Research Article

Existence of $W_0^{1,1}(\Omega)$ Solutions to Non-Coercivity Quasilinear Elliptic Problem

Jimao Xiawu ¹, Shuibo Huang ^{1,2}, Yingyuan Mi,¹ and Maoji Ri¹

¹School of Mathematics and Computer Science, Northwest Minzu University, Lanzhou, Gansu 730030, China

²Key Laboratory of Streaming Data Computing Technologies and Application, Northwest Minzu University, Lanzhou, Gansu 730030, China

Correspondence should be addressed to Shuibo Huang; huangshuibo2008@163.com

Received 19 June 2019; Revised 30 July 2019; Accepted 22 August 2019; Published 26 February 2020

Academic Editor: Pasquale Vetro

Copyright © 2020 Jimao Xiawu et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In this paper we consider the existence of $W_0^{1,1}(\Omega)$ solutions to following kind of problems $\left\{ -\operatorname{div}(|\nabla u|^{p-2}\nabla u/(1+|u|)^{\theta(p-1)}) = f(x), \quad x \in \Omega; \quad u(x) = 0, \quad x \in \partial\Omega \right\}$ where Ω is an open bounded subset of $\mathbb{R}^N$ ($N > 2$), $\{(p-2)N+1)/(p-1)(N-1), 0\} < \theta < 1$ and $1 < p \leq 1 + (N-1)/(N(1-\theta) + \theta)$, f is a function which belongs to a suitable integrable space.

1. Introduction

In this paper, we study the following kind of problems

$$\begin{aligned} -\operatorname{div}(a(x, u, \nabla u)) &= f(x), & x \in \Omega, \\ u(x) &= 0, & x \in \partial\Omega, \end{aligned} \quad (1)$$

where Ω is an open bounded subset of $\mathbb{R}^N$ ($N > 2$), f is a function which belongs to a suitable integrable space and $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a Carathéodory function, which satisfies the following assumptions: for almost every $x \in \Omega$ and every $s \in \mathbb{R}$

$$\frac{\alpha}{(1+|s|)^{\theta(p-1)}}|\xi|^p \leq a(x, s, \xi)\xi, \quad (2)$$

$$|a(x, s, \xi)| \leq \beta|\xi|^{p-1}, \quad (3)$$

$$[a(x, s, \xi) - a(x, s, \eta)] \cdot (\xi - \eta) > 0, \quad (4)$$

for every $\xi, \eta \in \mathbb{R}^N$ with $\xi \neq \eta$, where α, β are positive constants and $0 \leq \theta < 1$.

We note that there is a difficulty in dealing with (2) under the assumption (3). Indeed the differential operator

$$A(u) = -\operatorname{div}(a(x, u, \nabla u)), \quad (5)$$

is well defined in $W_0^{1,p}(\Omega)$, but is not coercive on the same space when u is unbounded. Therefore, the classical Leray—Lions theorem cannot be apply even in the case the datum f belongs to $W^{-1,p'}(\Omega)$ for $p' = p/(p-1)$. In order to overcome this difficulty,

we will proceed by approximation by means of truncatures in $a(x, s, \xi)$ to get a coercive differential operator on $W_0^{1,p}$. In particular in [1–3], the authors analyzed the problems

$$\begin{aligned} -\operatorname{div}(a(x, u)\nabla u) &= f, & x \in \Omega, \\ u(x) &= 0, & x \in \partial\Omega, \end{aligned} \quad (6)$$

where $f \in L^m(\Omega)$ with $m \geq 1$, $a(x, s) : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function such that

$$\frac{\alpha}{(1+|s|)^{\gamma}} \leq a(x, s) \leq \beta, \quad (7)$$

for all $s \in \mathbb{R}$ and $0 < \gamma \leq 1$. Denote

$$r = \frac{Nm(1-\gamma)}{N-2m}, \quad q = \frac{Nm(1-\gamma)}{N-m(1+\gamma)}. \quad (8)$$

Main results of [1–3] summarized as:

- (1) If $f \in L^m(\Omega)$ with $1 \leq m \leq \max\{N/(N+1-\gamma(N-1)), 1\}$, then there exists a solution u to problem (6) such that

$$|u|^s \in L^1(\Omega), \quad s < r \quad \text{and} \quad |\nabla u|^s \in L^1(\Omega), \quad s < q. \quad (9)$$

- (2) If $N/(N+1-\gamma(N-1)) < m \leq (2N)/(N(1-\gamma) + 2(\gamma+1))$, then $u \in W_0^{1,q}(\Omega)$;

- (3) If $2N/(N(1-\gamma) + 2(\gamma+1)) \leq m < N/2$, then $u \in H_0^1(\Omega) \cap L^r(\Omega)$;

- (4) If $m > N/2$, then $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

In the case where $\gamma > 1$, problem (6) may have no solution at all, even if the data f is a constant (see [1, 3]). Thus the non-coerciveness of operator A has great effect on the existence and regularity of the solution. Some other results of non-coercivity quasilinear elliptic problem, see [4–24].

Furthermore, in the literature [21], they proved the L^∞ -regularity for solutions to the following nonlinear elliptic equations with degenerate coercivity

$$\begin{aligned} -\operatorname{div}(a(x, u, \nabla u)) &= f, & x \in \Omega, \\ u(x) &= 0, & x \in \partial\Omega, \end{aligned} \quad (10)$$

where Ω is an open bounded subset of $\mathbb{R}^N$ ($N > 2$), $1 < p < N$, f satisfies

$$f \in L^{N/p \log^\alpha L} \quad (11)$$

with $\alpha > (N(p-1))/p$, and $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a Carathéodory function which satisfies the following assumptions

$$a(x, s, \xi) \geq h^{p-1}(|s|)|\xi|^p, \quad (12)$$

$$|a(x, s, \xi)| \leq a_0(x) + |s|^{p-1} + |\xi|^{p-1}, \quad (13)$$

$$(a(x, s, \xi) - a(x, s, \xi')) \cdot (\xi - \xi') > 0, \quad (14)$$

for almost every $x \in \Omega$, for every $s \in \mathbb{R}$, where a_0 is a nonnegative function in $L^{p'}$ with $p' = p/(p-1)$, and

$$h(t) = \frac{1}{(1 + |t|)^\theta}, \quad (15)$$

where θ is a real such that $0 \leq \theta \leq 1$ and $h : \mathbb{R}^+ \rightarrow [0, \infty]$ is a decreasing continuous function such that its primitive

$$H(s) = \int_0^s h(t) dt, \quad (16)$$

is unbounded. Under the hypothesis (11)–(14), the authors proved that (10) has at least a weak solution $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ in the sense that

$$\int_\Omega a(x, u, \nabla u) \cdot \nabla v dx = \int_\Omega f v dx \quad (17)$$

for all v in $C_0^\infty(\Omega)$.

On the other hand, as in [22], it has been shown the existence results of solutions for nonlinear elliptic boundary value problem

$$\begin{aligned} -\operatorname{div}(|\nabla u|^{p-2} \nabla u) &= f, & x \in \Omega, \\ u(x) &= 0, & x \in \partial\Omega, \end{aligned} \quad (18)$$

where $f \in L^m$ ($m \geq 1$), $1 < p < N$. We recall the following results contained in [22].

- (1) Let $f \in L^m(\Omega)$, $m = N/(N(p-1) + 1)$, $1 < p < 2 - (1/N)$. Then there exists a distributional solution $u \in W_0^{1,1}(\Omega)$ of (18).
- (2) Let $p = 2 - (1/N)$, and if $f \log(1 + |f|) \in L^1(\Omega)$, then there exists a distributional solution $u \in W_0^{1,1}(\Omega)$ of (18).

The aim of this article is to study the existence of $W_0^{1,1}(\Omega)$ solutions to the problem (1). Our main results are following:

Theorem 1. Assume (2)–(4) hold and $f \in L^m(\Omega)$ with

$$\begin{aligned} m &= \frac{N}{N(p-1) + 1 - \theta(p-1)(N-1)}, \\ \max \left\{ \frac{(p-2)N + 1}{(p-1)(N-1)}, 0 \right\} &< \theta < 1. \end{aligned} \quad (19)$$

Then problem (1) exists a distributional solution $u \in W_0^{1,1}(\Omega)$ provided

$$1 < p < 1 + \frac{N-1}{N(1-\theta) + \theta}. \quad (20)$$

Remark 2. We point out that the existence in $W_0^{1,1}(\Omega)$ is not so usual in the study of elliptic problems. The main difficulty of study is due to the a priori estimate of the sequence of the approximate solutions $\{u_n\}$ of (18) in the non-reflexive space $W_0^{1,1}(\Omega)$. For more results about existence of $W_0^{1,1}(\Omega)$ to elliptic equations, see [23–25].

Remark 3. Observe that, $m > 1$ if (20) holds.

Remark 4. It can be easily seen that, (20) implies $((p-2)N + 1)/((p-1)(N-1)) < \theta$, while $1 + (N-1)/(N(1-\theta) + \theta) < 2 - (1/N)$. Therefore, in (20), we should require $\max\{((p-2)N + 1)/((p-1)(N-1)), 0\} < \theta$, instead of $((p-2)N + 1)/((p-1)(N-1)) < \theta$.

Remark 5. It is worth pointing out that, the existence of $W_0^{1,1}(\Omega)$ solutions to (18) was established provided $1 < p < 2 - (1/N)$ and $f \in L^m(\Omega)$ with $m = N/(N(p-1) + 1)$. According to Remark 4, we know that problem (1) has $u \in W_0^{1,1}(\Omega)$ solution if p satisfies.

$$2 - \frac{1}{N} \leq p < 1 + \frac{N-1}{N(1-\theta) + \theta}. \quad (21)$$

This fact shows that the lack of coercivity of the differential operator $a(x, u, \nabla u)$ has the great influence on the existence of $W_0^{1,1}(\Omega)$ solutions to (1). Especially, for $p = 2 - (1/N)$, we obtain the existence of $W_0^{1,1}(\Omega)$ solutions for $f \in L^m(\Omega)$ with $m = (N^2)/(N^2 - \theta(N-1)^2) > 1$, rather than $f \log(1 + |f|) \in L^1(\Omega)$.

For $f \log(1 + |f|) \in L^1(\Omega)$, we have the following theorem.

Theorem 6. Assume (2)–(4) hold. Then problem (1) has a distributional solution $u \in W_0^{1,1}(\Omega)$ if

$$\int_\Omega |f| \log(1 + |f|) < \infty \text{ and } p = 1 + \frac{N-1}{N(1-\theta) + \theta}. \quad (22)$$

Remark 7. Note that if $\theta = 0$, then $p = 2 - 1/N$.

The paper is organized as follows. In Section 2 we collect some definitions and useful tools. The proof of Theorems 1 and 6 is given in Section 3.

2. Preliminaries

In order to prove our main results, we need to introduce some basic definitions and lemmas. We shall use the following notion of distributional solution for problem (1).

Definition 8. We say that $u \in W_0^{1,1}(\Omega)$ is a distributional solution to problem (1) if

$$\int_{\Omega} a(x, u, \nabla u) \cdot \nabla \varphi = \int_{\Omega} f \varphi, \quad (23)$$

for every $\varphi \in C_0^\infty(\Omega)$.

Definition 9. For all $k \geq 0$, the truncation function is defined by

$$T_k(s) = \max\{-k, \min\{k, s\}\}. \quad (24)$$

We recall the following lemma (see [26]).

Lemma 10. Let $u : \Omega \rightarrow \mathbb{R}$ be measurable function such that $T_k(u) \in H_0^1(\Omega)$ for $k > 0$. Then there exists a unique measurable $v : \Omega \rightarrow \mathbb{R}^N$ such that

$$\nabla T_k(u) = v \chi_{\{|u| < k\}} \text{ a.e. in } \Omega, \quad \forall k > 0, \quad (25)$$

the map v is called the weak gradient of u . Moreover if $u \in W_0^{1,1}(\Omega)$, then v coincides with the standard distributional gradient of u .

Definition 11. Let $p \geq 1$ and $\alpha > 0$, the Zygmund space $L^p \log^\alpha L$, is made up of all measurable functions g such that

$$\int_{\Omega} |g|^p \log^\alpha(1 + |g|) dx < \infty, \quad (26)$$

for g on Ω .

Remark 12. Observe that if $p = 1$ and $\alpha = 1$, then the (26) becomes

$$\int_{\Omega} |f| \log(1 + |f|) < \infty. \quad (27)$$

We obtain the existence solutions to (2) by considering the following approximation problems

$$\begin{aligned} -\operatorname{div}(a(x, T_n(u_n), \nabla u_n)) &= f, & x \in \Omega, \\ u_n(x) &= 0, & x \in \partial\Omega. \end{aligned} \quad (28)$$

Lemma 13. Let u_n be solutions to (28) and $f \in L^1(\Omega)$, $p > 1$. Then the sequence $\log(1 + |u_n|) \operatorname{sgn}(u_n)$ is bounded in $W_0^{1,p}(\Omega)$.

Proof. Let us take $\left[1 - (1 + |u_n|)^{-(1-\theta)(p-1)}\right] \operatorname{sgn}(u_n)$ as test function in (28), we get

$$\begin{aligned} C \int_{\Omega} |\nabla \log(1 + |u_n|)^p| &\leq C \int_{\Omega} \frac{|\nabla u_n|^p}{(1 + |u_n|)^p} \\ &\leq \int_{\Omega} |f| \left[1 - (1 + |u_n|)^{-(1-\theta)(p-1)}\right] \\ &\leq \|f\|_{L^1(\Omega)}. \end{aligned} \quad (29)$$

Then we arrive to the conclusion.

In order to prove the existence part of Theorems 1 and 6 we need the following preliminary result. This convergence of the gradients is a consequence of a classical result (see [22]).

Lemma 14. Let u_n be the sequence of solutions to (28), which are bounded in $W_0^{1,p}(\Omega)$. Assume $1 < p < N$, $f \in L^m(\Omega)$ ($m \geq 1$) and (2)–(4) hold. Suppose that

- (1) u_n is such that $\|u_n\|_{W_0^{1,1}(\Omega)} \leq M$ for constant M ;
- (2) u_n converges to u almost everywhere in Ω ;
- (3) $\nabla T_k(u_n)$ converges weakly to $\nabla T_k(u)$ in $W_0^{1,p}(\Omega)$.

Then, up to a subsequence, ∇u_n converges almost everywhere in Ω to ∇u .

3. Proof of Main Results

In this section, we prove the existence results of Theorems 1 and 6 by a classical approach. Consider the following approximate problems

$$\begin{aligned} -\operatorname{div}(a(x, T_n(u_n), \nabla u_n)) &= T_n(f), & x \in \Omega, \\ u_n(x) &= 0, & x \in \partial\Omega. \end{aligned} \quad (30)$$

Proof of Theorem 1. Step 1: Let

$$\sigma = \frac{pm' - p^* \theta}{p'(pm' - p^*)}, \quad (31)$$

and we know the function

$$v = \left[(1 + |u_n|)^{1-p(1-\sigma)}\right] \operatorname{sgn}(u_n) \quad (32)$$

is bounded since $1 - p(1 - \sigma) > 0$. Thus, choose v as a test function in the approximate problem (28), we find

$$\begin{aligned} C \int_{\Omega} \frac{|\nabla u_n|^p}{(1 + |u_n|)^{\theta(p-1)+p(1-\sigma)}} \\ \leq \|f\|_{L^m(\Omega)} \left(\int_{\Omega} \left((1 + |u_n|)^{1-p(1-\sigma)} \right)^{m'} \right)^{1/m'}, \end{aligned} \quad (33)$$

where C denotes a strictly positive constant, depending only on α, p, N, σ and $\|f\|_{L^m(\Omega)}$. By the Sobolev embedding theorem, we have

$$C \left(\int_{\Omega} (1 + |u_n|)^{(\sigma - (\theta/p'))p^*} \right)^{p/p^*} \leq \int_{\Omega} \frac{|\nabla u_n|^p}{(1 + |u_n|)^{\theta(p-1)+p(1-\sigma)}}. \quad (34)$$

Combining (34), (35), then

$$\begin{aligned}
& C \left(\int_{\Omega} (1 + |u_n|)^{(\sigma - (\theta/p'))p^*} \right)^{p/p^*} \\
& \leq C \int_{\Omega} \frac{|\nabla u_n|^p}{(1 + |u_n|)^{\theta(p-1)+p(1-\sigma)}} \\
& \leq \|f\|_{L^m(\Omega)} \left(\int_{\Omega} (1 + |u_n|)^{[1-p(1-\sigma)]m'} \right)^{1/m'}.
\end{aligned} \tag{35}$$

Note that $(p/p^*) > (1/m')$ since $m < (N/p)$. Moreover the choice of σ implies

$$\left(\sigma - \frac{\theta}{p'} \right) p^* = (1 - p(1 - \sigma))m' = \frac{N}{N-1}. \tag{36}$$

Thus we prove that

$$C \left(\int_{\Omega} |u_n|^{N/(N-1)} \right)^{(1/m)-(p/N)} \leq \|f\|_{L^m(\Omega)}. \tag{37}$$

This estimate implies that

$$\int_{\Omega} \frac{|\nabla u_n|^p}{(1 + |u_n|)^{\theta(p-1)+p(1-\sigma)}} < \infty. \tag{38}$$

Now we can estimate $\int_{\Omega} |\nabla u_n|$. Indeed, apply the Hölder inequality and we obtain

$$\begin{aligned}
\int_{\Omega} \nabla u_n &= \int_{\Omega} \frac{\nabla u_n}{(1 + |u_n|)^{(\theta/p)(p-1)+(1-\sigma)}} (1 + |u_n|)^{(\theta/p)(p-1)+(1-\sigma)} \\
&\leq \left(\int_{\Omega} \frac{|\nabla u_n|^p}{(1 + |u_n|)^{\theta(p-1)+p(1-\sigma)}} \right)^{(1/p)} \\
&\quad \cdot \left(\int_{\Omega} (1 + |u_n|)^{\theta+p'(1-\sigma)} \right)^{(1/p')}.
\end{aligned} \tag{39}$$

Note that $\theta + p'(1 - \sigma) = N/(N - 1)$, so the right hand side is bounded. Then we get that the sequence $\{u_n\}$ is uniformly bounded in $W_0^{1,1}(\Omega)$, subsequently there exist $M > 0$ such that

$$\|u_n\|_{W_0^{1,1}(\Omega)} \leq M. \tag{40}$$

As a consequence there exists a subsequence $\{u_n\}$ converging to u a.e. in $L^r(\Omega)$, $1 \leq r < N/(N - 1)$.

Step 2: Let us choose $T_k(u_n)$ as test function in (28)

$$\int_{\Omega} a(x, T_n(u_n), \nabla u_n) \cdot \nabla T_k(u_n) = \int_{\Omega} f_n T_k(u_n). \tag{41}$$

Using assumption (2) on a , we get

$$C \int_{\Omega} \frac{|\nabla u_n|^p}{(1 + T_k(u_n))^{\theta(p-1)}} \leq k \|f\|_{L^1(\Omega)}, \tag{42}$$

and so

$$\int_{\Omega} |\nabla T_k(u_n)|^p \leq C k^{\theta(p-1)+1}, \text{ for all } k > 0. \tag{43}$$

This implies that $T_k(u_n) \rightarrow T_k(u)$ weakly in $W_0^{1,p}(\Omega)$.

Now we are also going to estimate $\int_{\{k \leq |u_n|\}} |\nabla u_n|$. Choosing $\left[(1 + |u_n|)^{1-p(1-\sigma)} \right]^+ \text{sgn}(u_n)$ as a test function in (28), we have

$$\begin{aligned}
& \int_{\{k \leq |u_n|\}} \frac{|\nabla u_n|^p}{(1 + |u_n|)^{\theta(p-1)+p(1-\sigma)}} \\
& \leq \left(\int_{\{k \leq |u_n|\}} |f|^m \right)^{(1/m)} \left(\int_{\{k \leq |u_n|\}} (1 + |u_n|)^{[1-p(1-\sigma)]m'} \right)^{(1/m')} \\
& \leq C \left(\int_{\{k \leq |u_n|\}} |f|^m \right)^{(1/m)},
\end{aligned} \tag{44}$$

where we have use (37) in the last passage. By the Hölder inequality we get

$$\begin{aligned}
\int_{\{k \leq |u_n|\}} |\nabla u_n| &= \int_{\{k \leq |u_n|\}} \frac{\nabla u_n}{(1 + |u_n|)^{(\theta/p)(p-1)+(1-\sigma)}} (1 + |u_n|)^{(\theta/p)(p-1)+(1-\sigma)} \\
&\leq \left(\int_{\{k \leq |u_n|\}} \frac{|\nabla u_n|^p}{(1 + |u_n|)^{\theta(p-1)+p(1-\sigma)}} \right)^{(1/p)} \left(\int_{\{k \leq |u_n|\}} (1 + |u_n|)^{\theta+p'(1-\sigma)} \right)^{(1/p')} \\
&\leq C \left(\int_{\{k \leq |u_n|\}} |f|^m \right)^{(1/m)}.
\end{aligned} \tag{45}$$

Let E be a measurable subset of Ω , and let $i = 1, \dots, N$. Then by (43), (45) and Hölder inequality, we have

$$\begin{aligned}
\int_E \left| \frac{\partial u_n}{\partial x_i} \right| &\leq \int_E |\nabla u_n| \\
&\leq \int_E |\nabla T_k(u_n)| + \int_{\{k \leq |u_n|\}} |\nabla u_n| \\
&\leq \text{meas}(E)^{(1/p')} \left(\int_E |\nabla T_k(u_n)|^p \right)^{(1/p)} + \int_{\{k \leq |u_n|\}} |\nabla u_n| \\
&\leq C \text{meas}(E)^{(1/p')} k^{(1/p)[\theta(p-1)+1]} + C \left(\int_{\{k \leq |u_n|\}} |f|^m \right)^{(1/m)}.
\end{aligned} \tag{46}$$

The estimates (37) and (46) imply that the sequence $\{\partial u_n / \partial x_i\}$ is equiintegrable. Thus, by the Dunford—Pettis theorem, and up to subsequences, there exists Y_i in $L^1(\Omega)$ such that $\partial u_n / \partial x_i$ weakly converges to Y_i in $L^1(\Omega)$. Since $\partial u_n / \partial x_i$ is the distributional partial derivative of u_n , then for every n in $\mathbb{N}$, we have

$$\int_{\Omega} \partial_i u_n \varphi = - \int_{\Omega} u_n \partial_i \varphi, \tag{47}$$

where $\varphi \in C_0^\infty(\Omega)$. We now pass to the limit in the above identities, using that $\partial_i u_n$ weakly converges to Y_i in $L^1(\Omega)$, and that u_n strongly converges to u in $L^1(\Omega)$, we find

$$\int_{\Omega} Y_i \varphi = - \int_{\Omega} u \partial_i \varphi. \tag{48}$$

This implies that $Y_i = \partial u_n / \partial x_i$, and this result holds for every i . Since Y_i belongs to $L^1(\Omega)$ for every i , u belongs to $W_0^{1,1}(\Omega)$, as desired.

Step 3: We can apply Lemma 14 and conclude that $\nabla u_n \rightarrow \nabla u$ a.e., that is $a(x, T_n(u_n), \nabla u_n)$ almost everywhere converges to $a(x, u, \nabla u)$. Since u_n is bounded in $W_0^{1,p}(\Omega)$, the inequality

$$|a(x, u_n, \nabla u_n)| \leq \beta |\nabla u_n|^{p-1} \tag{49}$$

and the Vitali theorem imply that $a(x, u_n, \nabla u_n)$ converges to $a(x, u, \nabla u)$ in $(L^{1/(p-1)}(\Omega))^N$. Hence we are able to pass to the limit in (28). Thus we prove that $u \in W_0^{1,1}(\Omega)$ is a distributional solution of (2) and yields the conclusion of the proof of Theorem 1.

Proof of Theorem 6. Step 1: Let $[1 - (1 + |u_n|)^{1-\lambda}]$. Choosing $[1 - (1 + |u_n|)^{1-\lambda}] \operatorname{sgn}(u_n)$ as a test function in (28) and by (3), we get

$$C \int_{\Omega} \frac{|\nabla u_n|^p}{(1 + |u_n|^{\theta(p-1)+\lambda})} \leq \int_{\Omega} f_n [1 - (1 + |u_n|)^{1-\lambda}]. \quad (50)$$

Then using the Sobolev embedding theorem we have

$$\begin{aligned} & C \left(\int_{\Omega} \left((1 + |u_n|)^{1-(\theta(p-1)+\lambda)/p} - 1 \right)^{p^*} \right)^{p/p^*} \\ & \leq \int_{\Omega} \frac{|\nabla u_n|^p}{(1 + |u_n|^{\theta(p-1)+\lambda})}. \end{aligned} \quad (51)$$

So combining (50) and (51),

$$C \left(\int_{\Omega} \left((1 + |u_n|)^{1-(\theta(p-1)+\lambda)/p} - 1 \right)^{p^*} \right)^{p/p^*} \leq \|f\|_{L^1(\Omega)}. \quad (52)$$

Note that

$$\left(1 - \frac{\theta(p-1) + \lambda}{p} \right) p^* < \frac{N}{N-1} \quad (53)$$

since $\lambda > 1$, then we prove that the sequence $\{u_n\}$ is bounded in $L^r(\Omega)$, $1 \leq r < N/(N-1)$, that is

$$\|u_n\|_{L^r(\Omega)} \leq C. \quad (54)$$

Step 2: We use the following inequality: there exists $a_r > 0$ only depending on r such that

$$t \log(1 + s) \leq t \log(1 + t) + s^r + a_r, \quad (55)$$

for all $s, t \in \mathbb{R}_+$.

We take $[\log(1 + |u_n|)] \operatorname{sgn}(u_n)$ as a test function in the approximated problem (28) and by inequality (55), we obtain

$$\begin{aligned} & \alpha \int_{\Omega} \frac{|\nabla u_n|^p}{(1 + |u_n|^{\theta(p-1)+1})} \leq \int_{\Omega} |f| \log(1 + |u_n|) \\ & \leq \|f \log(1 + |f|)\|_{L^1(\Omega)} + \int_{\Omega} |u_n|^r + a_r \operatorname{meas}(\Omega). \end{aligned} \quad (56)$$

Now we can apply Hölder and Sobolev inequalities, since

$$\frac{p'}{p} [\theta(p-1) + 1] = \frac{N}{N-1} \text{ and } 1 - \frac{1}{p} = \frac{N-1}{(N-1)(1-\theta) + N}, \quad (57)$$

then

$$\begin{aligned} & S_{N,p} \alpha^{(1/p)} \left(\int_{\Omega} |u_n|^{N/(N-1)} \right)^{(N-1)/N} \leq \alpha^{(1/p)} \int_{\Omega} |\nabla u_n| \\ & = \int_{\Omega} \frac{|\nabla u_n|}{(1 + |u_n|)^{(1/p)[\theta(p-1)+1]}} \cdot (1 + |u_n|)^{(1/p)[\theta(p-1)+1]} \\ & \leq \left(\int_{\Omega} \frac{|\nabla u_n|^p}{(1 + |u_n|)^{\theta(p-1)+1}} \right)^{(1/p)} \\ & \quad \cdot \left(\int_{\Omega} (1 + |u_n|)^{(p'/p)[\theta(p-1)+1]} \right)^{(1/p')} \\ & \leq [\|f \log(1 + |f|)\|_{L^1(\Omega)} + C^r + a_r \operatorname{meas}(\Omega)]^{(1/p)} \\ & \quad \cdot \left(\int_{\Omega} (1 + |u_n|)^{N/(N-1)} \right)^{(N-1)/((N-1)(1-\theta)+N)}. \end{aligned} \quad (58)$$

Since $(N-1)/((N-1)(1-\theta)+N) < (N-1)/N$, we prove that the sequence $\{u_n\}$ is bounded in $W_0^{1,1}(\Omega)$ and so it is compact in $L^r(\Omega)$, $1 \leq r < N/(N-1)$.

Step 3: Choosing $[\log(1 + |u_n|) - \log(1 + k)] \operatorname{sgn}(u_n)$ as a test function in (28), we get

$$\begin{aligned} & \alpha \int_{\{k \leq |u_n|\}} \frac{|\nabla u_n|^p}{(1 + |u_n|^{\theta(p-1)+1})} \leq \int_{\{k \leq |u_n|\}} |f| \log(1 + |u_n|) \\ & \leq \int_{\{k \leq |u_n|\}} f \log(1 + |f|) + \int_{\{k \leq |u_n|\}} |u_n|^r + a_r \operatorname{meas}\{k \leq |u_n|\}. \end{aligned} \quad (59)$$

In addition,

$$\begin{aligned} & \int_{\{k \leq |u_n|\}} |\nabla u_n| = \int_{\{k \leq |u_n|\}} \frac{|\nabla u_n|}{(1 + |u_n|)^{(1/p)[\theta(p-1)+1]}} \\ & \quad \cdot (1 + |u_n|)^{(1/p)[\theta(p-1)+1]} \\ & \leq \left(\int_{\{k \leq |u_n|\}} \frac{|\nabla u_n|^p}{(1 + |u_n|)^{\theta(p-1)+1}} \right)^{(1/p)} \\ & \quad \cdot \left(\int_{\{k \leq |u_n|\}} (1 + |u_n|)^{(p'/p)[\theta(p-1)+1]} \right)^{(1/p')} \\ & \leq C \left[\int_{\{k \leq |u_n|\}} f \log(1 + |f|) \right. \\ & \quad \left. + \int_{\{k \leq |u_n|\}} |u_n|^r + a_r \operatorname{meas}\{k \leq |u_n|\} \right]^{(1/p)}. \end{aligned} \quad (60)$$

Therefore, for every measurable subset E , following (46) and we have

$$\begin{aligned} & \int_E \left| \frac{\partial u_n}{\partial x_i} \right| \leq \int_E |\nabla u_n| \leq \int_E |\nabla T_k(u_n)| \\ & \quad + \int_{\{k \leq |u_n|\}} |\nabla u_n| \leq C \operatorname{meas}(E)^{(1/p')} k^{(1/p)[\theta(p-1)+1]} \\ & \quad + C \left[\int_{\{k \leq |u_n|\}} f \log(1 + |f|) + \int_{\{k \leq |u_n|\}} |u_n|^r \right. \\ & \quad \left. + a_r \operatorname{meas}\{k \leq |u_n|\} \right]^{(1/p)}. \end{aligned} \quad (61)$$

Thus we prove that u_n weakly converges to u in $W_0^{1,1}(\Omega)$. Furthermore, the last part of the proof we follow the approach of the Theorem 1 and we can proof that $u \in W_0^{1,1}(\Omega)$ is a distributional solution of (1).

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This research was partially supported by the National Natural Science Foundation of China (No. 11761059), Fundamental Research Funds for the Central Universities (Nos. Yxm2019112, 31920200036), the Program for Young Talent of State Ethnic Affairs Commission of China (No. XBMU-2019-AB-34) and Key Subject of Gansu Province.

References

- [1] A. Alvino, L. Boccardo, V. Ferone, L. Orsina, and G. Trombetti, "Existence results for nonlinear elliptic equations with degenerate coercivity," *Annali di Matematica Pura ed Applicata*, vol. 182, no. 1, pp. 53–79, 2003.
- [2] L. Boccardo, "Some elliptic problems with degenerate coercivity," *Advanced Nonlinear Studies*, vol. 6, no. 1, pp. 1–12, 2006.
- [3] L. Boccardo, A. Dall'Aglio, and L. Orsina, "Existence and regularity results for some elliptic equations with degenerate coercivity," *Atti Del Seminario Matematico E Fisico Universita Di Modena*, vol. 46, pp. 51–81, 1998.
- [4] L. Boccardo and H. Brezis, "Some remarks on a class of elliptic equations with degenerate coercivity," *Bollettino della Unione Matematica Italiana-B*, vol. 6, pp. 521–530, 2003.
- [5] L. Boccardo, S. Buccheri, and G. R. Cirmi, "Two linear noncoercive Dirichlet problems in duality," *Milan Journal of Mathematics*, vol. 86, no. 1, pp. 97–104, 2018.
- [6] G. Croce, "The regularizing effects of some lower order terms in an elliptic equation with degenerate coercivity," *Ricerche di Matematica*, vol. 27, pp. 299–314, 2010.
- [7] C. De Coster and L. Jeanjean, "Multiplicity results in the non-coercive case for an elliptic problem with critical growth in the gradient," *Journal of Differential Equations*, vol. 262, no. 10, pp. 5231–5270, 2017.
- [8] L. Gasiński, L. Klimczak, and N. S. Papageorgiou, "Nonlinear noncoercive Neumann problems," *Communications on Pure and Applied Analysis*, vol. 15, no. 4, pp. 1107–1123, 2016.
- [9] L. Gasiński and N. S. Papageorgiou, "Positive solutions for nonlinear elliptic problems with dependence on the gradient," *Journal of Differential Equations*, vol. 263, no. 2, pp. 1451–1476, 2017.
- [10] D. Giachetti and M. M. Porzio, "Elliptic equations with degenerate coercivity gradient regularity," *Acta Mathematica Sinica, English Series*, vol. 19, no. 2, pp. 349–370, 2003.
- [11] S. Huang, T. Su, X. Du, and X. Zhang, "Entropy solutions to noncoercive nonlinear elliptic equations with measure data," *Electronic Journal of Differential Equations*, vol. 2019, no. 97, pp. 1–22, 2019.
- [12] S. Huang and Q. Tian, "Marcinkiewicz estimates for solution to fractional elliptic Laplacian equation," *Computers & Mathematics with Applications*, vol. 78, no. 5, pp. 1732–1738, 2019.
- [13] S. Huang, Q. Tian, J. Wang, and J. Mu, "Stability for noncoercive elliptic equations," *Electronic Journal of Differential Equations*, vol. 2016, no. 242, pp. 1–11, 2016.
- [14] H.-F. Huo, P. Yang, and H. Xiang, "Dynamics for an sirs epidemic model with infection age and relapse on a scale-free network," *Journal of the Franklin Institute*, vol. 356, no. 13, pp. 7411–7443, 2019.
- [15] A. Mercaldo, I. Peral, and A. Primo, "Results for degenerate nonlinear elliptic equations involving a hardy potential," *Journal of Differential Equations*, vol. 251, no. 11, pp. 3114–3142, 2011.
- [16] N. S. R. Papageorgiou and V. D. Rădulescu, "On noncoercive elliptic problems," *Nonlinear Differential Equations and Applications NoDEA*, vol. 23, no. 4, Article ID 42, 2016.
- [17] V. Rădulescu, "Noncoercive elliptic equations with subcritical growth," *Discrete and Continuous Dynamical Systems - Series S*, vol. 5, no. 4, pp. 857–864, 2012.
- [18] Q. Shi and S. Wang, "Klein-Gordon-Zakharov system in energy space: blow-up profile and subsonic limit," *Mathematical Methods in the Applied Sciences*, vol. 42, no. 9, pp. 3211–3221, 2019.
- [19] Q. Tian and Y. Xu, "Effect of the domain geometry on the solutions to fractional Brezis-Nirenberg problem," *Journal of Function Spaces*, vol. 2019, Article ID 1093804, 4 pages, 2019.
- [20] H. Xiang, M.-X. Zou, and H.-F. Huo, "Modeling the effects of health education and early therapy on tuberculosis transmission dynamics," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 20, no. 3–4, pp. 243–255, 2019.
- [21] X. Li and S. Huang, "Stability and bifurcation for a single-species model with delay weak kernel and constant rate harvesting," *Complexity*, vol. 2019, Article ID 1810385, 17 pages, 2019.
- [22] S. Huang, "Quasilinear elliptic equations with exponential nonlinearity and measure data," *Mathematical Methods in the Applied Sciences*, 2020.
- [23] H. Liu, Y. Ye, Y. Wei, W. Ma, M. Ma, and K. Zhang, "Pattern formation in a reaction-diffusion predator-prey model with weak Allee effect and delay," *Complexity*, vol. 2019, Article ID 6282958, 14 pages, 2019.
- [24] Y. Ye, H. Liu, Y. Wei, M. Ma, and K. Zhang, "Dynamic study of a predator-prey model with weak Allee effect and delay," *Advances in Mathematical Physics*, vol. 2019, Article ID 7296461, 15 pages, 2019.
- [25] B. L. Rita and G. Cirmi, " $W_0^{1,1}(\Omega)$ solution of some unilateral problems," *Nonlinear Analysis Theory Methods & Applications*, vol. 121, pp. 447–457, 2015.
- [26] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, and J. L. Vázquez, "An L^1 theory of existence and uniqueness of nonlinear elliptic equations," *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze*, vol. 22, no. 2, pp. 241–273, 1995.