Research Article

The Existence of Positive Solution of a Nonlinear Problem on Unit Circle

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We consider the following nonlinear problem $u''(x) + \lambda u(x) + f(x, u) = 0$, $x \in S^1$, (P) where S^1 is a unit circle, $\lambda \in \mathbb{R}$ is a parameter, and $f(x, u)$ is a nonlinearity. By using the variational method, we obtain the existence of a positive solution to (P) for all $\lambda \in \mathbb{R}$. This phenomenon is different from the corresponding boundary value problem.

1. Introduction

We consider the following nonlinear problem

$$u''(x) + \lambda u(x) + f(x, u) = 0, \quad x \in S^1, \tag{1}$$

where S^1 is a unit circle, λ is a parameter and the nonlinear term f satisfies.

(f_1) $f(x, t)$ is a continuous function on $S^1 \times \mathbb{R}$, and $f(x, t)$ is odd on t.

The second-order differential equation (1) is a nonlinear elliptic equation model arising from studying some physics processes or geometric problems. For example, (1) is the problem of oscillations [1, 2] of a spherical thick shell made of an elastic material when the nonlinearity f satisfies a superlinear condition as $t \longrightarrow +\infty$, namely,

(f_2) $\lim\limits_{t \to +\infty} f(x, t)/t = +\infty$ uniformly for $x \in S^1$.

For $\lambda = 1$, $f(x, t) = g(x)|t|^{p-1}t$ with $p \in \mathbb{R}$ and $g(x)$: $S^1 \longrightarrow \mathbb{R}$, (1) is also used to model the planar Minkowski problem [3–5].

As is well known, the strong maximum principle is a powerful tool for studying positive solutions of a nonlinear elliptic equation, see for example [6, 7]. In [8], the existence of positive solutions to the nonlinear elliptic equation can also be obtained by a standard argument, based on trans-forming the partial differential equation to the equivalent ordinary differential equation. The iterative method can also be used in the existence of a solution to nonlinear elliptic equation (see [9]). If the parameter $\lambda > 0$ is small, the related Green function of the operator $d/dx^2 + \lambda$ on S^1 is positive. When the Green function is positive, the upper and lower solution method and Schauder's fixed point theorem are also suitable to obtain the positive solution to some nonlinear elliptic equations, see for example [10–12]. However, the Green function of operator $d/dx^2 + \lambda$ on S^1 may be sign-changing for some $\lambda > 0$.

In this paper, we aim at studying the existence of a positive solution of nonlinear equation (1) for all $\lambda \in \mathbb{R}$ and the effect of asymptotic properties of nonlinearity $f(x, t)$ when $t \longrightarrow 0$, to the positivity of solution of (1). Our contribution includes two aspects. Firstly, we establish the following Theorem 1 to show that a nonnegative solution is strictly positive by the analysis based on the Taylor expansion theorem.

Theorem 1. *Assume $\lambda \in \mathbb{R}$, f satisfies (f_1) and $f(x, t)/t$ is bounded as $t \longrightarrow 0$. Let $v \in C^2(S^1)$ be a nonnegative solution of (1). If $v \not\equiv 0$, then v is strictly positive.*

Indeed, we can also apply the Green function or the strong maximum principle to study the positive of the solution. However, the direct application of Green function for

studying the strictly positive of solution often needs more assumptions on parameter λ and the information of sign of the non-homogeneous term. By using Theorem 1, we could estimate the strictly positive of solution to (1) for all $\lambda \in \mathbb{R}$ as the following applications.

Secondly, as an application, we give a strictly positive solution of (1) for all $\lambda \in \mathbb{R}$ by using Theorem 1 and the variational method. To study the existence of positive solution to (1) under a general nonlinearity, we also need the following general assumptions.

$(f_3)\lim_{t \to 0} f(x, t)/t = 0$ uniformly for $x \in S^1$;

(f_4) there exists $\alpha > 2$ such that, for every $x \in S^1$ and $t \in \mathbb{R}$

$$\alpha F(x, t) \le t f(x, t) ; \qquad (2)$$

(f_5) there exists $T > 0$ such that

$$\inf_{x \in S^1, |t| > T} F(x, t) > 0 ; \qquad (3)$$

(f_6) for every $x \in S^1$, $f(x, t)/t$ is an increasing function of t on $[0, +\infty)$.

Theorem 2. *Assume (f_1) - (f_6) hold. For $\lambda < 0$, (1) has a positive solution.*

We see that the existence of a positive solution in Theorem 2 depends on the assumption that $\lambda < 0$. It is natural to ask whether there exist nontrivial solutions of (1) for some $\lambda \ge 0$. To shed some light to this problem, we give the existence of positive solutions to (1) with some special nonlinearity as follows.

Theorem 3. *Assume $f(x, u) = g(x)|u(x)|^{p-1}u(x)$ with $p > 1$. Let $\lambda \ge 0$. If $g \in C(S^1)$ is nonnegative with $\int_{S^1} |g(x)|^{2/(1-p)} dx < +\infty$, then (1) has a positive solution.*

From Theorems 2 and 3, we obtain that (1) may own a positive solution for all $\lambda \in \mathbb{R}$, which is different from the boundary value problem [13–16]. To illustrate that the conclusions above for (1) is a new phenomenon that different from the existence of a positive solution to boundary value problem, we consider the following problem

$$u''(x) + \lambda u(x) + g(x)|u(x)|^{p-1}u(x) = 0 \quad x \in (0, 2\pi),$$
$$u(0) = u(2\pi) = 0. \qquad (4)$$

Let $u \in C^2([0, 2\pi])$ be a solution of (4), we have

$$\left(\lambda - \frac{1}{4}\right) \int_0^{2\pi} \sin\left(\frac{x}{2}\right) u(x) dx + \int_0^{2\pi} g(x)|u(x)|^{p-1} \sin\left(\frac{x}{2}\right) dx = 0.$$
$$(5)$$

If $\lambda > 1/4$, it follows that there is no positive solution to the boundary value problem (4) with $g(x) \ge 0$.

Since the proof of Theorem 2 depends on (f_1) and $\lambda < 0$, it is impossible to deduce Theorem 3 by a similar method for Theorem 2. In the next section, we apply a reverse Hölder's inequality and constrained variational method to prove Theorem 3.

2. Proof of Theorems

We first give our new method to estimate the positivity of the solution of (1) as follows. By assumption, we obtain

$$-v''(x) - \lambda v(x) = f(x, v(x)), x \in [0, 2\pi]. \qquad (6)$$

We will prove that v is strictly positive by the method of contradiction. Assume $v(\theta_0) = 0$ for some $\theta_0 \in [0, 2\pi]$. We choose $\theta_0 = \inf \{x \in [0, 2\pi]: v(x) = 0\}$. Since $v \not\equiv 0$, we have two points θ_r and θ_l in $[0, 2\pi]$ such that $\theta_0 \le \theta_r < \theta_l$ and

$$v(x) = 0 \text{ for } x \in [\theta_0, \theta_r] ; \text{ and } v(x) > 0 \text{ for } x \in (\theta_r, \theta_l). \qquad (7)$$

Since 0 is the minimum value of v on $[0, 2\pi]$, we have

$$v'(x) = 0 \text{ for } x \in [\theta_0, \theta_r]. \qquad (8)$$

By (f_1), we see that $f(x, 0) = 0$. It follows from (6) and (7) that

$$v''(x) = 0 \text{ for } x \in [\theta_0, \theta_r]. \qquad (9)$$

By Taylor's theorem, we give a formula of $v(x)$ for all $x \in (\theta_r, \theta_l)$ as follows,

$$v(x) = \frac{v''(\xi_x)}{2}(x - \theta_r)^2 \quad \text{for} \quad \text{some} \quad \xi_x \in (\theta_r, x). \qquad (10)$$

We claim that there exists a strictly monotone decreasing sequence $\{\theta_n\} \subset (\theta_r, \theta_l)$ such that

$$\theta_n \xrightarrow{n} \theta_r \quad \text{and} \quad |v''(\theta_n)| > 0. \qquad (11)$$

Otherwise, there exists a small $\delta > 0$ such that $|v''(x)| = 0$ for all $x \in (\theta_r, \theta_r + \delta)$. By applying (10), we derive that

$$v(x) = 0 \text{ for } x \in (\theta_r, \theta_r + \delta), \qquad (12)$$

which contradicts to the second part of (7). By (9), we see that $v''(\theta_r) = 0$. Let θ_1 be given by (11), then we have

$$\rho_1 = \sup \left\{ x \in [\theta_r, \theta_1]: \ |v''(x)| = \sup_{\rho \in [\theta_r, \theta_1]} |v''(\rho)| \right\} > \theta_r.$$
$$(13)$$

For $k \in \mathbb{N}$, we can define a sequence of ρ_k by

$$\rho_{k+1} = \sup \left\{ x \in (\theta_r, \rho_k - \delta_k): \ |v''(x)| = \sup_{\rho \in (\theta_r, \rho_k - \delta_k)} |v''(\rho)| \right\}, \tag{14}$$

where $\delta_k = (\rho_k - \theta_r)/2$. From (11), we see that $\theta_n \xrightarrow{n} \theta_r$. Fix $k \in \mathbb{N}$, if $i \in \mathbb{N}$ is large enough, then $\theta_i \in (\theta_r, \rho_k)$; and it follows from (14) that $|v''(\rho_k)| \geq |v''(\theta_i)| > 0$. Following this way, we obtain a strictly monotone decreasing sequence $\{\rho_n\} \subset (\theta_r, \theta_l)$ such that $\rho_n \xrightarrow{n} \theta_r$ and

$$|v''(\rho_n)| > 0, \ \ |v''(x)| \leq |v''(\rho_n)| \text{ for } x \in (\theta_r, \rho_n). \tag{15}$$

For each $n \in \mathbb{N}$, via applying (10) with $x = \rho_n$, there exists $\xi_n \in (\theta_r, \rho_n)$ such that

$$v(\rho_n) = \frac{v(\xi_n)}{2}(\rho_n - \theta_r)^2 \text{ and } |v''(\xi_n)| \leq |v''(\rho_n)|. \tag{16}$$

Let $x = \rho_n$ in (6), we have

$$-v''(\rho_n) - \lambda v(\rho_n) = f(\rho_n, v(\rho_n)). \tag{17}$$

Substituting $v(\rho_n)$ in (17) by using the first part of (16), and then multiplying (17) by $1/v''(\rho_n)$, we deduce that

$$-1 - \lambda \frac{v''(\xi_n)}{2v''(\rho_n)}(\rho_n - \theta_r)^2 = \frac{f(\rho_n, v(\rho_n))}{v(\rho_n)} \frac{v''(\xi_n)}{2v''(\rho_n)}(\rho_n - \theta_r)^2. \tag{18}$$

By using the second part of (16) and the assumption that $f(\theta, t)/t$ is bounded, we derive that

$$\frac{f(\rho_n, v(\rho_n))}{v(\rho_n)} = O(1) \text{ and } \frac{v''(\xi_n)}{2v''(\rho_n)}(\rho_n - \theta_r)^2 \tag{19}$$
$$= o(1) \text{ as } n \longrightarrow +\infty.$$

From (18), we get a contradiction that $1 = o(1)$ as $n \longrightarrow +\infty$.

To prove Theorem 2, we firstly obtain a nonnegative solution of (1) via a minimum of I on manifold N; then, we prove that it is strictly positive by using Theorem 1. The process is standard for studying the existence of nonnegative solutions to nonlinear elliptic equations by the method involving the Nehari manifold. For completeness, we give the main steps. The following deformation lemma is needed.

Lemma 1. *(see, for example, Lemma 2.3 in [17]). Let X be a Banach space, $I \in C^1(X, \mathbb{R})$, $S \subset X$, $c \in \mathbb{R}$, $\varepsilon, \delta > 0$ such that*

$$\inf_{\phi = 0} \frac{I'(u)\phi}{\|\phi\|} \geq 8\varepsilon/\delta \text{ for all } u \tag{20}$$
$$\in \{u : \|u - v\| \leq 2\delta, c - 2 \leq \varepsilon I(u) \leq c + 2\varepsilon\}.$$

We denote by I^d the set $\{u \in X : I(u) \leq d\}$ for any $d \in \mathbb{R}$. Then, there exists $\eta \in C([0, 1] \times X, X)$ such that

(i) $\eta(t, u) = u$, if $t = 0$ or if $u \in \{u \in X : \inf_{w \in S} \|u - w\| \leq 2\delta,$
$c - 2\varepsilon \leq I(u) \leq c + 2\varepsilon\}$,

(ii) $\eta(1, I^{c+\varepsilon} \cap S) \subset I^{c-\varepsilon}$

(iii) $\eta(t, \cdot)$ is an homeomorphism of X for any $t \in [0, 1]$

(iv) $I(\eta(\cdot, u))$ is non increasing for any $u \in X$

Let S be parameterized by angle x, $H := H^1(S)$ be the Sobolev space equipped with the usual normal

$$\|u\| = \left\{ \int_S \left(u'\right)^2 + u^2 dx \right\}^{1/2}. \tag{21}$$

Define $F(x, u) = \int_0^u f(x, t)dt$. Under the assumptions $(f_1) - (f_3)$, we have the following C^1 variational function of (1).

$$I(u) = \frac{1}{2} \int_{S^1} u'^2 dx - \frac{\lambda}{2} \int_{S^1} u^2 dx - \int_{S^1} F(x, u)dx. \tag{22}$$

The related Nehari manifold is defined by

$$N := \left\{ u \in H^1(S^1) \setminus \{0\}: \int_{S^1} u'^2 dx - \lambda \int_{S^1} u^2 dx \right.$$
$$\left. - \int_{S^1} f(x, u)u dx = 0 \right\}. \tag{23}$$

Similar to Lemma 4.1 in [17], we have the following lemma.

Lemma 2. *Assume $(f_1) - (f_6)$ hold. Let $u \in H$ be nontrivial and $h(t) = I(tu)$ for $t > 0$. If $\lambda < 0$, there exists $t_u > 0$ such that $t_u u \in N$, $h(t_u) = \max_{t>0} h(t)$. The map $u \longrightarrow t_u$ is continuous, and $u \longrightarrow t_u u$ defines a homeomorphism of the unit sphere of H with N.*

Proof. If $\lambda < 0$, we see that $1/2 \int_{S^1} u'^2 dx - \lambda/2 \int_{S^1} u^2 dx > 0$ for the nontrivial $u \in H$. For any $u \in N$, let $g(t) = 1/t \int_{S^1} f(x, tu) dx$. Then g is an increasing function of t by (f_6). By the definition of $h(t) = I(tu)$, we see that $h(0) = 0$, and $t_u u \in N$ if and only if $\int_{S^1} u'^2 dx - \lambda \int_{S^1} u^2 dx = 1/t_u \int_{S^1} f(x, t_u u)u dx$, which is equivalent to $h'(t_u) = 0$. Under the assumptions $(f_3) - (f_5)$,

we see that $h(t) > 0$ for small $t > 0$ and $h(t) < 0$ for large t. Therefore, there exists a unique t_u such that $h'(t_u) = 0$ and $t_u u \in N$. By (f_4), there exists a constant $C_0 > 0$ such that

$$C_0(|u|^\alpha - 1) \leq u f(x, u), \tag{24}$$

where $\alpha > 2$ is given by (f_4).

Assume that there exists a nontrivial sequence $u_n \xrightarrow{n} u$ in $H^1(S^1)$ with $u=0$, we have a sequence $\{t_{u_n}\}$ that $t_{u_n} u_n \in N$. This and (f_4) imply that

$$
\begin{aligned}
C t_{u_n} \|u_n\|^2 &\geq t_{u_n} \left(\int_{S^1} u_n'^2 dx - \lambda \int_{S^1} u_n^2 dx \right) \\
&= \int_{S^1} f\left(x, t_{u_n} u_n\right) u_n dx \\
&\geq \int_{S^1} C_0 \left(t_{u_n}^\alpha |u_n|^\alpha - 1 \right) dx.
\end{aligned}
\tag{25}
$$

It follows that $\{t_{u_n}\}$ is bounded. And a subsequence of $\{t_{u_n}\}$ converges to a number t_0. By using (25) and the uniqueness of map $t_u = t(u)$, we see that $t_0 = t_u$. That is, t_{u_n} converge to t_u. The inverse map of $u \longrightarrow t_u u$ is the retraction $u \longrightarrow u/\|u\|$.

By using the properties of I in Lemma 2. We obtain the following Lemma 3.

Lemma 3. Assume $(f_1) - (f_6)$ hold and $\lambda < 0$. Let $c_1 = \inf_{u \in N} I(u)$, then $c_1 > 0$. If $v \in N$ and $I(v) = c_1$, then v is a critical point of I on H.

Proof. Let $c_2 := \inf_{u \in H, u=0} \max_{t \geq 0} I(tu)$ and $c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t))$, where $\Gamma := \{\gamma \in C([0,1], H): I(\gamma(1)) < 0, \gamma(0) = 0\}$. We prove this Lemma by the following two steps.

Step 1. $c_1 = c_2 = c > 0$.

By Lemma 2, we see that $I(t_u u) = \max_{t \geq 0} I(tu)$ if and only if $t_u u \in N$. This and the definitions of c_1, c_2 imply that $c_1 = c_2$. By $(f_1) - (f_2)$, we see that $I(tu) < 0$ for $u=0$ and $t > 0$ large. Hence, $c \leq c_2$. From Lemma 2, we see that the Nehari manifold N separates H into two parts. By assumptions $(f_1) - (f_3)$, we deduce that $\inf_{u \in N} \|u\| > 0$ and there exists $r > 0$ such that

$$\min_{\|u\| \leq r} I(u) = 0, \qquad \inf_{\|u\| = r} I(u) > 0. \tag{26}$$

It follows that the pass $\gamma \in \Gamma$ has to cross N and $c \geq \inf_{\|u\|=r} I(u) > 0$. These facts imply that $c_1 \leq c$.

Step 2. v is a critical point of I.

Assume that $I'(v) = 0$. Then there exists $\delta > 0$ and $\varepsilon > 0$ such that

$$\inf_{\phi=0} \frac{I'(u)\phi}{\|\phi\|} \geq \varepsilon \quad \text{for} \quad \text{all} \quad \|u - v\| \leq 3\delta. \tag{27}$$

For $\varepsilon := \{c/2, \varepsilon\delta/8\}$ and $S := B(v, \delta)$, Lemma 1 yields a deformation η such that

(i) $\eta(1, u) = u$ if $u \in \{u \in H : c - \varepsilon \leq I(u) \leq c + 2\varepsilon\}$,

(ii) $\eta(1, I^{c+\varepsilon} \cap B(v, \delta)) \subset I^{c-\varepsilon}$

(iii) $I(\eta(1, u)) \leq I(u), u \in H$

We deduce that $\max_{t>0} I(\eta(1, tv)) \leq c - \varepsilon < c = c_2$, which contradicts the definition of c_2.

By Lemma 3, we see that $c_1 = \inf_{u \in N} I(u) > 0$. There exists a sequence of $\{u_n\} \subset N$ such that $I(u_n) \longrightarrow c_1$ as $n \longrightarrow +\infty$, that is,

$$\frac{1}{2} \int_{S^1} u'^2 dx - \frac{\lambda}{2} \int_{S^1} u^2 dx - \int_{S^1} F(x, u) dx = c_1 + o(1), \text{ as } n \longrightarrow +\infty, \tag{28}$$

$$\int_{S^1} u'^2 dx - \lambda \int_{S^1} u^2 dx - \int_{S^1} f(x, u) u dx = 0, \tag{29}$$

Let $\alpha > 2$ be given by (f_4), it follows from (28) and (29) that

$$\frac{\alpha - 2}{2} \left(\int_{S^1} u'^2 dx - \lambda \int_{S^1} u^2 dx \right) \leq 2c_1 \quad \text{for} \quad \text{large} \quad n. \tag{30}$$

Since $\lambda < 0$, we have that $\{\|u_n\|\}$ is bounded. Then, there exists a weak limitation v of the sequence $\{u_n\}$ in H, i.e., up to a subsequence, $u_n \rightharpoonup v$ weakly in H and $u_n \longrightarrow v$ uniformly on S^1 as $n \longrightarrow +\infty$. This together with the assumptions $(f_1) - (f_3)$ imply that

$$\lim_{n \to +\infty} \int_{S^1} f(x, u_n) u_n dx = \int_{S^1} f(x, v) v dx. \tag{31}$$

It follows from (29) and (31) that $\lim_{n \to +\infty} \|u_n - v\| = 0$, which implies that $v \in N$ and $c_1 = \inf_{u \in N} I(u) = I(v)$. By Lemma 3, we see that $I'(v) = 0$. Let $v^+ = \max\{0, v\}$ and $v^- = -\max\{0, -v\}$. Then, we deduce that $I(v^+)v^+ = I(v)v^+ = I(v^-)v^- = I(v)v^- = 0$. If v is a sign-changing function, then v^- and v^+ are nontrivial. It follows that $v^-, v^+ \in N$. We thus deduce the following contradiction

$$c_1 = I(v) = I(v^+) + I(v^-) \geq 2 \inf_{u \in N} I(u) = 2c_1 > 0. \tag{32}$$

So, v is not a sign-changing function. By the second part of (f_1), we see that $I(v) = I(-v)$. Without loss of generality, we assume that v is nonnegative. A standard regularity

shows that v is C^2 on S^1. Since the nonlinear term f satisfies assumptions (f_1) and (f_3), we can apply Theorem 1 to get that v is strictly positive.

Let us consider the constrained optimal problem

$$c = \inf_{F(u)=1} I(u), \tag{33}$$

where

$$I(u) = \int_S \left(u'\right)^2 - \lambda u^2 dx, \tag{34}$$

$$F(u) = \int_S g(x)|u|^{p+1} dx. \tag{35}$$

For $\lambda \geq 0$, it follows from reverse Hölder inequality and (34) that

$$
\begin{aligned}
\inf_{F(u)=1} I(u) &= \inf_{F(u)=0} \frac{I(u)}{\{F(u)\}^{2/p+1}} \\
&\geq \inf_{F(u)=0} \frac{-\lambda \int_{S^1} |u|^2 dx}{\left\{\int_{S^1} g(x)^{2/(1-p)} d\theta\right\}^{(1-p)/2} \int_{S^1} |u|^2 dx} \\
&= -\lambda \left\{\int_{S^1} g(x)^{2/(1-p)} d\theta\right\}^{(p-1)/2} > -\infty,
\end{aligned}
\tag{36}
$$

which implies that $I(u)$ is bounded in $H^1(S)$ from below under the constrained condition that $F(u) = 1$. By the definition of I, F in (34) and (35), there exists a nonnegative minimizing sequence $\{u_n\} \subset H^1(S)$ such that $F(u_n) = 1$ and

$$I(u_n) \longrightarrow \inf_{F(u)=1} I(u) \text{ as } n \longrightarrow +\infty. \tag{37}$$

It follows from $2/(p-1) > (p-1)/(p+1)$ and the Hölder inequality that

$$\int_{S^1} g(x)^{-(p-1)/(p+1)} dx \leq C \left|\int_{S^1} g(x)^{2/(1-p)} dx\right|^{\frac{(p-1)^2}{2(p+1)}} < +\infty. \tag{38}$$

Since $F(u_n) = 1$, by Hölder inequality we see that

$$
\begin{aligned}
\int_{S^1} u_n^2 dx &\leq \left(\int_{S^1} g(x)^{-(p-1)/(p+1)} dx\right)^{(p+1)/(p+2)} \left(\int_{S^1} g(x) u_n^{p+1} dx\right)^{2/(p+1)} \\
&= \left(\int_{S^1} g(x)^{-(p-1)/(p+1)} dx\right)^{(p+1)/(p+2)} < +\infty.
\end{aligned}
\tag{39}
$$

This together with $I(u_n) \overset{(n \to +\infty)}{\longrightarrow} \inf_{F(u)=1} I(u)$ imply that $\{u_n\}$ is bounded in $H^1(S)$. So, there is a subsequence such that

$$u_n \overset{n}{\longrightarrow} v \quad \text{weakly in} \quad H^1(S^1). \tag{40}$$

By the compactness of Sobolev imbedding, up to a subsequence, we have

$$u_n \overset{n}{\longrightarrow} v \quad \text{strongly in} \quad C^\alpha(S^1) \quad \text{for} \quad \alpha \in (0, 1/2). \tag{41}$$

It follows that $F(|v|) = 1$, and therefore $v=0$.

$$\int_0^{2\pi} g(\theta)|v|^{p+1} d\theta = \lim_{n \to +\infty} \int_0^{2\pi} g(\theta)|u_n|^{p+1} d\theta = 1. \tag{42}$$

By the Fatou lemma, we get

$$\inf_{F(u)=1} I(u) \leq I(|v|) \leq \lim_{n \to +\infty} I(u_n) = \inf_{F(u)=1} I(u), \tag{43}$$

So, $|v|$ is a minimizer of I under constrained condition that $F(u) = 1$. There exists $\omega \in \mathbb{R}$ such that

$$I'(|v|) = \omega F'(|v|) \quad \text{in} \quad H^{-1}(S). \tag{44}$$

It follows from the regularity theorem that $|v| \in C^2(S)$ is a classical nonnegative solution of

$$-u'' - \lambda u = \frac{\omega(p+1)}{2} g(x)|u|^{p-1} u, x \in S^1. \tag{45}$$

Let $w = (\omega(p+1)/2)^{1/(p-1)} |v|$, we see that w is nonnegative solution of (1) with $f(x, u) = g(x)|u|^{p-1} u$. By applying Theorem 1, we see that w is positive.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

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