$w=1+e^{\eta_1}+e^{\eta_2}+\frac{\left(P_1-P_2\right)^2}{\left(P_1-P_2\right)^2}e^{\eta_1+\eta_2}, \tag{27}$$

where $\eta_1=P_1x-(a_{00}P_1+\delta P_1^3)t+\xi_1^0$, $\eta_2=P_2x-(a_{00}P_2+\delta P_2^3)t+\xi_2^0$, $P_i,\xi_i^0\ (i=1,2)$, and a_{00} are arbitrary constants.

Substituting Equation (27) into Equation (11), 2-soliton solution of Equation (6) can be obtained. Similarly, we can get N-soliton solution of Equation (6).

Remark 2. Obviously, setting $a_{00}=0$ in Equations (23) and (27), 1-soliton and 2-soliton solutions of Equation (6) are identical to Hirota's results [22–24].

Remark 3. By using the properties of D-operators [22–24], a Bäcklund transformation of Equation (17) can be obtained as follows:

$$\begin{aligned}\left(D_t+(a_{00}+\alpha)D_x+\delta D_x^3\right)w^*\cdot w&=0,\\\left(D_x^2-\beta D_x-\alpha\right)w^*\cdot w&=0,\end{aligned} \tag{28}$$

where w^* and w satisfy Equation (17) and α, β, and a_{00} are arbitrary constants.

(ii) Now, we discuss Equation (16) by using the sub-ODE method.

Using transformations $w(x,t)=w(\xi)$, $\xi=x-Vt$, Equation (16) is reduced to

$$\left(a_{00}-V\right)\left(w''w-w'^2\right)+\delta\left(w''''w-4w'w'''+3w''^2\right)=0, \tag{29}$$

where the prime denotes the derivation with respect to ξ and V is a constant to be determined later.

Noticing the bilinear property of Equation (16), suppose that w satisfies the following ODE:

$$w''+\lambda w'+\mu w=0, \tag{30}$$

where λ and μ are parameters.

Substituting Equation (30) into Equation (16), we get

$$l_1w^2+l_2ww'+l_3w'^2=0, \tag{31}$$

where

$$\begin{aligned}l_1&=\mu\left(V-a_{00}+\delta\left(4\mu-\lambda^2\right)\right),\\l_2&=\lambda\left(V-a_{00}+\delta\left(4\mu-\lambda^2\right)\right),\\l_3&=V-a_{00}+\delta\left(4\mu-\lambda^2\right).\end{aligned} \tag{32}$$

Setting $l_1=l_2=l_3=0$ yields a set of algebraic equations for V, λ, and μ. Solving this set of algebraic equations, we get

$$V=a_{00}+\delta\left(\lambda^2-4\mu\right), \tag{33}$$

where λ, μ, and a_{00} are arbitrary constants.

Substituting Equation (30) into Equation (11), we get

$$u=12\delta\left(\frac{w_x}{w}+\frac{\lambda}{2}\right)^2+3\delta\left(\lambda^2-4\mu\right)+a_{00}. \tag{34}$$

Substituting the general solutions of Equation (30) into Equation (34), we get three types of traveling wave solutions of Equation (6) as follows.

When $\lambda^2-4\mu>0$,

$$\begin{aligned}u_1(x,t)=u_1(\xi)=&-3\delta A\left(\frac{C_1e^{(\sqrt{A}/2)\xi}-C_2e^{-(\sqrt{A}/2)\xi}}{C_1e^{(\sqrt{A}/2)\xi}-C_2e^{-(\sqrt{A}/2)\xi}}\right)^2\\&+3\delta A+a_{00},\end{aligned} \tag{35}$$

where

$$A = \lambda^2 - 4\mu,$$
$$V = a_{00} + A\delta, \quad (36)$$
$$\xi = x - (a_{00} + A\delta)t,$$

and λ, μ, C_1, C_2, and a_{00} are arbitrary constants.

Taking $C_1 = C_3 + C_4/2$ and $C_2 = C_3 - C_4/2$, Equation (35) can be rewritten as

$$u_2(x,t) = u_2(\xi) = -3\delta A \left(\frac{C_3 \sinh\left(\sqrt{A}/2\right)\xi + C_4 \cosh\left(\sqrt{A}/2\right)\xi}{C_3 \cosh\left(\sqrt{A}/2\right)\xi + C_4 \sinh\left(\sqrt{A}/2\right)\xi} \right)^2 + 3\delta A + a_{00}, \quad (37)$$

where C_3, C_4, and a_{00} are arbitrary constants and A, V, and ξ are given by Equation (36).

In particular, if $|C_4/C_3| < 1$, then Equation (37) is reduced to

$$u_3(x,t) = u_3(\xi) = 3\delta A \operatorname{sech}^2\left(\frac{\sqrt{A}}{2}\xi + \xi_0\right) + a_{00}, \quad (38)$$

where C_3, C_4, and a_{00} are arbitrary constants and A, V, and ξ are given by Equation (36), $\xi_0 = \operatorname{arctanh}(C_4/C_3)$.

When $\lambda^2 - 4\mu < 0$,

$$u_4(x,t) = u_4(\xi) = 3\delta A \left(\frac{-C_1 \sin\left(\sqrt{-A}/2\right)\xi + C_2 \cos\left(\sqrt{-A}/2\right)\xi}{C_1 \cos\left(\sqrt{-A}/2\right)\xi + C_2 \sin\left(\sqrt{-A}/2\right)\xi} \right)^2 + 3\delta A + a_{00}, \quad (39)$$

where C_1, C_2, and a_{00} are arbitrary constants and A, V, and ξ are given by Equation (36).

Obviously, Equation (39) can be written as

$$u_5(x,t) = u_5(\xi) = 3\delta A \sec^2\left(\frac{\sqrt{-A}}{2}\xi + \xi_0\right) + a_{00}, \quad (40)$$

where C_1, C_2, and a_{00} are arbitrary constants and A, V, and ξ are given by Equation (36), $\xi_0 = -\arctan(C_2/C_1)$.

When $\lambda^2 - 4\mu = 0$,

$$u_6(x,t) = u_6(\xi) = -12\delta \left(\frac{C_2}{C_1 + C_2\xi}\right)^2 + a_{00}, \quad (41)$$

where $V = a_{00}$, $\xi = x - a_{00}t$, C_1, C_2, and a_{00} are arbitrary constants.

(iii) Now, we discuss Equation (16) from the compatible condition. Equation (16) can be written as

$$(\delta w_{xxxx} + w_{xt} + a_{00}w_{xx})w + \left(\delta\left(3w_{xx}^2 - 4w_{xxx}w_x\right) - a_{00}w_x^2 - w_x w_t\right) = 0. \quad (42)$$

Notice $w_x \neq 0$; otherwise, we can only get a trivial solution. Setting the second term of Equation (42) to zero and solving w_t yield

$$w_t = \frac{\delta\left(3w_{xx}^2 - 4w_{xxx}w_x\right) - a_{00}w_x^2}{w_x}. \quad (43)$$

Substituting Equation (43) into Equation (42), we get

$$w_{xxxx} + \frac{w_{xx}^3 - 2w_x w_{xx} w_{xxx}}{w_x^2} = 0. \quad (44)$$

Integrating Equation (44) once with respect to x, we get

$$w_{xxx} - \frac{w_{xx}^2}{w_x} = b(t), \quad (45)$$

where $b(t)$ is an arbitrary function of t.

Using transformations $Y = w_{xx}^2$ and $X = w_x$, Equation (45) is reduced to

$$\frac{dY}{dX} = \frac{2Y}{X} + 2b(t). \quad (46)$$

Solving the above equation, we get

$$Y = c(t)X^2 - 2b(t)X, \quad (47)$$

namely,

$$w_{xx} = \sqrt{c(t)w_x^2 - 2b(t)w_x}, \quad (48)$$

where $b(t)$ and $c(t)$ are arbitrary functions of t.

Case 1. When $b(t) = c(t) = 0$, from Equation (48), we get

$$w = c_1(t)x + c_2(t), \quad (49)$$

where $c_1(t)$ and $c_2(t)$ are arbitrary functions of t.

Substituting the above equation into Equation (48), we get

$$x\frac{dc_1(t)}{dt} + \frac{dc_2(t)}{dt} = -a_{00}c_1(t). \quad (50)$$

Setting the coefficients of $x^i (i = 0, 1)$ to zero in the above equation, we get

$$\frac{dc_1(t)}{dt} = 0,$$
$$\frac{dc_2(t)}{dt} = -a_{00}c_1(t). \tag{51}$$

Solving the above equations, we get

$$c_1(t) = C_1,$$
$$c_2(t) = C_2 - a_{00}C_1 t, \tag{52}$$

where C_i $(i = 1, 2)$ are arbitrary constants. Then, we get

$$w = C_1(x - a_{00}t) + C_2. \tag{53}$$

Substituting Equation (53) into Equation (11), we get an exact solution of Equation (6) as follows:

$$u_7(x, t) = a_{00} - \frac{12\delta C_1^2}{(C_1(x - a_{00}t) + C_2)^2}, \tag{54}$$

where C_i $(i = 1, 2)$ and a_{00} are arbitrary constants.

Case 2. When $b(t) = 0$ and $c(t) > 0$, similar to Case 1, we get

$$w = C_3 + C_2 e^{C_1\left(x - \left(a_{00} + \delta C_1{}^2\right)t\right)}, \tag{55}$$

and an exact solution of Equation (6) as follows:

$$u_8(x, t) = \frac{12\delta C_1^2 C_2 C_3 e^{C_1\left(x-\left(a_{00}+\delta C_1^2\right)t\right)}}{\left(C_3 + C_2 e^{C_1\left(x-\left(a_{00}+\delta C_1^2\right)t\right)}\right)^2} + a_{00}, \tag{56}$$

where C_i $(i = 1, 2, 3)$ and a_{00} are arbitrary constants.

Case 3. When $b(t) \neq 0$ and $c(t) = 0$ similar to Case 1, we get

$$w = C_1(x - a_{00}t + C_2)^3 + 12\delta C_1 t + C_3, \tag{57}$$

and an exact solution of Equation (6) as follows:

$$u_9(x, t) = a_{00} - \frac{36\delta C_1(x - a_{00}t + C_2)\left(C_1(x - a_{00}t + C_2)^3 - 24\delta C_1 t - 2C_3\right)}{\left(C_1(x - a_{00}t + C_2)^3 - 12\delta C_1 t + C_3\right)^2}, \tag{58}$$

where C_i $(i = 1, 2, 3)$ and a_{00} are arbitrary constants.

Case 4. When $b(t) \neq 0$ and $c(t) > 0$, similar to Case 1, we get

$$w = C_4 e^{C_1\left(x-\left(a_{00}+\delta C_1^2\right)t\right)} + C_3 e^{-C_1\left(x-\left(a_{00}+\delta C_1^2\right)t\right)} + C_2 x - C_2\left(a_{00} + 3\delta C_1^2\right)t + C_5, \tag{59}$$

and an exact solution of Equation (6) as follows:

$$u_{10}(x, t) = 12\delta(A_1 - B_1) + a_{00}, \tag{60}$$

where

$$A_1 = \frac{C_1^2 C_4 e^{C_1\left(x-\left(a_{00}+\delta C_1^2\right)t\right)} + C_1^2 C_3 e^{-C_1\left(x-\left(a_{00}+\delta C_1^2\right)t\right)}}{C_4 e^{C_1\left(x-\left(a_{00}+\delta C_1^2\right)t\right)} + C_3 e^{-C_1\left(x-\left(a_{00}+\delta C_1^2\right)t\right)} + C_2 x - C_2\left(a_{00} + 3\delta C_1^2\right)t + C_5},$$
$$B_1 = \frac{\left(C_1 C_4 e^{C_1\left(x-\left(a_{00}+\delta C_1^2\right)t\right)} - C_1 C_3 e^{-C_1\left(x-\left(a_{00}+\delta C_1^2\right)t\right)} + C_2\right)^2}{\left(C_4 e^{C_1\left(x-\left(a_{00}+\delta C_1^2\right)t\right)} + C_3 e^{-C_1\left(x-\left(a_{00}+\delta C_1^2\right)t\right)} + C_2 x - C_2\left(a_{00} + 3\delta C_1^2\right)t + C_5\right)^2}, \tag{61}$$

where $C_2^2 + 4C_1^2 C_3 C_4 = 0$, C_i $(i = 1, 2, 3, 4, 5)$, and a_{00} are arbitrary constants.

Case 5. When $b(t) \neq 0$ and $c(t) < 0$, similar to Case 1, we get

$$w = -\frac{C_1 C_2 x + C_1 \cos\left(C_2\left(x + \delta C_2^2 t - a_{00}t + C_3\right)\right) + 3\delta C_1 C_2^3 t - a_{00}C_1 C_2 t - C_4 C_2^3}{C_2^3}, \tag{62}$$

and an exact solution of Equation (6) as follows:

$$u_{11}(x,t) = 12\delta(A_2 - B_2) + a_{00}, \tag{63}$$

where

$$\begin{aligned} A_2 &= \frac{-C_1C_2^2 \cos\left(C_2\left(x+\delta C_2^2 t - a_{00}t + C_3\right)\right)}{C_1C_2x + C_1 \cos\left(C_2\left(x+\delta C_2^2 t - a_{00}t + C_3\right)\right) + 3\delta C_1 C_2^3 t - C_1C_2a_{00}t - C_2^3C_4}, \\ B_2 &= \frac{C_1^2C_2^2\left(-1 + \sin\left(C_2\left(x+\delta C_2^2 t - a_{00}t + C_3\right)\right)\right)^2}{\left(C_1C_2x + C_1 \cos\left(C_2\left(x+\delta C_2^2 t - a_{00}t + C_3\right)\right) + 3\delta C_1 C_2^3 t - a_{00}C_1C_2t - C_2^3C_4\right)^2}, \end{aligned} \tag{64}$$

where C_i $(i = 1, 2, 3, 4)$ and a_{00} are arbitrary constants.

So far, based on the bilinear equation which is derived by using the HB of undetermined coefficient method, many exact solutions of the KdV are obtained by applying the perturbation method, sub-ODE method, and compatible condition. Our results can compare with the (G'/G)-expansion method, Hirota's method, and HB method [20–30].

4. Application to the Burgers Equation

Let us consider the Burgers equation in the form

$$u_t + uu_x + \delta u_{xx} = 0, \tag{65}$$

where δ is a constant.

Suppose that the solution of Equation (65) is of the form

$$u = a_{mn}(\ln w)_{m,n} + \sum_{\substack{i,j=0 \\ i+j\neq 0,m+n}}^{i=m,j=n} a_{ij}(\ln w)_{i,j} + a_{00}, \tag{66}$$

where m, n, and a_{ij} $(i = 0, 1, \cdots, m, j = 0, 1, \cdots, n)$ are constants to be determined later.

Balancing u_{xx} and uu_x in Equation (65), it is required that $m + 2 = 2m + 1$ and $n = 2n$. Then, Equation (66) can be written as

$$u = a_{10}(\ln w)_x + a_{00}. \tag{67}$$

From Equation (67), one can calculate the following derivatives:

$$\begin{aligned} u &= a_{10}\frac{w_x}{w} + a_{00}, \\ u_x &= a_{10}\left(\frac{w_{xx}}{w} - \frac{w_x^2}{w^2}\right), \\ u_t &= a_{10}\left(\frac{w_{xt}}{w} - \frac{w_x w_t}{w^2}\right), \\ u_{xx} &= a_{10}\left(\frac{w_{xxx}}{w} - \frac{3w_{xx}w_x}{w^2} + \frac{2w_x^3}{w^3}\right). \end{aligned} \tag{68}$$

Equating the coefficients of $(w_x/w)^3$ on the left-hand side of Equation (65) to zero yields an algebraic equation for a_{10} as follows:

$$-a_{10}^2 + 2a_{10}\delta = 0. \tag{69}$$

Solving the above algebraic equation, we get $a_{10} = 2\delta$. Substituting a_{10} back into Equation (67), we get

$$u = 2\delta(\ln w)_x + a_{00}, \tag{70}$$

where a_{00} is an arbitrary constant.

Substituting Equation (70) into Equation (65), we get

$$2\delta(K_1 + K_2 + K_3) = 0, \tag{71}$$

where

$$\begin{aligned} K_1 &= \frac{w_{xt}w - w_x w_t}{w^2}, \\ K_2 &= a_{00}\left(\frac{w_{xx}w - w_x^2}{w^2}\right), \\ K_3 &= \delta\left(\frac{ww_{xxx} - w_x w_{xx}}{w^2}\right). \end{aligned} \tag{72}$$

Simplifying Equation (71), we get

$$(w_{xt}w - w_x w_t) + a_{00}\left(w_{xx}w - w_x^2\right) + \delta(ww_{xxx} - w_x w_{xx}) = 0. \tag{73}$$

Equation (73) can be written concisely in terms of D-operators as

$$D_x(w_t + \delta w_{xx} + a_{00}w_x) \cdot w = 0. \tag{74}$$

Equation (74) is identical to

$$w_t + \delta w_{xx} + a_{00}w_x - C(t)w = 0, \tag{75}$$

where $C(t)$ is an arbitrary function of t and a_{00} is an arbitrary constant.

In particular, taking $C(t)$ as constant C in Equation (75), we get

$$w_t + \delta w_{xx} + a_{00} w_x - Cw = 0. \tag{76}$$

Remark 4. Applying Hirota's method [22–24], the bilinear equation of Equation (74) can be written as

$$D_x(w_t + \delta w_{xx}) \cdot w = 0. \tag{77}$$

Equation (77) is obtained by setting $a_{00} = 0$ in Equation (74). Obviously, Equation (77) is a special case of Equation (74).

Remark 5. Equations (70) and (76) are general Cole-Hopf transformations. In fact, setting $a_{00} = C = 0$ in Equations (70) and (76), we get the famous Cole-Hopf transformations

$$u = 2\delta(\ln w)_x, w_t + \delta w_{xx} = 0. \tag{78}$$

(i) Now, we apply the perturbation method to Equation (74) to derive N-soliton solution of Equation (65). Suppose that w can be expanded as follows:

$$w = 1 + \varepsilon w_1 + \varepsilon^2 w_2 + \cdots \varepsilon^N w_N + \cdots, \tag{79}$$

where ε is a parameter and $w_i = w_i(x, t)$ $(i = 1, 2, \cdots)$.

Substituting Equation (79) into Equation (74) and arranging it at each order of ε, we get

$$\begin{aligned}
&\varepsilon : \frac{1}{2}\left(D_{xt} + a_{00}D_x^2\right)(w_1 \cdot 1 + 1 \cdot w_1) + \delta D_x w_{1xx} \cdot 1 = 0,\\
&\varepsilon^2 : \frac{1}{2}\left(D_{xt} + a_{00}D_x^2\right)(w_2 \cdot 1 + w_1 \cdot w_1 + 1 \cdot w_2)\\
&\quad + \delta D_x(w_{1xx} \cdot w_1 + w_{2xx} \cdot 1) = 0,\\
&\varepsilon^3 : \frac{1}{2}\left(D_{xt} + a_{00}D_x^2\right)(w_3 \cdot 1 + w_2 \cdot w_1 + w_1 \cdot w_2 + 1 \cdot w_3)\\
&\quad + \delta D_x(w_{1xx} \cdot w_2 + w_{2xx} \cdot w_1 + w_{3xx} \cdot 1) = 0,\\
&\cdots
\end{aligned} \tag{80}$$

The order-ε equation can be rewritten as a linear differential equation for w_1 as follows:

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} + a_{00}\frac{\partial}{\partial x} + \delta\frac{\partial^2}{\partial x^2}\right)w_1 = 0. \tag{81}$$

Solving Equation (81), we get

$$w_1 = e^{P_1 x - \left(a_{00}P_1 + \delta P_1^2\right)t}, \tag{82}$$

where P_1 is an arbitrary constant.

The coefficient of ε^2 can be rearranged as follows:

$$\begin{aligned}
&\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} + a_{00}\frac{\partial}{\partial x} + \delta\frac{\partial^2}{\partial x^2}\right)w_2\\
&\quad = -\frac{1}{2}\left(D_{xt} + a_{00}D_x^2\right)w_1 \cdot w_1 - \delta D_x w_{1xx} \cdot w_1.
\end{aligned} \tag{83}$$

Substituting Equation (82) into Equation (83), the right-hand side of Equation (83) equals zero. Therefore, we can choose

$$w_2 = 0. \tag{84}$$

Substituting Equations (82) and (84) into Equation (79), we get

$$w = 1 + e^{P_1 x - \left(a_{00}P_1 + \delta P_1{}^2\right)t + \xi_1^0}, \tag{85}$$

where P_1, a_{00}, and $\xi_1{}^0$ are arbitrary constants.

Substituting Equation (85) into Equation (70), 1-soliton solution of Equation (65) can be obtained.

If we choose $w_1 = e^{P_1 x - (a_{00}P_1 + \delta P_1^2)t} + e^{P_2 x - (a_{00}P_2 + \delta P_2^2)t}$ in Equation (81), Equation (74) has no 2-soliton solution. But there exists a solution as follows:

$$w = 1 + \sum_{i=1}^{n} e^{\eta_1}, \tag{86}$$

where $\eta_i = P_i x - (a_{00}P_i + \delta P_i^2)t + \xi_1^0$, P_i, ξ_1^0 $(i = 1, 2, \cdots, n)$, and a_{00} are arbitrary constants.

(ii) Now, we discuss Equation (73) by using the sub-ODE method.

Using transformations $w(x, t) = w(\xi)$, $\xi = x - Vt$, Equation (73) is reduced to

$$(a_{00} - V)\left(w'' w - w'^2\right) + \delta\left(w w''' - w' w''\right) = 0, \tag{87}$$

where the prime denotes the derivation with respect to ξ and V is a constant to be determined later.

Noticing the bilinear property of Equation (87), suppose that w satisfies the following ODE:

$$w'' + \lambda w' + \mu w = 0, \tag{88}$$

where λ and μ are parameters.

Substituting Equation (88) into Equation (87), we get

$$l_1 w^2 + l_2 w w' + l_3 w'^2 = 0, \tag{89}$$

where

$$\begin{aligned} l_1 &= \mu(\delta\lambda + V - a_{00}), \\ l_2 &= \lambda(\delta\lambda + V - a_{00}), \\ l_3 &= \delta\lambda + V - a_{00}. \end{aligned} \tag{90}$$

Setting $l_1 = l_2 = l_3 = 0$ yields a set of algebraic equations for V, λ, and μ. Solving the set of algebraic equations, we get

$$V = a_{00} - \delta\lambda, \tag{91}$$

where λ, μ, and a_{00} are arbitrary constants.

Substituting the general solutions of Equation (88) into Equation (70), we get three types of traveling wave solutions of Equation (65) as follows.

When $\lambda^2 - 4\mu > 0$,

$$u_1(x,t) = u_1(\xi) = \delta\sqrt{A}\left(\frac{C_1 e^{(\sqrt{A}/2)\xi} - C_2 e^{-(\sqrt{A}/2)\xi}}{C_1 e^{(\sqrt{A}/2)\xi} + C_2 e^{-(\sqrt{A}/2)\xi}}\right) + a_{00} - \delta\lambda, \tag{92}$$

where

$$\begin{aligned} A &= \lambda^2 - 4\mu, \\ V &= a_{00} - \delta\lambda, \\ \xi &= x - (a_{00} - \delta\lambda)t, \end{aligned} \tag{93}$$

where C_1, C_2, λ, μ, and a_{00} are arbitrary constants.

Taking $C_1 = (C_3 + C_4)/2$ and $C_2 = (C_3 - C_4)/2$, Equation (93) can be written as

$$\begin{aligned} u_2(x,t) &= u_2(\xi) = \delta\sqrt{A} \\ &\cdot \left(\frac{C_3 \sinh\left(\sqrt{A}/2\right)\xi + C_4 \cosh\left(\sqrt{A}/2\right)\xi}{C_3 \cosh\left(\sqrt{A}/2\right)\xi + C_4 \sinh\left(\sqrt{A}/2\right)\xi}\right) \\ &+ a_{00} - \delta\lambda, \end{aligned} \tag{94}$$

where C_3, C_4, and a_{00} are arbitrary constants and A and ξ are given by Equation (94).

In particular, if $|C_4/C_3| < 1$, then Equation (94) is reduced to

$$u_3(x,t) = u_3(\xi) = \delta\sqrt{A}\tanh\left(\frac{\sqrt{A}}{2}\xi + \xi_0\right) + a_{00} - \delta\lambda, \tag{95}$$

where C_3, C_4, and a_{00} are arbitrary constants and A and ξ are given by Equation (94), $\xi_0 = \operatorname{arctanh}(C_4/C_3)$.

When $\lambda^2 - 4\mu < 0$,

$$\begin{aligned} u_4(x,t) &= u_4(\xi) = \delta\sqrt{-A} \\ &\cdot \left(\frac{-C_3 \sin\left(\sqrt{-A}/2\right)\xi + C_4 \cos\left(\sqrt{-A}/2\right)\xi}{C_3 \cos\left(\sqrt{-A}/2\right)\xi + C_4 \sin\left(\sqrt{-A}/2\right)\xi}\right) \\ &+ a_{00} - \delta\lambda, \end{aligned} \tag{96}$$

where C_3, C_4, and a_{00} are arbitrary constants and A and ξ are given by Equation (94).

Obviously, Equation (96) can be written as

$$u_5(x,t) = u_5(\xi) = -\delta\sqrt{-A}\tan\left(\frac{\sqrt{-A}}{2}\xi - \xi_0\right) + a_{00} - \delta\lambda, \tag{97}$$

where λ, μ, and a_{00} are arbitrary constants and A and ξ are given by Equation (94), $\xi_0 = -\arctan(C_4/C_3)$.

When $\lambda^2 - 4\mu = 0$,

$$u_6(x,t) = u_6(\xi) = \frac{2\delta C^2}{C_1 + C_2\xi} + a_{00} - \delta\lambda, \tag{98}$$

where $\xi = x - (a_{00} - \delta\lambda)t$, C_1, C_2, λ, μ, and a_{00} are arbitrary constants.

(iii) Now, we discuss Equation (73) from the compatible condition.

Using the compatible condition, we can get nothing but Equation (76). Using transformations $w(x,t) = w(\xi)$, $\xi = x - Vt$, Equation (76) is reduced to

$$\delta w'' + (a_{00} - V)w' - Cw = 0. \tag{99}$$

Substituting the general solutions of Equation (99) into Equation (70), we get three types of traveling wave solutions of Equation (65) as follows.

When $(a_{00} - V_1)^2 + 4C\delta > 0$,

$$u_7(x,t) = u_7(\xi) = \sqrt{\Delta}\left(\frac{C_1 e^{(\sqrt{\Delta}/2\delta)\xi} - C_2 e^{-(\sqrt{\Delta}/2\delta)\xi}}{C_1 e^{(\sqrt{\Delta}/2\delta)\xi} + C_2 e^{-(\sqrt{\Delta}/2\delta)\xi}}\right) + V_1, \tag{100}$$

where

$$\begin{aligned} \Delta &= (a_{00} - V_1)^2 + 4C\delta, \\ \xi &= x - V_1 t, \end{aligned} \tag{101}$$

where V_1, C, C_1, C_2, and a_{00} are arbitrary constants.

Taking $C_1 = (C_3 + C_4)/2$ and $C_2 = (C_3 + C_4)/2$, Equation (101) can be written as

$$u_8(x,t) = u_8(\xi) = \sqrt{\Delta} \cdot \left(\frac{C_3 \sinh\left(\sqrt{\Delta}/2\delta\right)\xi + C_4 \cosh\left(\sqrt{\Delta}/2\delta\right)\xi}{C_3 \cosh\left(\sqrt{\Delta}/2\delta\right)\xi + C_4 \sinh\left(\sqrt{\Delta}/2\delta\right)\xi} \right) + V_1, \tag{102}$$

where V_1, C, C_3, C_4, and a_{00} are arbitrary constants and Δ and ξ are given by Equation (101).

In particular, if $C_4/C_3 < 1$, then Equation (102) is reduced to

$$u_9(x,t) = u_9(\xi) = \sqrt{\Delta} \tanh\left(\frac{\sqrt{\Delta}}{2\delta}\xi + \xi_0 \right) + V_1, \tag{103}$$

where $\xi_0 = \operatorname{arctanh}(C_4/C_3)$, V_1, C, C_3, C_4, and a_{00} are arbitrary constants and Δ and ξ are given by Equation (101).

When $(a_{00} - V_2)^2 + 4C\delta < 0$,

$$u_{10}(x,t) = u_{10}(\xi) = \sqrt{-\Delta} \cdot \left(\frac{-C_3 \sin\left(\sqrt{-\Delta}/2\delta\right)\xi + C_4 \cos\left(\sqrt{-\Delta}/2\delta\right)\xi}{C_3 \cos\left(\sqrt{-\Delta}/2\delta\right)\xi + C_4 \sin\left(\sqrt{-\Delta}/2\delta\right)\xi} \right) + V_2, \tag{104}$$

where V_2, C, C_3, C_4, and a_{00} are arbitrary constants and Δ and ξ are given by Equation (101).

Obviously, Equation (104) can be written as

$$u_{11}(x,t) = u_{11}(\xi) = -\sqrt{-\Delta} \tan\left(\sqrt{\frac{\sqrt{-\Delta}}{2\delta}}\xi - \xi_0 \right) + V_2, \tag{105}$$

where $\xi_0 = -\arctan(C_4/C_3)$, V_2, C, C_3, C_4, and a_{00} are arbitrary constants and Δ and ξ are given by Equation (101).

When $(a_{00} - V_3)^2 + 4C\delta = 0$,

$$u_{12}(x,t) = u_{12}(\xi) = 2\delta \frac{C_2}{C_1 + C_2\xi} + V_3, \tag{106}$$

where $\xi = x - V_3 t$, V_3, C_1, C_2, and a_{00} are arbitrary constants.

Moreover, note that Equation (76) is linear, so we can get the solution of Equation (65) as follows:

$$w = \sum_{i=1}^{n_1} w_{1i} + \sum_{i=1}^{n_2} w_{2i} + \sum_{i=1}^{n_3} w_{3i} u_{13} = 2\delta(\ln w) + a_{00}, \tag{107}$$

where

$$\begin{aligned} w_{1i} &= C_{1,1i} e^{\left(\left(-(a_{00}-V_{1i})+\sqrt{\Delta_{1i}}\right)/2\right)\xi_{1i}} + C_{2,1i} e^{\left(\left(-(a_{00}-V_{1i})-\sqrt{\Delta_{1i}}\right)/2\right)\xi_{1i}}, \\ w_{2i} &= \left(C_{1,2i} \cos\left(\frac{\sqrt{-\Delta_{2i}}\xi_{2i}}{2} \right) + C_{2,2i} \sin\left(\frac{\sqrt{-\Delta_{2i}}\xi_{2i}}{2} \right) \right) e^{\left(\left(-(a_{00}-V_{2i})\xi_{2i}\right)/2\right)}, \\ w_{3i} &= (C_{1,3i} + C_{2,3i}\xi_{3i}) e^{\left(\left(-(a_{00}-V_{3i})\xi_{3i}\right)/2\right)}, \end{aligned} \tag{108}$$

where $\xi_{1i} = x - V_{1i}t$, $\Delta_{1i} = (a_{00} - V_{1i})^2 + 4C\delta > 0$ $(i = 1, \cdots, n_1)$; $\xi_{2i} = x - V_{2i}t$, $\Delta_{2i} = (a_{00} - V_{2i})^2 + 4C\delta < 0$ $(i = 1, \cdots, n_2)$; $\xi_{3i} = x - V_{3i}t$, $\Delta_{3i} = (a_{00} - V_{3i})^2 + 4C\delta = 0$ $(i = 1, \cdots, n_3)$; $C_{1,1i}, C_{2,1i}$ $(i = 1, \cdots, n_1)$; $C_{1,2i}, C_{2,2i}$ $(i = 1, \cdots, n_2)$; and $C_{1,3i}, C_{2,3i}$ $(i = 1, \cdots, n_3)$ are arbitrary constants, and n_1, n_2, and n_3 are arbitrary but finite integers.

Remark 6. We can deal with Equation (76) by using some assumptions. For example, when we suppose that $w = -\alpha t + W(x)$ and $C = 0$, we get

$$\begin{aligned} w &= -\alpha t + \frac{\alpha x}{a_{00}} + C_1 + C_2 e^{-a_{00}x/\delta}, \\ u_{14}(x,t) &= \frac{\alpha\left(2\delta - t a_{00}^2 + a_{00}x\right) + a_{00}^2\left(C_1 - C_2 e^{-a_{00}x/\delta}\right)}{\alpha(x - a_{00}t) + a_{00}\left(C_1 + C_2 e^{-a_{00}x/\delta}\right)}, \end{aligned} \tag{109}$$

where C_1, C_2, α, and a_{00} are arbitrary constants.

When we suppose that $w = -\alpha t + W(x)$ and $C = a_{00} = 0$, we get

$$\begin{aligned} w &= -\alpha t + \frac{\alpha x^2}{2\delta} + C_1 x + C_2, \\ u_{15}(x,t) &= \frac{4\delta(\alpha x + \delta C_1)}{2\delta(C_1 x + C_2 - \alpha t) + \alpha x^2}, \end{aligned} \tag{110}$$

where C_1, C_2, and α are arbitrary constants.

Similarly, we can assume that $w = \sum_{i=1}^{n} p_i(x) q_i(t)$; then, a new solution of Equation (65) can be obtained. Being similar to above process, we omit it.

So far, applying the HB of undetermined coefficient method to the Burgers equation, we get the bilinear equation of Burgers equation. Moreover, we reduce the Burgers to a linear equation. Based on them, many exact solutions of the Burgers equation are obtained by applying the perturbation method, sub-ODE method, and compatible condition. Our results can compare with the (G'/G)-expansion method, Hirota's method, and HB method [20–30].

5. Bilinear Equation of the Boussinesq Equation and Sawada-Kotera Equation

In this section, we derive the bilinear equations of the Boussinesq equation and Sawada-Kotera equation by using the HB of undetermined coefficient method. Being similar to Section 4, we omit the process of solving exact solutions.

Example 1. The generalized Boussinesq equation reads

$$u_{tt} + 2\alpha u_{xt} + \left(\alpha^2 + \beta\right)u_{xx} + \gamma u u_{xx} + \delta u_{xxxx} = 0, \quad (111)$$

where α, β, γ, and δ are known constants.

In order to balance u_{xxxx} and uu_x in Equation (111), it is required that $m+4=2m+2$ and $n=2n$. Then, we suppose that the solution of Equation (111) is of the form

$$u = a_{20}(\ln w)_{xx} + a_{10}(\ln w)_x + a_{00}, \quad (112)$$

where a_{i0} $(i=0,1,2)$ are constants to be determined later.

Substituting Equation (112) into Equation (111) and equating the coefficients of $(w_x/w)^6$ and $(w_x/w)^5$ on the left-hand side of Equation (111) to zero yield a set of algebraic equations for a_{20} and a_{10}. Solving the algebraic equations, we get $a_{20} = 6\delta/\gamma$ and $a_{10}=0$. Substituting a_{20} and a_{10} back into Equation (112), we get

$$u = \frac{6\delta}{\gamma}(\ln w)_{xx} + a_{00}. \quad (113)$$

where a_{00} is an arbitrary constant.

Substituting Equation (113) into Equation (111), we get

$$\frac{6\delta}{\gamma}\left(K_1 + K_2 + K_3\right) = 0, \quad (114)$$

where

$$\begin{aligned} K_1 = {} & \beta\left(\frac{w_{xxxx}}{w} - \frac{3w_{xx}^2 + 4w_x w_{xxx}}{w^2} + \frac{12w_{xx}w_x^2}{w^3} - \frac{6w_x^4}{w^4}\right) \\ & + \alpha^2\left(\frac{12w_{xx}w_x^2}{w^3} - \frac{3w_{xx}^2 + 4w_x w_{xxx}}{w^2} + \frac{w_{xxxx}}{w} - \frac{6w_x^4}{w^4}\right) \\ & + \alpha\left(\frac{2w_{xxxt}}{w} - \frac{6w_{xx}w_{xt} + 6w_{xxt}w_x + 2w_t w_{xxx}}{w^2}\right. \\ & \left. + \frac{12w_x w_t w_{xx} + 12w_x^2 w_{xt}}{w^3} - \frac{12w_t w_x^3}{w^4}\right) + \frac{w_{xxtt}}{w} \\ & - \frac{2w_x w_{xtt} + w_{xx}w_{tt} + 2w_t w_{xxt} + 2w_{xt}^2}{w^2} \\ & + \frac{2w_x^2 w_{tt} + 2w_{xx}w_t^2 + 8w_x w_t w_{xt}}{w^3} - \frac{6w_x^2 w_t^2}{w^4}, \\ K_2 = {} & a_{00}\left(\frac{2\gamma w_{xxxx}}{w} - \frac{8\gamma w_x w_{xxx} + 6\gamma w_{xx}^2}{w^2}\right. \\ & \left. + \frac{24\gamma w_{xx}w_x^2}{w^3} - \frac{12\gamma w_x^4}{w^4}\right), \\ K_3 = {} & \delta\left(\frac{w_{xxxxxx}}{w} + \frac{2w_{xxx}^2 - 6w_x w_{xxxxx} - 3w_{xx}w_{xxxx}}{w^2}\right. \\ & \left. + \frac{18w_x^2 w_{xxxx} - 6w_{xx}^3}{w^3} + \frac{18w_x^2 w_{xx}^2 - 24w_x^3 w_{xxx}}{w^4}\right). \end{aligned} \quad (115)$$

Simplifying Equation (114) and integrating twice with respect to x, we get

$$\begin{aligned} & \frac{\partial^2}{\partial x^2}\left(\left(\alpha^2 + \beta + 2\gamma a_{00}\right)\left(\frac{ww_{xx} - w_x^2}{w^2}\right) + 2\alpha\left(\frac{ww_{xt} - w_x w_t}{w^2}\right)\right. \\ & \left. + \left(\frac{ww_{tt} - w_t^2}{w^2}\right) + \delta\left(\frac{ww_{xxxx} - 4w_x w_{xxx} 3w_{xx}^2}{w^2}\right)\right) = 0. \end{aligned} \quad (116)$$

Equation (116) is identical to

$$\begin{aligned} & \left(\alpha^2 + \beta + 2\gamma a_{00}\right)\left(ww_{xx} - w_x^2\right) + 2\alpha\left(ww_{xt} - w_x w_t\right) \\ & + \left(ww_{tt} - w_t^2\right) + \delta\left(ww_{xxxx} - 4w_x w_{xxx} + 3w_{xx}^2\right) \\ & - \left(C_1(t)x + C_2(t)\right)w^2 = 0, \end{aligned} \quad (117)$$

where $C_1(t)$ and $C_2(t)$ are arbitrary functions of t and a_{00} is an arbitrary constant.

In particular, letting $C_1(t) = C_2(t) = 0$ in Equation (117), we get the bilinear equation of Equation (111) as follows:

$$\begin{aligned} & \left(\alpha^2 + \beta + 2\gamma a_{00}\right)\left(ww_{xx} - w_x^2\right) + 2\alpha\left(ww_{xt} - w_x w_t\right) \\ & + \left(ww_{tt} - w_t^2\right) + \delta\left(ww_{xxxx} - 4w_x w_{xxx} + 3w_{xx}^2\right) = 0. \end{aligned} \quad (118)$$

Equation (118) can be written concisely in terms of D-operators as

$$\left(2\alpha D_x D_t + \delta D_x^4 + \left(\alpha^2 + \beta + 2\gamma a_{00}\right)D_x^2 + D_t^2\right)w \cdot w = 0, \quad (119)$$

where a_{00} is an arbitrary constant.

Example 2. The Sawada-Kotera equation reads

$$u_t + 15\left(u^3 + uu_{xx}\right)_x + u_{xxxxx} = 0. \quad (120)$$

In order to balance u_{xxxxx} and u^3 in Equation (120), it is required that $m+5=3m+1$ and $n=3n$. Then, we suppose that the solution of Equation (120) is of the form

$$u = a_{20}(\ln w)_{xx} + a_{10}(\ln w)_x + a_{00}, \quad (121)$$

where a_{i0} $(i=0,1,2)$ are constants to be determined later.

Substituting Equation (121) into Equation (120) and equating the coefficients of $(w_x/w)^7$ and $(w_x/w)^6$ on the left-hand side of Equation (120) to zero yield a set of algebraic equations for a_{20} and a_{10}. Solving the algebraic equations, we get $a_{20} = 2$ and $a_{10} = 0$. Substituting a_{20} and a_{10} back into Equation (121), we get

$$u = 2(\ln w)_{xx} + a_{00}, \tag{122}$$

where a_{00} is an arbitrary constant.

Substituting Equation (122) into Equation (120), we get

$$2(K_1 + K_2 + K_3) = 0, \tag{123}$$

where

$$\begin{aligned} K_1 &= 45a_{00}^2\left(\frac{w_{xxx}}{w} - \frac{3w_x w_{xx}}{w^2} + \frac{2w_x^3}{w^3}\right), \\ K_2 &= 15a_{00}\left(\frac{w_{xxxxx}}{w} + \frac{2w_{xx}w_{xxx} - 5w_x w_{xxxx}}{w^2}\right. \\ &\quad \left. + \frac{8w_{xxx}w_x^2 - 6w_x w_{xx}^2}{w^3}\right), \\ K_3 &= \left(\frac{w_{xxt}}{\mathrm{w}} - \frac{2w_x w_{xt} + w_{xx}w_t}{w^2} + \frac{2w_x^2 w_t}{w^3}\right) \\ &\quad + \left(\frac{w_{xxxxxxx}}{w} + \frac{9w_{xx}w_{xxxxx} - 5w_{xxx}w_{xxxx} - 7w_x w_{xxxxxx}}{w^2}\right. \\ &\quad \left. + \frac{20w_x w_{xxx}^2 - 30w_x w_{xx} w_{xxxx} + 12w_x^2 w_{xxxxx}}{w^3}\right). \end{aligned} \tag{124}$$

Simplifying Equation (123) and integrating once with respect to x, we get

$$\begin{aligned} &\frac{\partial}{\partial x}\left(45a_{00}^2\left(\frac{ww_{xx} - w_x^2}{w^2}\right)\right. \\ &\quad + 15a_{00}\left(\frac{ww_{xxxx} - 4w_x w_{xxx} + 3w_{xx}^2}{w^2}\right) + \frac{ww_{xt} - w_x w_t}{w^2} \\ &\quad \left. + \frac{ww_{xxxxxx} - 6w_x w_{xxxxx} + 15w_{xx}w_{xxxx} - 10w_{xxx}^2}{w^2}\right) = 0. \end{aligned} \tag{125}$$

Equation (125) is identical to

$$\begin{aligned} &45a_{00}^2\left(ww_{xx} - w_x^2\right) + 15a_{00}\left(ww_{xxxx} - 4w_x w_{xxx} + 3w_{xx}^2\right) \\ &\quad + \left(ww_{xt} - w_x w_t\right) + \left(ww_{xxxxxx} - 6w_x w_{xxxxx}\right. \\ &\quad \left. + 15w_{xx}w_{xxxx} - 10w_{xxx}^2\right) - C(t)w^2 = 0, \end{aligned} \tag{126}$$

where $C(t)$ is an arbitrary function of t and a_{00} is an arbitrary constant.

In particular, letting $C(t) = 0$ in Equation (126), we get the bilinear equation of Equation (120) as follows:

$$\begin{aligned} &45a_{00}^2\left(ww_{xx} - w_x^2\right) + 15a_{00}\left(ww_{xxxx} - 4w_x w_{xxx} + 3w_{xx}^2\right) \\ &\quad + \left(ww_{xt} - w_x w_t\right) + \left(ww_{xxxxxx} - 6w_x w_{xxxxx}\right. \\ &\quad \left. + 15w_{xx}w_{xxxx} - 10w_{xxx}^2\right) = 0. \end{aligned} \tag{127}$$

Equation (127) can be written concisely in terms of D-operators as

$$\left(45a_{00}^2 D_x^2 + 15a_{00}D_x^4 + D_x D_t + D_x^6\right)w \cdot w = 0, \tag{128}$$

where a_{00} is an arbitrary constant.

6. Conclusions

The HB of undetermined coefficient method is successfully used to establish the bilinear equation of NLPDE. By applying the perturbation method, sub-ODE method, and compatible condition to the bilinear equation, more exact solutions of NLPDE are obtained. We illustrate the real meaning of balance numbers. We show the underlying relations among the (G'/G)-expansion method, Hirota's method, and HB method. Many well-known NLPDE can be handled by the HB of undetermined coefficient method. The performance of our method is found to be simple and efficient. The availability of computer systems like Maple facilitates the tedious algebraic calculations. Our method is also a standard and computable method, which allows us to solve complicated and tedious algebraic calculations.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare no competing interests.

Authors' Contributions

Xiao-Feng Yang analyzed and interpreted the data and wrote the manuscript; Yi Wei designed and optimized the algorithm and program. All authors read the manuscript.

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