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Research Article

On New Unified Bounds for a Family of Functions via Fractional q-Calculus Theory

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The present article deals with the new estimates in q-calculus and fractional q-calculus on a time scale $\mathbb{T}_{t_0} = \{0\} \cup \{t : t = t_0 q^n, n$ is a nonnegative integer$\}$, where $t_0 \in \mathbb{R}$ and $0 < q < 1$. The role of fractional time scale q-calculus can be found as one of the prominent techniques to generate some variants for a class of positive functions $n\,(n \in \mathbb{N})$.Finally, our work will provide foundation and motivation for further investigation on time-fractional q-calculus systems that have an intriguing application in quantum theory and special relativity theory.

1. Introduction

Fractional differential equations were executed to demonstrate tremendous innovations for different issues in the physical sciences [1–15]. Since most frameworks involve recollections, the scientists are agreeing with the nonlocality of the fractional operators make it progressively functional in demonstrating the classical derivatives. Recently, nonlocal fractional derivatives without the singular kernel have been exhibited and contemplated [16, 17]. However, there are no solid numerical defenses of the new sorts of fractional derivatives; their applications were demonstrated by numerous analysts [18, 19]. Furthermore, presently we have the utilization of fractional calculus in fields like science, material science, and building and among different zones. It is a stunner of the fractional calculus that we have such a large number of valuable meanings of differential and integral operators, for instance, Saigo, conformable, Riemann-Liouville, Katugampola, Hadamard, Erdélyi-Kober, Liouville, local, and Weyl types. These operators are having their significance and applications in picture handling, science, hydrodynamics, and viscoelastic. For a detailed depiction of the origination of fractional calculus, advancement, and applications, we refer the interested readers to the notable books and research articles [20–22].

Hilger [23] began the theories of time scales in his doctoral dissertation and combined discrete and continuous analysis [24, 25]. At that time, this theory has received a lot of attention. In the book written by Bohner and Peterson [26] on the issues of time scale, a brief summary is given and several time calculations are performed. Over the past decade, many analysts working in special applications have proved a reasonable number of dynamic inequalities on a time scale [27, 28]. Several researchers have created various results relating to fractional calculus on time scales to obtain the corresponding dynamic variants [29].

In the eighteenth century (1707–1783), Euler initiated calculus with no limits refer to as quantum calculus. Jackson began a deliberate investigation of q-calculus and presented the q-definite integrals. Additionally, he was the first to create q-calculus in an efficient manner. Few selected branches of pure and applied mathematics, such as combinatorics, Gauss hypergeometric functions, orthogonal polynomials, dynamic, and quantum theory, have been enhanced by the exploration work of different researchers.

Motivated, by what we mentioned above, we extend the idea of fractional q-calculus type operators with a time scale to arbitrary positive order, provide several bounds for a family of $n \in \mathbb{N}$, and finally prove several variants for time-fractional q-calculus theory. These new results have utilities in the monotonicity for this nabla continuous fractional operator with singular and nonsingular kernel and compare them to the discrete classical ones. The time-fractional q-calculus under consideration in this paper have kernels different from classical nabla fractional differences with kernels depending on the rising factorial powers, and we believe that they bring new kernels with new memories, which may be of different interest for applications. The idea is quite new and seems to have opened new doors of investigation towards various scientific fields of research including engineering, fluid dynamics, meteorology, analysis, and aerodynamics.

Inequalities have wild applications in pure and applied mathematics [30–33]. Very recently, many new inequalities such as Hermite-Hadamard type inequality [34–38], Petrović type inequality [39], Pólya-Szegö type inequality [40], Ostrowski type inequality [41], reverse Minkowski inequality [42], Jensen type inequality [43, 44], Bessel function inequality [45], trigonometric and hyperbolic functions inequalities [46], fractional integral inequality [47–51], complete and generalized elliptic integrals inequalities [52–57], generalized convex function inequality [58–60], and mean values inequality [61–63] have been discovered by many researchers.

Variants regarding fractional integral operators are the use of noteworthy significant strategies amongst researchers and accumulate fertile functional applications in various areas of science [64, 65]. We state some of them, that is, the variants of Minkowski, Hardy, Opial, Hermite-Hadamard, Grüss, Lyenger, Ostrowski, C ebyšev, and Pólya-Szegö, and others. Such applications of fractional integral operators compelled us to show the generalization by using a family of n positive functions involving time-fractional q-calculus integrals operators.

Owing to the above phenomena, the key aim of this research is to demonstrate the notations and primary definitions of our noteworthy time-fractional q-calculus operator. Also, we present the results concerning for a class of family of $n(n \in \mathbb{N})$ continuous positive decreasing functions on $[\varsigma_1, \varsigma_2]$ by employing a time-fractional q-calculus operator. Finally, it is emphasized that combining these two approaches, q-fractional calculus and time scale analysis, could be the most efficient way of incorporating inequalities into both times and q-components for quantum theory and special relativity theory.

2. Preliminaries

Let us recall some necessary definitions and preliminary results that are used for further discussion. For more details, we may refer to [33].

Definition 1 (See [33]). The particular time scale $\mathbb{T}_{t_0}$ is defined by

$$\mathbb{T}_{t_0} = \{t : t = t_0 q^n, n \text{ is a nonnegative integer}\} \cup \{0\}, 0 < q < 1. \tag{1}$$

If there is no confusion concerning t_0, we will denote $\mathbb{T}_{t_0}$ by $\mathbb{T}$.

Definition 2. The q-factorial function is defined in the following way

$$\begin{aligned} (\zeta - \varphi)^{\underline{(n)}} &= (\zeta - \varphi)(\zeta - q\varphi)(\zeta - q^{\varphi}) \cdots (\zeta - q^n \varphi), n \in \mathbb{N}, \\ (\zeta - \varphi)^{\underline{(n)}} &= \zeta^n \prod_{\kappa=0}^{\infty} \frac{1 - (\varphi/\zeta) q^{\kappa}}{1 - (\varphi/\zeta) q^{n+\kappa}}, \quad n \notin \mathbb{N}. \end{aligned} \tag{2}$$

Definition 3. The q-derivative of the q-factorial function with respect to ζ is defined by

$$\nabla_q (\zeta - \varphi)^{\underline{(n)}} = \frac{1 - q^n}{1 - q} (\zeta - \varphi)^{\underline{(n-1)}}, \tag{3}$$

and the q-derivative of the q-factorial function with respect to s is defined by

$$\nabla_q (\zeta - \varphi)^{\underline{(n)}} = -\frac{1 - q^n}{1 - q} (\zeta - q\varphi)^{\underline{(n-1)}}. \tag{4}$$

Definition 4. The q-exponential function is defined as

$$e_q(\zeta) = \prod_{\kappa=0}^{\infty} (1 - q^{\kappa} \zeta), \quad e_q(0) = 1. \tag{5}$$

Definition 5. The q-Gamma function is defined by

$$\Gamma_q(\beta) = \frac{1}{1 - q} \int_0^1 \left(\frac{\zeta}{1 - q} \right)^{\beta - 1} e_q(q\zeta) \nabla \zeta, \beta \in \mathbb{R}^+. \tag{6}$$

Remark 6. We observe that

$$\Gamma_q(\beta + 1) = [\beta]_q \Gamma_q(\beta), \quad \beta \in \mathbb{R}^+, \tag{7}$$

and $[\beta]_q = 1 - q^{\beta}/1 - q$.

Definition 7. The fractional q-integral is defined as

$$\nabla_q^{-\beta}\Psi(\zeta) = \frac{1}{\Gamma_q(\beta)}\int_{\varsigma_1}^{\zeta} (\zeta - q\varphi)^{\underline{\beta-1}}\Psi(\varphi)\nabla\varphi. \quad (8)$$

Remark 8. Let $\Psi(\zeta) = 1$. Then Definition 7 gives

$$\nabla_q^{-\beta}(1) = \frac{1}{\Gamma_q(\beta)}\frac{q-1}{q^\beta - 1}\zeta^{\underline{\beta}} = \frac{1}{\Gamma_q(\beta+1)}\zeta^{\underline{\beta}}. \quad (9)$$

3. Main Results

Now we demonstrate the left fractional q integral operator on an arbitrary time scale $\mathbb{T}$ to derive the generalization of some classical inequalities.

Theorem 9. *Let $\alpha > 0$, $\eta \geq \delta > 0$, $\beta \in \mathbb{C}$ with $\Re(\beta) > 0$, and Ψ be a continuous positive decreasing function defined on $\mathbb{T}_{t_0}$. Then, one has*

$$\frac{\nabla_{\varsigma_1^+,q}^{-\beta}[\Psi^\eta(\zeta)]}{\nabla_{\varsigma_1^+,q}^{-\beta}[\Psi^\delta(\zeta)]} \geq \frac{\nabla_{\varsigma_1^+,q}^{-\beta}[(\zeta-\varsigma_1)^\alpha\Psi^\eta(\zeta)]}{\nabla_{\varsigma_1^+,q}^{-\beta}[(\zeta-\varsigma_1)^\alpha\Psi^\delta(\zeta)]}. \quad (10)$$

Proof. Using the hypothesis given in Theorem 9, we have

$$((\omega-\varsigma_1)^\alpha - (\varphi-\varsigma_1)^\alpha)\left(\Psi^{\eta-\delta}(\varphi) - \Psi^{\eta-\delta}(\omega)\right) \geq 0, \quad (11)$$

where $\alpha > 0$, $\eta \geq \delta > 0$, and $\varphi, \omega \in [\varsigma_1, \zeta]$.

It follows from (11) that

$$\begin{aligned}&(\omega-\varsigma_1)^\alpha\Psi^{\eta-\delta}(\varphi) - (\varphi-\varsigma_1)^\alpha\Psi^{\eta-\delta}(\omega) - (\omega-\varsigma_1)^\alpha\Psi^{\eta-\delta}(\omega)\\&\quad + (\varphi-\varsigma_1)^\alpha\Psi^{\eta-\delta}(\varphi) \geq 0.\end{aligned} \quad (12)$$

Multiplying (12) by $1/\Gamma_q(\beta)(\zeta-q\varphi)^{\underline{\beta-1}/}\Psi^\delta(\varphi), \varphi \in (\varsigma_1, \zeta)$, we have

$$\begin{aligned}&\frac{1}{\Gamma_q(\beta)}(\zeta-q\varphi)^{\underline{\beta-1}}\Big[(\omega-\varsigma_1)^\alpha\Psi^{\eta-\delta}(\varphi) - (\varphi-\varsigma_1)^\alpha\Psi^{\eta-\delta}(\omega)\\&\quad - (\omega-\varsigma_1)^\alpha\Psi^{\eta-\delta}(\omega) + (\varphi-\varsigma_1)^\alpha\Psi^{\eta-\delta}(\varphi)\Big]\Psi^\delta(\varphi)\\&\quad = (\omega-\varsigma_1)^\alpha\frac{1}{\Gamma_q(\beta)}(\zeta-q\varphi)^{\underline{\beta-1}}\Psi^\delta(\varphi)\Psi^{\eta-\delta}(\varphi)\\&\quad - (\varphi-\varsigma_1)^\alpha\frac{1}{\Gamma_q(\beta)}(\zeta-q\varphi)^{\underline{\beta-1}}\Psi^\delta(\varphi)\Psi^{\eta-\delta}(\omega)\\&\quad - (\omega-\varsigma_1)^\alpha\frac{1}{\Gamma_q(\beta)}(\zeta-q\varphi)^{\underline{\beta-1}}\Psi^\delta(\varphi)\Psi^{\eta-\delta}(\omega)\\&\quad + (\varphi-\varsigma_1)^\alpha\frac{1}{\Gamma_q(\beta)}(\zeta-q\varphi)^{\underline{\beta-1}}\Psi^\delta(\varphi)\Psi^{\eta-\delta}(\varphi) \geq 0.\end{aligned} \quad (13)$$

Integrating on both sides of (13) for φ over (ς_1, ζ), we have

$$\begin{aligned}&(\omega-\varsigma_1)^\alpha\int_{\varsigma_1}^{\zeta}\frac{1}{\Gamma_q(\beta)}(\zeta-q\varphi)^{\underline{\beta-1}}\Psi^\delta(\varphi)\Psi^{\eta-\delta}(\varphi)\nabla\varphi - (\varphi-\varsigma_1)^\alpha\\&\quad\cdot\int_{\varsigma_1}^{\zeta}\frac{1}{\Gamma_q(\beta)}(\zeta-q\varphi)^{\underline{\beta-1}}\Psi^\delta(\varphi)\Psi^{\eta-\delta}(\omega)\nabla\varphi - (\omega-\varsigma_1)^\alpha\\&\quad\cdot\int_{\varsigma_1}^{\zeta}\frac{1}{\Gamma_q(\beta)}(\zeta-q\varphi)^{\underline{\beta-1}}\Psi^\delta(\varphi)\Psi^{\eta-\delta}(\omega)\nabla\varphi + (\varphi-\varsigma_1)^\alpha\\&\quad\cdot\int_{\varsigma_1}^{\zeta}\frac{1}{\Gamma_q(\beta)}(\zeta-q\varphi)^{\underline{\beta-1}}\Psi^\delta(\varphi)\Psi^{\eta-\delta}(\varphi)\nabla\varphi \geq 0,\end{aligned} \quad (14)$$

that is

$$\begin{aligned}&(\omega-\varsigma_1)^\alpha\left(\nabla_{\varsigma_1^+,q}^{-\beta}[\Psi^\eta(\zeta)]\right) + \Psi^{\eta-\delta}(\omega)\left(\nabla_{\varsigma_1^+,q}^{-\beta}\left[(\zeta-\varsigma_1)^\alpha\Psi^\delta(\zeta)\right]\right)\\&\quad - (\omega-\varsigma_1)^\alpha\Psi^{\eta-\delta}(\omega)\left(\nabla_{\varsigma_1^+,q}^{-\beta}\left[\Psi^\delta(\zeta)\right]\right) - \left(\left[(\zeta-\varsigma_1)^\alpha\Psi^\eta(\zeta)\right]\right) \geq 0.\end{aligned} \quad (15)$$

Multiplying (15) by $1/\Gamma_q(\beta)(\zeta-q\omega)^{\beta-1/}\Psi^\delta(\omega), \omega \in (\varsigma_1, \zeta)$, and integrating for ω over (ς_1, ζ) shows

$$\begin{aligned}&\left(\nabla_{\varsigma_1^+,q}^{-\beta}[\Psi^\eta(\zeta)]\right)\left(\nabla_{\varsigma_1^+,q}^{-\beta}\left[(\zeta-\varsigma_1)^\alpha\Psi^\delta(\zeta)\right]\right)\\&\quad - \left(\nabla_{\varsigma_1^+,q}^{-\beta}\left[(\zeta-\varsigma_1)^\alpha\Psi^\eta(\zeta)\right]\right)\left(\left[\Psi^\delta(\zeta)\right]\right) \geq 0.\end{aligned} \quad (16)$$

Dividing the above inequality by $(\nabla_{\varsigma_1^+,q}^{-\beta}[(\zeta-\varsigma_1)^\alpha\Psi^\delta(\zeta)])$ $(\nabla_{\varsigma_1^+,q}^{-\beta}[\Psi^\delta(\zeta)])$, we get the desired inequality (10).

Theorem 10. *Let $\alpha > 0$, and $\eta \geq \delta > 0$, $\beta, \lambda \in \mathbb{C}$ with $\Re(\beta) > 0$ and $\Re(\lambda) > 0$, and Ψ be a continuous positive decreasing function defined on $\mathbb{T}_{t_0}$. Then the time-fractional q-integral satisfies the inequality*

$$\frac{\left(\nabla_{\varsigma_1^+,q}^{-\beta}[\Psi^\eta(\zeta)]\right)\left(\nabla_{\varsigma_1^+,q}^{-\lambda}[(\zeta-\varsigma_1)^\alpha\Psi^\delta(\zeta)]\right) + \left(\nabla_{\varsigma_1^+,q}^{-\lambda}[\Psi^\eta(\zeta)]\right)\left(\nabla_{\varsigma_1^+,q}^{-\beta}[(\zeta-\varsigma_1)^\alpha\Psi^\delta(\zeta)]\right)}{\left(\nabla_{\varsigma_1^+,q}^{-\lambda}[\Psi^\delta(\zeta)]\right)\left(\nabla_{\varsigma_1^+,q}^{-\beta}[(\zeta-\varsigma_1)^\alpha\Psi^\eta(\zeta)]\right) + \left(\nabla_{\varsigma_1^+,q}^{-\beta}[\Psi^\delta(\zeta)]\right)\left(\nabla_{\varsigma_1^+,q}^{-\lambda}[(\zeta-\varsigma_1)^\alpha\Psi^\eta(\zeta)]\right)} \geq 1. \quad (17)$$

Proof. Multiplying both sides of (15) by $1/\Gamma_q(\lambda)(\zeta-q\omega)^{\lambda-1/}$ $\Psi^\delta(\omega)$, $\omega \in (\varsigma_1, \zeta)$ and integrating for ω over (ς_1, ζ) shows

$$\begin{aligned}&\left(\nabla_{\varsigma_1^+,q}^{-\beta}[\Psi^\eta(\zeta)]\right)\left(\nabla_{\varsigma_1^+,q}^{-\lambda}\left[(\zeta-\varsigma_1)^\alpha\Psi^\delta(\zeta)\right]\right) + \nabla_{\varsigma_1^+,q}^{-\lambda}[\Psi^\eta(\zeta)]\Big)\\&\quad\cdot\left(\nabla_{\varsigma_1^+,q}^{-\beta}\left[(\zeta-\varsigma_1)^\alpha\Psi^\delta(\zeta)\right]\right) - \left(\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\Psi^\delta(\zeta)\right]\right)\\&\quad\cdot\left(\nabla_{\varsigma_1^+,q}^{-\beta}\left[(\zeta-\varsigma_1)^\alpha\Psi^\eta(\zeta)\right]\right) - \left(\nabla_{\varsigma_1^+,q}^{-\beta}\left[\Psi^\delta(\zeta)\right]\right)\\&\quad\cdot\left(\nabla_{\varsigma_1^+,q}^{-\lambda}\left[(\zeta-\varsigma_1)^\alpha\Psi^\eta(\zeta)\right]\right) \geq 0.\end{aligned} \quad (18)$$

Dividing (18) by

$$\left(\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\Psi^{\delta}(\zeta)\right]\right)\left(\nabla_{\varsigma_1^+,q}^{-\beta}\left[(\zeta-\varsigma_1)^{\alpha}\Psi^{\eta}(\zeta)\right]\right) - \left(\nabla_{\varsigma_1^+,q}^{-\beta}\left[\Psi^{\delta}(\zeta)\right]\right)\left(\nabla_{\varsigma_1^+,q}^{-\lambda}\left[(\zeta-\varsigma_1)^{\alpha}\Psi^{\eta}(\zeta)\right]\right), \tag{19}$$

we get the desired inequality (17).

Theorem 11. *Let $\alpha>0$, $\eta\ge\delta>0$, $\beta\in\mathbb{C}$ with $\mathfrak{R}(\beta)>0$, Ψ be a continuous positive decreasing function defined on $\mathbb{T}_{t_0}$, and $\hbar$ be a continuous positive increasing function on $\mathbb{T}_{t_0}$. Then the time-fractional q -integral satisfies the inequality*

$$\frac{\left(\nabla_{\varsigma_1^+,q}^{-\beta}\left[\Psi^{\eta}(\zeta)\right]\right)\left(\nabla_{\varsigma_1^+,q}^{-\beta}\left[\Psi^{\delta}(\zeta)\hbar^{\alpha}(\zeta)\right]\right)}{\left(\nabla_{\varsigma_1^+,q}^{-\beta}\left[\Psi^{\delta}(\zeta)\right]\right)\left(\nabla_{\varsigma_1^+,q}^{-\beta}\left[\Psi^{\eta}(\zeta)\hbar^{\alpha}(\zeta)\right]\right)}\ge 1. \tag{20}$$

Proof. Using the hypothesis given in Theorem 11, we have

$$(\hbar^{\alpha}(\omega)-\hbar^{\alpha}(\varphi))\left(\Psi^{\eta-\delta}(\varphi)-\Psi^{\eta-\delta}(\omega)\right)\ge 0, \tag{21}$$

where $\alpha>0$, $\eta\ge\delta>0$, and $\varphi,\omega\in[\varsigma_1,\zeta]$. From (21), we have

$$\hbar^{\alpha}(\omega)\Psi^{\eta-\delta}(\varphi)-\hbar^{\alpha}(\varphi)\Psi^{\eta-\delta}(\omega) + \hbar^{\alpha}(\omega)\Psi^{\eta-\delta}(\omega)-\hbar^{\alpha}(\varphi)\Psi^{\eta-\delta}(\varphi)\ge 0. \tag{22}$$

Taking product of (22) by $1/\Gamma_q(\beta)(\zeta-q\varphi)^{\beta-1/}\Psi^{\delta}(\varphi)$, $\varphi\in(\varsigma_1,\zeta)$, we get

$$\begin{aligned}
&\frac{1}{\Gamma_q(\beta)}(\zeta-q\varphi)^{\underline{\beta-1}}\Psi^{\delta}(\varphi)\left[\hbar^{\alpha}(\omega)\Psi^{\eta-\delta}(\varphi)-\hbar^{\alpha}(\varphi)\Psi^{\eta-\delta}(\omega)\right.\\
&\quad\left.+\hbar^{\alpha}(\omega)\Psi^{\eta-\delta}(\omega)-\hbar^{\alpha}(\varphi)\Psi^{\eta-\delta}(\varphi)\right]\\
&\quad=\hbar^{\alpha}(\omega)\frac{1}{\Gamma_q(\beta)}(\zeta-q\varphi)^{\underline{\beta-1}}\Psi^{\eta}(\varphi)-\hbar^{\alpha}(\varphi)\frac{1}{\Gamma_q(\beta)}\\
&\quad\cdot(\zeta-q\varphi)^{\underline{\beta-1}}\Psi^{\eta-\delta}(\omega)\Psi^{\delta}(\varphi)+\hbar^{\alpha}(\omega)\frac{1}{\Gamma_q(\beta)}(\zeta-q\varphi)^{\underline{\beta-1}}\Psi^{\eta-\delta}\\
&\quad\cdot(\omega)\Psi^{\delta}(\varphi)-\hbar^{\alpha}(\varphi)\frac{1}{\Gamma_q(\beta)}(\zeta-q\varphi)^{\underline{\beta-1}}\Psi^{\eta}(\varphi)\ge 0.
\end{aligned} \tag{23}$$

Integrating (23) for φ over (ς_1,ζ), we obtain

$$\begin{aligned}
&\frac{\hbar^{\alpha}(\omega)}{\Gamma_q(\beta)}\int_{\varsigma_1}^{\zeta}(\zeta-q\varphi)^{\underline{\beta-1}}\Psi^{\eta}(\varphi)\nabla\varphi-\frac{\hbar^{\alpha}(\varphi)}{\Gamma_q(\beta)}\int_{\varsigma_1}^{\zeta}(\zeta-q\varphi)^{\underline{\beta-1}}\Psi^{\eta-\delta}\\
&\quad\cdot(\omega)\Psi^{\delta}(\varphi)\nabla\varphi+\frac{\hbar^{\alpha}(\omega)}{\Gamma_q(\beta)}\int_{\varsigma_1}^{\zeta}(\zeta-q\varphi)^{\underline{\beta-1}}\Psi^{\eta-\delta}(\omega)\Psi^{\delta}(\varphi)\nabla\varphi\\
&\quad-\frac{\hbar^{\alpha}(\varphi)}{\Gamma_q(\beta)}\int_{\varsigma_1}^{\zeta}(\zeta-q\varphi)^{\underline{\beta-1}}\Psi^{\eta}(\varphi)\nabla\varphi\ge 0.
\end{aligned} \tag{24}$$

It follows that

$$\begin{aligned}
&\hbar^{\alpha}(\omega)\left(\nabla_{\varsigma_1^+,q}^{-\beta}\left[\Psi^{\eta}(\zeta)\right]\right)+\Psi^{\eta-\delta}(\omega)\left(\nabla_{\varsigma_1^+,q}^{-\beta}\left[\hbar^{\alpha}(\zeta)\Psi^{\delta}(\zeta)\right]\right)\\
&\quad-\hbar^{\alpha}(\omega)\Psi^{\eta-\delta}(\omega)\left(\nabla_{\varsigma_1^+,q}^{-\beta}\left[\Psi^{\delta}(\zeta)\right]\right)-\left(\nabla_{\varsigma_1^+,q}^{-\beta}\left[\hbar^{\alpha}(\zeta)\Psi^{\delta}(\zeta)\right]\right)\ge 0.
\end{aligned} \tag{25}$$

Again, taking the product (15) by $1/\Gamma_q(\beta)(\zeta-q\omega)^{\beta-1/}\Psi^{\delta}(\omega)$, $\omega\in(\varsigma_1,\zeta)$, and integrating for ω over (ς_1,ζ) gives

$$\left(\nabla_{\varsigma_1^+,q}^{-\beta}\left[\Psi^{\eta}(\zeta)\right]\right)\left(\nabla_{\varsigma_1^+,q}^{-\beta}\left[\Psi^{\delta}(\zeta)\hbar^{\alpha}(\zeta)\right]\right) - \left(\nabla_{\varsigma_1^+,q}^{-\beta}\left[\Psi^{\delta}(\zeta)\right]\right)\left(\nabla_{\varsigma_1^+,q}^{-\beta}\left[\Psi^{\eta}(\zeta)\hbar^{\alpha}(\zeta)\right]\right)\ge 0, \tag{26}$$

which completes the proof of the desired inequality (20).

Theorem 12. *Let $\alpha>0$, $\eta\ge\delta>0$, $\mathfrak{R}(\lambda),\mathfrak{R}(\beta)>0$ with $\mathfrak{R}(\lambda),\mathfrak{R}(\beta)>0$,$\Psi$ be a continuous positive decreasing function defined on $\mathbb{T}_{t_0}$ and $\hbar$ be a continuous positive increasing function on $\mathbb{T}_{t_0}$. Then, one has*

$$\frac{\left(\nabla_{\varsigma_1^+,q}^{-\beta}\left[\Psi^{\eta}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\hbar^{\alpha}(\zeta)\Psi^{\delta}(\zeta)\right]\right)+\left(\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\Psi^{\eta}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\beta}\left[\hbar^{\theta}(\zeta)\Psi^{\delta}(\zeta)\right]\right)}{\left(\nabla_{\varsigma_1^+,q}^{-\beta}\left[\hbar^{\alpha}(\zeta)\Psi^{\eta}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\Psi^{\delta}(\zeta)\right]\right)+\left(\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\hbar^{\alpha}(\zeta)\Psi^{\eta}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\Psi^{\delta}(\zeta)\right]\right)}\ge 1. \tag{27}$$

Proof. Multiplying both sides of (25) by $1/\Gamma_q(\beta)(\zeta-q\omega)^{\beta-1/}\Psi^{\delta}(\omega)$, $\omega\in(\varsigma_1,\zeta)$, and integrating for ω over (ς_1,ζ) leads to the conclusion that

$$\begin{aligned}
&\left(\nabla_{\varsigma_1^+,q}^{-\beta}\left[\Psi^{\eta}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\hbar^{\alpha}(\zeta)\Psi^{\delta}(\zeta)\right]\right)\\
&\quad+\left(\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\Psi^{\eta}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\beta}\left[\hbar^{\theta}(\zeta)\Psi^{\delta}(\zeta)\right]\right)\\
&\quad-\left(\nabla_{\varsigma_1^+,q}^{-\beta}\left[\hbar^{\alpha}(\zeta)\Psi^{\eta}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\Psi^{\delta}(\zeta)\right]\right)\\
&\quad-\left(\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\hbar^{\alpha}(\zeta)\Psi^{\eta}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\Psi^{\delta}(\zeta)\right]\right)\ge 0.
\end{aligned} \tag{28}$$

It follows that

$$\begin{aligned}
&\left(\nabla_{\varsigma_1^+,q}^{-\beta}\left[\Psi^{\eta}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\hbar^{\alpha}(\zeta)\Psi^{\delta}(\zeta)\right]\right)\\
&\quad+\left(\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\Psi^{\eta}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\beta}\left[\hbar^{\theta}(\zeta)\Psi^{\delta}(\zeta)\right]\right)\\
&\quad\ge\left(\nabla_{\varsigma_1^+,q}^{-\beta}\left[\hbar^{\alpha}(\zeta)\Psi^{\eta}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\Psi^{\delta}(\zeta)\right]\right)\\
&\quad+\left(\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\hbar^{\alpha}(\zeta)\Psi^{\eta}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\Psi^{\delta}(\zeta)\right]\right).
\end{aligned} \tag{29}$$

Dividing above inequality by

$$\left(\nabla_{\varsigma_1^+,q}^{-\beta}\left[\hbar^{\alpha}(\zeta)\Psi^{\eta}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\Psi^{\delta}(\zeta)\right]\right)+\left(\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\hbar^{\alpha}(\zeta)\Psi^{\eta}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\Psi^{\delta}(\zeta)\right]\right), \tag{30}$$

we get the desired inequality (27).

Now, we demonstrate the fractional q-integral to derive some inequalities for a class of n-decreasing positive functions.

Theorem 13. *Let $\alpha > 0$, $\eta \geq \delta_\kappa > 0$ for any fixed $\kappa \in \{1, 2, 3, \cdots, n\}$, $\beta \in \mathbb{C}$ with $\Re(\beta) > 0$, and $\{\Psi_j, j = 1, 2, 3, \cdots, n\}$ be a sequence of continuous positive decreasing functions defined on $\mathbb{T}_{t_0}$. Then, the time-fractional q -integral satisfies the inequality*

$$\frac{\nabla^{-\beta}_{\varsigma_1^+,q}\left[\prod_{j\neq\kappa}^n \Psi_j^{\delta j}\Psi_\kappa^\eta(\zeta)\right]}{\nabla^{-\beta}_{\varsigma_1^+,q}\left[\prod_{j=1}^n \Psi_j^{\delta j}(\zeta)\right]} \geq \frac{\nabla^{-\beta}_{\varsigma_1^+,q}\left[(\zeta-\varsigma_1)^\alpha\prod_{j\neq\kappa}^n \Psi_j^{\delta j}\Psi_\kappa^\eta(\zeta)\right]}{\nabla^{-\beta}_{\varsigma_1^+,q}\left[(\zeta-\varsigma_1)^\alpha\prod_{j=1}^n \Psi_j^{\delta j}(\zeta)\right]}. \tag{31}$$

Proof. Since $\{\Psi_j, j = 1, 2, 3, \cdots, n\}$ is a sequence of continuous positive decreasing functions on $[\varsigma_1, \zeta]$, we have

$$\left((\omega-\varsigma_1)^\alpha - (\varphi-\varsigma_1)^\alpha\right)\left(\Psi_\kappa^{\eta-\delta\kappa}(\varphi) - \Psi_\kappa^{\eta-\delta\kappa}(\omega)\right) \geq 0, \tag{32}$$

for any fixed $\kappa \in \{1, 2, 3, \cdots, n\}$, $\alpha > 0$, $\eta \geq \delta_\kappa > 0$ and $\varphi, \omega \in [\varsigma_1, \zeta]$.

It follows from (32) that

$$\begin{aligned}&(\omega-\varsigma_1)^\alpha\Psi_\kappa^{\eta-\delta\kappa}(\varphi) + (\varphi-\varsigma_1)^\alpha\Psi_\kappa^{\eta-\delta\kappa}(\omega)\\ &\quad\geq (\omega-\varsigma_1)^\alpha\Psi_\kappa^{\eta-\delta\kappa}(\omega) + (\varphi-\varsigma_1)^\alpha\Psi_\kappa^{\eta-\delta\kappa}(\varphi).\end{aligned} \tag{33}$$

Taking the product of (22) by $1/\Gamma_q(\beta)(\zeta-q\varphi)^{\beta-1/}\prod_{j=1}^n \Psi_j^{\delta_j}(\varphi)$, $\varphi \in (\varsigma_1, \zeta)$, and integrating for φ over (ς_1, ζ), we have

$$\begin{aligned}&\frac{1}{\Gamma_q(\beta)}(\zeta-q\varphi)^{\underline{\beta-1}}\Big[(\omega-\varsigma_1)^\alpha\Psi_\kappa^{\eta-\delta\kappa}(\varphi) + (\varphi-\varsigma_1)^\alpha\Psi_\kappa^{\eta-\delta\kappa}(\omega)\\
&\quad - (\omega-\varsigma_1)^\alpha\Psi_\kappa^{\eta-\delta\kappa}(\omega) - (\varphi-\varsigma_1)^\alpha\Psi_\kappa^{\eta-\delta\kappa}(\varphi)\Big]\prod_{j=1}^n \Psi_j^{\delta_j}(\varphi)\\
&\quad = (\omega-\varsigma_1)^\alpha\frac{1}{\Gamma_q(\beta)}(\zeta-q\varphi)^{\underline{\beta-1}}\prod_{j=1}^n \Psi_j^{\delta_j}(\varphi)\Psi_\kappa^{\eta-\delta\kappa}(\varphi)\\
&\quad + (\varphi-\varsigma_1)^\alpha\frac{1}{\Gamma_q(\beta)}(\zeta-q\varphi)^{\underline{\beta-1}}\prod_{j=1}^n \Psi_j^{\delta_j}(\varphi)\Psi_\kappa^{\eta-\delta\kappa}(\omega)\\
&\quad - (\omega-\varsigma_1)^\alpha\frac{1}{\Gamma_q(\beta)}(\zeta-q\varphi)^{\underline{\beta-1}}\prod_{j=1}^n \Psi_j^{\delta_j}(\varphi)\Psi_\kappa^{\eta-\delta\kappa}(\omega)\\
&\quad - (\varphi-\varsigma_1)^\alpha\frac{1}{\Gamma_q(\beta)}(\zeta-q\varphi)^{\underline{\beta-1}}\prod_{j=1}^n \Psi_j^{\delta_j}(\varphi)\Psi_\kappa^{\eta-\delta\kappa}(\varphi) \geq 0.\end{aligned} \tag{34}$$

Integrating (34) for φ over (ς_1, ζ), we get

$$\begin{aligned}&(\omega-\varsigma_1)^\alpha\frac{1}{\Gamma_q(\beta)}\int_{\varsigma_1}^{\zeta}(\zeta-q\varphi)^{\underline{\beta-1}}\prod_{j=1}^n \Psi_j^{\delta_j}(\varphi)\Psi_\kappa^{\eta-\delta\kappa}(\varphi)\nabla\varphi\\
&\quad + (\varphi-\varsigma_1)^\alpha\frac{1}{\Gamma_q(\beta)}\int_{\varsigma_1}^{\zeta}(\zeta-q\varphi)^{\underline{\beta-1}}\prod_{j=1}^n \Psi_j^{\delta_j}(\varphi)\Psi_\kappa^{\eta-\delta\kappa}(\omega)\nabla\varphi\\
&\quad - (\omega-\varsigma_1)^\alpha\frac{1}{\Gamma_q(\beta)}\int_{\varsigma_1}^{\zeta}(\zeta-q\varphi)^{\underline{\beta-1}}\prod_{j=1}^n \Psi_j^{\delta_j}(\varphi)\Psi_\kappa^{\eta-\delta\kappa}(\omega)\nabla\varphi\\
&\quad - (\varphi-\varsigma_1)^\alpha\frac{1}{\Gamma_q(\beta)}\int_{\varsigma_1}^{\zeta}(\zeta-q\varphi)^{\underline{\beta-1}}\prod_{j=1}^n \Psi_j^{\delta_j}(\varphi)\Psi_\kappa^{\eta-\delta\kappa}(\varphi)\nabla\varphi \geq 0.\end{aligned} \tag{35}$$

It follows from (35) that

$$\begin{aligned}&(\omega-\varsigma_1)^\alpha\nabla^{-\beta}_{\varsigma_1^+,q}\left[\prod_{j\neq\kappa}^n \Psi_j^{\delta j}\Psi_\kappa^\eta(\zeta)\right] + \Psi_\kappa^{\eta-\delta\kappa}(\omega)\,\nabla^{-\beta}_{\varsigma_1^+,q}\left[(\zeta-\varsigma_1)^\alpha\prod_{j=1}^n \Psi_j^{\delta j}(\zeta)\right]\\
&\quad\geq (\omega-\varsigma_1)^\alpha\Psi_\kappa^{\eta-\delta\kappa}(\omega)\nabla^{-\beta}_{\varsigma_1^+,q}\left[\prod_{j=1}^n \Psi_j^{\delta j}(\zeta)\right]\\
&\quad + \nabla^{-\beta}_{\varsigma_1^+,q}\left[(\zeta-\varsigma_1)^\alpha\prod_{j\neq\kappa}^n \Psi_j^{\delta j}\Psi_\kappa^\eta(\zeta)\right].\end{aligned} \tag{36}$$

Again, taking the product of (36) by $1/\Gamma_q(\beta)(\zeta-q\omega)^{\beta-1/}\prod_{j=1}^n \Psi_j^{\delta_j}(\omega)$, $\omega \in (\varsigma_1, \zeta)$, and integrating for ω over (ς_1, ζ), we obtain

$$\begin{aligned}&\nabla^{-\beta}_{\varsigma_1^+,q}\left[\prod_{j\neq\kappa}^n \Psi_j^{\delta j}\Psi_\kappa^\eta(\zeta)\right]\nabla^{-\beta}_{\varsigma_1^+,q}\left[(\zeta-\varsigma_1)^\alpha\prod_{j=1}^n \Psi_j^{\delta j}(\zeta)\right]\\
&\quad\geq \nabla^{-\beta}_{\varsigma_1^+,q}\left[(\zeta-\varsigma_1)^\alpha\prod_{j\neq\kappa}^n \Psi_j^{\delta j}\Psi_\kappa^\eta(\zeta)\right]\nabla^{-\beta}_{\varsigma_1^+,q}\left[\prod_{j=1}^n \Psi_j^{\delta j}(\zeta)\right],\end{aligned} \tag{37}$$

which gives the desired inequality (31).

Theorem 14. *Let $\alpha > 0$, $\eta \geq \delta_\kappa > 0$ for any fixed $\kappa \in \{1, 2, 3, \cdots, n\}$, $\beta, \lambda \in \mathbb{C}$ with $\Re(\beta) > 0$, $\Re(\lambda) > 0$, and $\{\Psi_j, j = 1, 2, 3, \cdots, n\}$ be a sequence of continuous positive decreasing functions defined on $\mathbb{T}_{t_0}$. Then, we have the inequality*

$$\begin{aligned}&\left(\nabla^{-\beta}_{\varsigma_1^+,q}\left[\prod_{j\neq\kappa}^n \Psi_j^{\delta j}\Psi_\kappa^\eta(\zeta)\right]\nabla^{-\lambda}_{\varsigma_1^+,q}\left[(\zeta-\varsigma_1)^\alpha\prod_{j=1}^n \Psi_j^{\delta j}(\zeta)\right]\right.\\
&\quad\left. + \nabla^{-\lambda}_{\varsigma_1^+,q}\left[\prod_{j\neq\kappa}^n \Psi_j^{\delta j}\Psi_\kappa^\eta(\zeta)\right]\nabla^{-\beta}_{\varsigma_1^+,q}\left[(\zeta-\varsigma_1)^\alpha\prod_{j=1}^n \Psi_j^{\delta j}(\zeta)\right]\right)/\\
&\quad\left(\nabla^{-\beta}_{\varsigma_1^+,q}\left[(\zeta-\varsigma_1)^\alpha\prod_{j\neq\kappa}^n \Psi_j^{\delta j}\Psi_\kappa^\eta(\zeta)\right]\nabla^{-\lambda}_{\varsigma_1^+,q}\left[\prod_{j=1}^n \Psi_j^{\delta j}(\zeta)\right]\right.\\
&\quad\left. + \nabla^{-\lambda}_{\varsigma_1^+,q}\left[(\zeta-\varsigma_1)^\alpha\prod_{j\neq\kappa}^n \Psi_j^{\delta j}\Psi_\kappa^\eta(\zeta)\right]\nabla^{-\beta}_{\varsigma_1^+,q}\left[\prod_{j=1}^n \Psi_j^{\delta j}(\zeta)\right]\right) \geq 1.\end{aligned} \tag{38}$$

Proof. Taking product on both sides of (36) by $1/\Gamma_q(\lambda)(\zeta-q\theta)^{\lambda-1/}\prod_{j=1}^{n}\Psi_j^{\delta_j}(\omega), \omega\in(\varsigma_1,\zeta)$, and integrating for ω over (ς_1,ζ), we get

$$\begin{aligned}&\nabla_{\varsigma_1^+,q}^{-\beta}\left[\prod_{j\neq\kappa}^{n}\Psi_j^{\delta j}\Psi_\kappa^{\eta}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\lambda}\left[(\zeta-\varsigma_1)^{\alpha}\prod_{j=1}^{n}\Psi_j^{\delta j}(\zeta)\right]\\&+\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\prod_{j\neq\kappa}^{n}\Psi_j^{\delta j}\Psi_\kappa^{\eta}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\beta}\left[(\zeta-\varsigma_1)^{\alpha}\prod_{j=1}^{n}\Psi_j^{\delta j}(\zeta)\right]\\&\geq\nabla_{\varsigma_1^+,q}^{-\beta}\left[(\zeta-\varsigma_1)^{\alpha}\prod_{j\neq\kappa}^{n}\Psi_j^{\delta j}\Psi_\kappa^{\eta}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\prod_{j=1}^{n}\Psi_j^{\delta j}(\zeta)\right]\\&+\nabla_{\varsigma_1^+,q}^{-\lambda}\left[(\zeta-\varsigma_1)^{\alpha}\prod_{j\neq\kappa}^{n}\Psi_j^{\delta j}\Psi_\kappa^{\eta}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\beta}\left[\prod_{j=1}^{n}\Psi_j^{\delta j}(\zeta)\right].\end{aligned}\tag{39}$$

Dividing the above inequality by

$$\begin{aligned}&\nabla_{\varsigma_1^+,q}^{-\beta}\left[(\zeta-\varsigma_1)^{\alpha}\prod_{j\neq\kappa}^{n}\Psi_j^{\delta j}\Psi_\kappa^{\eta}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\prod_{j=1}^{n}\Psi_j^{\delta j}(\zeta)\right]\\&+\nabla_{\varsigma_1^+,q}^{-\lambda}\left[(\zeta-\varsigma_1)^{\alpha}\prod_{j\neq\kappa}^{n}\Psi_j^{\delta j}\Psi_\kappa^{\eta}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\beta}\left[\prod_{j=1}^{n}\Psi_j^{\delta j}(\zeta)\right],\end{aligned}\tag{40}$$

gives the desired inequality (38).

Theorem 15. *Let $\alpha>0$, $\eta\geq\delta_\kappa>0$ for any fixed $\kappa\in\{1,2,3,\cdots,n\}$, $\beta\in\mathbb{C}$ with $\mathfrak{R}(\beta)>0$, and $\hbar$ and $\Psi_j(j=1,2,3,\cdots,n)$ be the continuous positive decreasing functions defined on $\mathbb{T}_{t_0}$. Then, the time-fractional q -integral satisfies the inequality*

$$\frac{\nabla_{\varsigma_1^+,q}^{-\beta}\left[\prod_{j\neq\kappa}^{n}\Psi_j^{\delta j}\Psi_\kappa^{\eta}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\beta}\left[\hbar^{\alpha}(\zeta)\prod_{j=1}^{n}\Psi_j^{\delta j}(\zeta)\right]}{\nabla_{\varsigma_1^+,q}^{-\beta}\left[\hbar^{\alpha}(\zeta)\prod_{j\neq\kappa}^{n}\Psi_j^{\delta j}\Psi_\kappa^{\eta}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\beta}\left[\prod_{j=1}^{n}\Psi_j^{\delta j}(\zeta)\right]}\geq 1.\tag{41}$$

Proof. It follows from the given hypothesis that

$$(\hbar^{\alpha}(\omega)-\hbar^{\alpha}(\varphi))\left(\Psi_\kappa^{\eta-\delta\kappa}(\varphi)-\Psi_\kappa^{\eta-\delta\kappa}(\omega)\right)\geq 0,\tag{42}$$

for any fixed $\kappa\in\{1,2,3,\cdots,n\}, \alpha>0,\ \eta\geq\delta_\kappa>0$, and $\varphi,\omega\in[\varsigma_1,\zeta]$.

Inequality (42) leads to

$$\begin{aligned}&\hbar^{\alpha}(\omega)\Psi_\kappa^{\eta-\delta\kappa}(\varphi)+\hbar^{\alpha}(\varphi)\Psi_\kappa^{\eta-\delta\kappa}(\omega)-\hbar^{\alpha}(\omega)\Psi_\kappa^{\eta-\delta\kappa}(\omega)\\&-\hbar^{\alpha}(\varphi)\Psi_\kappa^{\eta-\delta\kappa}(\varphi)\geq 0.\end{aligned}\tag{43}$$

Taking the product on both sides of (43) by $1/\Gamma_q(\beta)(\zeta-q\varphi)^{\beta-1/}\prod_{j=1}^{n}\Psi_j^{\delta_j}(\varphi), \varphi\in(\varsigma_1,\zeta)$, and integrating for φ over (ς_1,ζ), we obtain

$$\begin{aligned}&\hbar^{\alpha}(\omega)\frac{1}{\Gamma_q(\beta)}(\zeta-q\varphi)^{\overline{\beta-1}}\prod_{j=1}^{n}\Psi_j^{\delta_j}(\varphi)\Psi_\kappa^{\eta-\delta\kappa}(\varphi)\\&+\hbar^{\alpha}(\varphi)\frac{1}{\Gamma_q(\beta)}(\zeta-q\varphi)^{\overline{\beta-1}}\prod_{j=1}^{n}\Psi_j^{\delta_j}(\varphi)\Psi_\kappa^{\eta-\delta\kappa}(\omega)\\&-\hbar^{\alpha}(\omega)\frac{1}{\Gamma_q(\beta)}(\zeta-q\varphi)^{\overline{\beta-1}}\prod_{j=1}^{n}\Psi_j^{\delta_j}(\varphi)\Psi_\kappa^{\eta-\delta\kappa}(\omega)\\&-\hbar^{\alpha}(\varphi)\frac{1}{\Gamma_q(\beta)}(\zeta-q\varphi)^{\overline{\beta-1}}\prod_{j=1}^{n}\Psi_j^{\delta_j}(\varphi)\Psi_\kappa^{\eta-\delta\kappa}(\varphi)\geq 0.\end{aligned}\tag{44}$$

Integrating (44) for φ over (ς_1,ζ), we have

$$\begin{aligned}&\hbar^{\alpha}(\omega)\frac{1}{\Gamma_q(\beta)}\int_{\varsigma_1}^{\zeta}(\zeta-q\varphi)^{\overline{\beta-1}}\prod_{j=1}^{n}\Psi_j^{\delta_j}(\varphi)\Psi_\kappa^{\eta-\delta\kappa}(\varphi)\nabla\varphi\\&+\hbar^{\alpha}(\varphi)\frac{1}{\Gamma_q(\beta)}\int_{\varsigma_1}^{\zeta}(\zeta-q\varphi)^{\overline{\beta-1}}\prod_{j=1}^{n}\Psi_j^{\delta_j}(\varphi)\Psi_\kappa^{\eta-\delta\kappa}(\omega)\nabla\varphi\\&-\hbar^{\alpha}(\omega)\frac{1}{\Gamma_q(\beta)}\int_{\varsigma_1}^{\zeta}(\zeta-q\varphi)^{\overline{\beta-1}}\prod_{j=1}^{n}\Psi_j^{\delta_j}(\varphi)\Psi_\kappa^{\eta-\delta\kappa}(\omega)\nabla\varphi\\&-\hbar^{\alpha}(\varphi)\frac{1}{\Gamma_q(\beta)}\int_{\varsigma_1}^{\zeta}(\zeta-q\varphi)^{\overline{\beta-1}}\prod_{j=1}^{n}\Psi_j^{\delta_j}(\varphi)\Psi_\kappa^{\eta-\delta\kappa}(\varphi)\nabla\varphi\geq 0.\end{aligned}\tag{45}$$

From (43), we clearly see that

$$\begin{aligned}&\hbar^{\alpha}(\omega)\nabla_{\varsigma_1^+,q}^{-\beta}\left[\prod_{j\neq\kappa}^{n}\Psi_j^{\delta j}\Psi_\kappa^{\eta}(\zeta)\right]+\Psi_\kappa^{\eta-\delta\kappa}(\zeta)\nabla_{\varsigma_1^+,q}^{-\beta}\left[\hbar^{\alpha}(\zeta)\prod_{j=1}^{n}\Psi_j^{\delta j}(\zeta)\right]\\&-\hbar^{\alpha}(\omega)\Psi_\kappa^{\eta-\delta\kappa}(\omega)\nabla_{\varsigma_1^+,q}^{-\beta}\left[\prod_{j=1}^{n}\Psi_j^{\delta j}(\zeta)\right]-\nabla_{\varsigma_1^+,q}^{-\beta}\left[\hbar^{\alpha}(\zeta)\prod_{j\neq\kappa}^{n}\Psi_j^{\delta j}\Psi_\kappa^{\eta}(\zeta)\right]\geq 0.\end{aligned}\tag{46}$$

Again, taking the product on both sides of (46) by $1/\Gamma_q(\beta)(\zeta-q\theta)\prod_{j=1}^{n}\Psi_j^{\delta_j}(\omega), \omega\in(\varsigma_1,\zeta)$, and integrating for ω over (ς_1,ζ), we have

$$\begin{aligned}&\nabla_{\varsigma_1^+,q}^{-\beta}\left[\prod_{j\neq\kappa}^{n}\Psi_j^{\delta j}\Psi_\kappa^{\eta}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\beta}\left[\hbar^{\alpha}(\zeta)\prod_{j=1}^{n}\Psi_j^{\delta j}(\zeta)\right]\\&-\nabla_{\varsigma_1^+,q}^{-\beta}\left[\hbar^{\alpha}(\zeta)\prod_{j\neq\kappa}^{n}\Psi_j^{\delta j}\Psi_\kappa^{\eta}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\beta}\left[\prod_{j=1}^{n}\Psi_j^{\delta j}(\zeta)\right]\geq 0,\end{aligned}\tag{47}$$

which completes the proof of the desired inequality (41).

Theorem 16. *Let $\alpha>0,\eta\geq\delta_\kappa>0$ for any fixed $\kappa\in\{1,2,3,\cdots,n\}$, $\beta,\lambda\in\mathbb{C}$, with $\mathfrak{R}(\beta)>0,\mathfrak{R}(\lambda)>0$, $\{\Psi_j, j=1,2,3,\cdots,n\}$ be a sequence of continuous positive decreasing functions defined on $\mathbb{T}_{t_0}$ and $\hbar$ be a continuous positive increasing functions defined on $\mathbb{T}_{t_0}$. Then*

$$\left(\nabla_{\varsigma_1^+,q}^{-\beta}\left[\prod_{j\neq\kappa}^{n}\Psi_j^{\delta_j}\Psi_\kappa^{\eta}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\hbar^{\alpha}(\zeta)\prod_{j=1}^{n}\Psi_j^{\delta_j}(\zeta)\right] + \nabla_{\varsigma_1^+,q}^{-\beta}\left[\hbar^{\alpha}(\zeta)\prod_{j=1}^{n}\Psi_j^{\delta_j}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\prod_{j\neq\kappa}^{n}\Psi_j^{\delta_j}\Psi_\kappa^{\eta}(\zeta)\right]\right)/ \left(\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\prod_{j=1}^{n}\Psi_j^{\delta_j}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\beta}\left[\hbar^{\alpha}(\zeta)\prod_{j\neq\kappa}^{n}\Psi_j^{\delta_j}\Psi_\kappa^{\eta}(\zeta)\right] + \nabla_{\varsigma_1^+,q}^{-\lambda}\left[\hbar^{\alpha}(\zeta)\prod_{j\neq\kappa}^{n}\Psi_j^{\delta_j}\Psi_\kappa^{\eta}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\beta}\left[\prod_{j=1}^{n}\Psi_j^{\delta_j}(\zeta)\right]\right) \geq 1. \tag{48}$$

Proof. Multiplying both sides of (46) by $1/\Gamma_q(\lambda)(\zeta - q\omega)^{\lambda-1}/\prod_{j=1}^{n}\Psi_j^{\delta_j}(\omega), \omega \in (\varsigma_1, \zeta)$, and integrating for ω over (ς_1, ζ), we have

$$\nabla_{\varsigma_1^+,q}^{-\beta}\left[\prod_{j\neq\kappa}^{n}\Psi_j^{\delta_j}\Psi_\kappa^{\eta}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\hbar^{\alpha}(\zeta)\prod_{j=1}^{n}\Psi_j^{\delta_j}(\zeta)\right] + \nabla_{\varsigma_1^+,q}^{-\beta}\left[\hbar^{\alpha}(\zeta)\prod_{j=1}^{n}\Psi_j^{\delta_j}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\prod_{j\neq\kappa}^{n}\Psi_j^{\delta_j}\Psi_\kappa^{\eta}(\zeta)\right] - \nabla_{\varsigma_1^+,q}^{-\lambda}\left[\prod_{j=1}^{n}\Psi_j^{\delta_j}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\beta}\left[\hbar^{\alpha}(\zeta)\prod_{j\neq\kappa}^{n}\Psi_j^{\delta_j}\Psi_\kappa^{\eta}(\zeta)\right] - \nabla_{\varsigma_1^+,q}^{-\lambda}\left[\hbar^{\alpha}(\zeta)\prod_{j\neq\kappa}^{n}\Psi_j^{\delta_j}\Psi_\kappa^{\eta}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\beta}\left[\prod_{j=1}^{n}\Psi_j^{\delta_j}(\zeta)\right] \geq 0. \tag{49}$$

It follows that

$$\nabla_{\varsigma_1^+,q}^{-\beta}\left[\prod_{j\neq\kappa}^{n}\Psi_j^{\delta_j}\Psi_\kappa^{\eta}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\hbar^{\alpha}(\zeta)\prod_{j=1}^{n}\Psi_j^{\delta_j}(\zeta)\right] + \nabla_{\varsigma_1^+,q}^{-\beta}\left[\hbar^{\alpha}(\zeta)\prod_{j=1}^{n}\Psi_j^{\delta_j}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\prod_{j\neq\kappa}^{n}\Psi_j^{\delta_j}\Psi_\kappa^{\eta}(\zeta)\right] \geq \nabla_{\varsigma_1^+,q}^{-\lambda}\left[\prod_{j=1}^{n}\Psi_j^{\delta_j}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\beta}\left[\hbar^{\alpha}(\zeta)\prod_{j\neq\kappa}^{n}\Psi_j^{\delta_j}\Psi_\kappa^{\eta}(\zeta)\right] + \nabla_{\varsigma_1^+,q}^{-\lambda}\left[\hbar^{\alpha}(\zeta)\prod_{j\neq\kappa}^{n}\Psi_j^{\delta_j}\Psi_\kappa^{\eta}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\beta}\left[\prod_{j=1}^{n}\Psi_j^{\delta_j}(\zeta)\right]. \tag{50}$$

Dividing both sides of the above inequality by

$$\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\prod_{j=1}^{n}\Psi_j^{\delta_j}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\beta}\left[\hbar^{\alpha}(\zeta)\prod_{j\neq\kappa}^{n}\Psi_j^{\delta_j}\Psi_\kappa^{\eta}(\zeta)\right] + \nabla_{\varsigma_1^+,q}^{-\lambda}\left[\hbar^{\alpha}(\zeta)\prod_{j\neq\kappa}^{n}\Psi_j^{\delta_j}\Psi_\kappa^{\eta}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\beta}\left[\prod_{j=1}^{n}\Psi_j^{\delta_j}(\zeta)\right], \tag{51}$$

gives desired inequality (48).

4. Conclusion

In this note, we have derived certain variants by using the time-fractional q-calculus operator related to a class of n positive continuous, and decreasing functions on the interval $[\varsigma_1, \varsigma_2]$ are elaborated. In [66], Liu et al. investigated thought-provoking integral inequalities for continuous functions on $[\varsigma_1, \varsigma_2]$. Recently, Dahmani [67] has presented the more generalizations of the work of [66] by utilizing the Riemann-Liouville fractional integral operators. If we take into account $\mathbb{T} = \mathbb{R}$ and $q = 1$, then our findings are the special cases of the results proposed by Dahmani [67]. The established relationship highlighted the importance of selecting appropriate combinations and validated q-fractional time scale approaches for special relativity theory and quantum mechanics. From the existence and uniqueness viewpoint, it is found that the q-fractional order controls potentially provide the tools to better represent measured that cannot be fit to the classical model.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

L. Xu provided the main ideas of the article and carried out the proof of Theorem 9. Y.-M. Chu drafted the manuscript and carried out the proof of Theorem 10. S. Rashid carried out the proof of Theorem 11 and Theorem 12, completed the final revision, and submitted the article. A. A.El-Deeb carried out the proof of Theorems 13 and 14. K. S. Nisar carried out the proof of Theorems 15 and 16. All authors read and approved the final manuscript.

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