Research Article

Rate of Approximation for Modified Lupaş-Jain-Beta Operators

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The main intent of this paper is to innovate a new construction of modified Lupaş-Jain operators with weights of some Beta basis functions whose construction depends on σ such that $\sigma(0) = 0$ and $\inf_{x \in [0, \infty)} \sigma'(x) \geq 1$. Primarily, for the sequence of operators, the convergence is discussed for functions belong to weighted spaces. Further, to prove pointwise convergence Voronovskaya type theorem is taken into consideration. Finally, quantitative estimates for the local approximation are discussed.

1. Introduction

In 1972, Jain [1] with the help of Poisson distribution introduced a famous linear positive operators as follows:

$$P_m^{[\alpha]}(f; x) = \sum_{j=0}^{\infty} \mathcal{W}_{\alpha}(j, mx) f\left(\frac{j}{m}\right), \quad x \geq 0, \quad (1)$$

where $m \geq 1$, f defined on $[0, \infty)$, and

$$\mathcal{W}_{\alpha}(j, mx) = mx(mx + j\alpha)^{j-1} \frac{e^{-(mx+j\alpha)}}{j!}, \quad 0 \leq \alpha < 1, \quad (2)$$

$$\sum_{j=0}^{\infty} \mathcal{W}_{\alpha}(j, mx) = 1.$$

If we put $\alpha = 0$ in (1), then it becomes Szász-Mirakyan-type.

In 1995, Lupaş [2] introduced a sequence of linear positive operators; later on in 1999, it was modified by Agratini [3] as follows:

$$L_m(f; x) = 2^{-mx} \sum_{j=0}^{\infty} \frac{(mx)_j}{j! 2^j} f\left(\frac{j}{m}\right), \quad x \geq 0, \quad (3)$$

and also discussed the Kantorovich and Durrmeyer variant of operator (1).

In 2018, Tunca et al. [4] modified operator (3) in such a way that in the construction, authors take the negative subscript -1 of the Pochhammer symbol into consideration; due to this, the calculations become simpler in a remarkable degree just as

$$L_m^{\alpha}(f; x) = 2^{-(mx+j\alpha)} \sum_{j=0}^{\infty} \frac{mx(mx+1+j\alpha)_{j-1}}{j! 2^j} f\left(\frac{j}{m}\right), \quad x \geq 0. \quad (4)$$

In order to approximate Lebesgue integrable functions, the most important modifications are Kantorovich and Durrmeyer integral operators. The Durrmeyer variant of operator (1) is introduced by Tarabie [5] and Mishra and Patel [6] with some beta basis functions.

In 2011, Cárdenas-Morales et al. [7] defined the Bernstein-type operators by $B_m(f \circ \sigma^{-1}) \circ \sigma$ and also presents a better degree of approximation depending on σ . This type of approximation operators generalizes the Korovkin set from $\{e_0, e_1, e_2\}$ to $\{e_0, \sigma, \sigma^2\}$. The Durrmeyer variant of $B_m(f \circ \sigma^{-1}) \circ \sigma$ is defined in [8]. In 2014, Aral et al. [9] defined a similar modification of Szász-Mirakyan-type operators by using a suitable function σ .

Motivated by the above mentioned work very recently, Bodur [10] introduced a new modification of operator (4) by using a suitable function σ , which satisfies the following properties:

$$(\sigma_1)\sigma \text{ be a continuously differentiable function on } [0, \infty) \\ (\sigma_2)\sigma(0) = 0 \text{ and } \inf_{x \in [0, \infty)} \sigma'(x) \geq 1$$

The new formulated operators are defined as

$$\mathcal{L}_{m,\sigma}^\alpha(f; x) = 2^{-(m\sigma(x)+j\alpha)} \sum_{j=0}^{\infty} \frac{m\sigma(x)(m\sigma(x) + 1 + j\alpha)_{j-1}}{2^j j!} \cdot (f \circ \sigma^{-1})\left(\frac{j}{m}\right), \quad x \geq 0, \quad (5)$$

for $m \geq 1$, $x \geq 0$, and suitable functions f defined on $[0, \infty)$.

As we know, in order to approximate Lebesgue integrable functions, the most important modifications are Kantorovich and Durrmeyer integral operators. Motivated by the above mentioned Durrmeyer type generalizations of various operators and also from [11–23], in this paper, Durrmeyer-type modification of generalized Lupaş-Jain operators (5) by taking weights of some beta basis function is defined as follows:

$$\mathcal{D}_{m,\sigma}^\alpha(f; x) = 2^{-(m\sigma(x)+j\alpha)} \sum_{j=0}^{\infty} \frac{m\sigma(x)(m\sigma(x) + 1 + j\alpha)_{j-1}}{2^j j!} \int_0^\infty b_{m,j}(\vartheta) (f \circ \sigma^{-1})(\vartheta) d\vartheta, \quad (6)$$

where $m \in \mathbb{N}$ and $b_{m,j}(\vartheta)$ is defined as

$$b_{m,j}(\vartheta) = \frac{1}{\beta(m+1, j)} \frac{\vartheta^{j-1}}{(1+\vartheta)^{m+j+1}}, \quad (7)$$

where $\beta(m+1, j)$ is the beta basis function and σ is a function satisfying the conditions (σ_1) and (σ_2) given above.

The rest of the work is organized as follows: in the second section, moments and central moments for $\mathcal{D}_{m,\sigma}^\alpha$ are calculated. In the third section, we study convergence properties of $\mathcal{D}_{m,\sigma}^\alpha$ in the light of weighted space. In the fourth section, we obtain the order of approximation of new constructed operators associated with the weighted modulus of continuity. In the fifth section, we shall prove Voronovskaya-type theorem in quantitative form. These kinds of results are very useful to describe the rate of point-wise convergence. Finally, in the last section, we obtain some local approximation results related to $\mathcal{K}$ -functional.

2. Basic Results

In this section, we prove some lemmas for $\mathcal{D}_{m,\sigma}^\alpha$ which are required to prove our main results.

Lemma 1. Let $\mathcal{D}_{m,\sigma}^\alpha$ be given by (6). Then for each $x \geq 0$, $m \in \mathbb{N}$, and $0 \leq \alpha < 1$, we have

$$(i) \mathcal{D}_{m,\sigma}^\alpha(1; x) = 1$$

$$(ii) \mathcal{D}_{m,\sigma}^\alpha(\sigma; x) = \sigma(x)/1 - \alpha$$

$$(iii) \mathcal{D}_{m,\sigma}^\alpha(\sigma^2; x) = (1/(m-1)(1-\alpha)^2)(m\sigma^2(x) + (\alpha^2 - 2\alpha + 3/(1-\alpha))\sigma(x))$$

$$(iv) \mathcal{D}_{m,\sigma}^\alpha(\sigma^3; x) = (1/(m-1)(m-2)(1-\alpha)^3)(m^2\sigma^3(x) + ((6m+3m(1-\alpha)^2)/1-\alpha)\sigma^2(x) + (6(\alpha+1)+6(1-\alpha)^2+2(1-\alpha)^4/(1-\alpha)^2)\sigma(x))$$

$$(v) \mathcal{D}_{m,\sigma}^\alpha(\sigma^4; x) = (1/(m-1)(m-2)(m-3)(1-\alpha)^4)(m^3\sigma^4(x) + ((12m^2+6m^2(1-\alpha)^2)/1-\alpha)\sigma^3(x) + (24m\alpha+36m+36m(1-\alpha)^2+11m(1-\alpha)^4/(1-\alpha)^2)\sigma^2(x) + (26\alpha^2+68\alpha+24+(36\alpha+36)(1-\alpha)^2+22(1-\alpha)^4+6(1-\alpha)^6/(1-\alpha)^3)\sigma(x))$$

By using beta function and Lemma 2.1 in [4], it can be proved. So we omit it. Now, from the linearity of the operators $\mathcal{D}_{m,\sigma}^\alpha$, we can state Lemma 2.

Lemma 2. For operators $\mathcal{D}_{m,\sigma}^\alpha$, we have the following properties:

$$(i) \mathcal{D}_{m,\sigma}^\alpha(\sigma(\vartheta) - \sigma(x); x) = (\alpha/1 - \alpha)\sigma(x)$$

$$(ii) \mathcal{D}_{m,\sigma}^\alpha((\sigma(\vartheta) - \sigma(x))^2; x) = (m/(m-1)(1-\alpha)^2 - 1 + \alpha/1 - \alpha)\sigma^2(x) + (\alpha^2 - 2\alpha + 3/(m-1)(1-\alpha)^3)\sigma(x)$$

$$(iii) \mathcal{D}_{m,\sigma}^\alpha((\sigma(\vartheta) - \sigma(x))^4; x) = (1/(m-1)(m-2)(m-3)(1-\alpha)^4)(\{m^3+6m(m-2)(m-3)(1-\alpha)^2-4m^2(m-3)(1-\alpha)-4(m-1)(m-2)(m-3)(1-\alpha)^3+(m-1)(m-2)(m-3)(1-\alpha)^4\}\sigma^4(x) + \{6\alpha^2-12\alpha+18(m-2)(m-3)(1-\alpha)-(24m+12m(1-\alpha)^2)(m-3)+(12m^2+6m^2(1-\alpha)^2/(1-\alpha))\}\sigma^3(x) + \{24m\alpha+36m+36m(1-\alpha)^2+11m(1-\alpha)^4/(1-\alpha)^2-(24(\alpha+1)+24(1-\alpha)^2+8(1-\alpha)^4)(m-3)/1-\alpha\}\sigma^2(x) + (26\alpha^2+68\alpha+24+(36\alpha+36)(1-\alpha)^2+22(1-\alpha)^4+6(1-\alpha)^6/(1-\alpha)^3)\sigma(x))$$

3. Convergence of $\mathcal{D}_{m,\sigma}^\alpha$

Here, we prove the convergence of $\mathcal{D}_{m,\sigma}^\alpha$ by using weight function. Let $\lambda(x)$ be a function satisfying the conditions (σ_1) and (σ_2) given above. Also, let $\lambda(x) = 1 + \sigma^2(x)$ be a weight function and the weighted space is defined as follows:

$$\mathcal{B}_\lambda[0, \infty) = \{f : [0, \infty) \longrightarrow \mathbb{R} \mid |f(x)| \leq \mathcal{M}_f \lambda(x), x \geq 0\}, \quad (8)$$

where $\mathcal{M}_f$ is a constant which depends only on f , with the norm

$$\|f\|_\lambda = \sup_{x \in [0, \infty)} \frac{|f(x)|}{\lambda(x)}. \quad (9)$$

Also, we mention some subspaces of $\mathcal{B}_\lambda[0, \infty)$ as

$$\begin{aligned}\mathcal{C}_\lambda[0, \infty) &= \{f \in \mathcal{B}_\lambda[0, \infty) : f \text{ is continuous on } [0, \infty)\}, \\ \mathcal{C}_\lambda^*[0, \infty) &= \left\{f \in \mathcal{C}_\lambda[0, \infty) : \lim_{x \rightarrow \infty} \frac{f(x)}{\lambda(x)} = \mathcal{M}_f = \text{constant}\right\}, \\ U_\lambda[0, \infty) &= \left\{f \in \mathcal{C}_\lambda[0, \infty) : \frac{f(x)}{\lambda(x)} \text{ is uniformly continuous on } [0, \infty)\right\}.\end{aligned}\quad (10)$$

It is obvious that $\mathcal{C}_\lambda^*[0, \infty) \subset U_\lambda[0, \infty) \subset \mathcal{C}_\lambda[0, \infty) \subset \mathcal{B}_\lambda[0, \infty)$.

We have the following results for the weighted Korovkin-type theorems due to Gadjeiev [24]

Lemma 3. [24].

The positive linear operators $\mathcal{T}_m, m \geq 1$ act from $\mathcal{C}_\lambda[0, \infty)$ to $\mathcal{B}_\lambda[0, \infty)$ if and only if the inequality

$$|\mathcal{T}_m(\lambda; x)| \leq \mathcal{M}_m \lambda(x), \quad x \geq 0 \quad (11)$$

holds, where $\mathcal{M}_m > 0$ is a constant depending on m .

Theorem 4 (see [24]). Let the sequence of positive linear operators $\mathcal{T}_m, m \geq 1$ acting from $\mathcal{C}_\lambda[0, \infty)$ to $\mathcal{B}_\lambda[0, \infty)$ and satisfying

$$\lim_{m \rightarrow \infty} \|\mathcal{T}_m \sigma^r - \sigma^r\|_\lambda = 0, \quad r = 0, 1, 2. \quad (12)$$

Then, for each $f \in C_\lambda^*[0, \infty)$, we have

$$\lim_{m \rightarrow \infty} \|\mathcal{T}_m(f) - f\|_\lambda = 0. \quad (13)$$

Remark 5. Examining Lemma 1 based on the famous Korovkin theorem [25], it is clear that $(\mathcal{D}_{m,\sigma}^\alpha)_{m \geq 1}$ does not form an approximation process. Now, in order to obtain convergence properties, we replace the constant α by $\alpha_m \in [0, 1)$ such that $\lim_{m \rightarrow \infty} \alpha_m = 0$.

Theorem 6. Let $0 \leq \alpha_m < 1$ such that $\lim_{m \rightarrow \infty} \alpha_m = 0$, and also, let $\mathcal{D}_{m,\sigma}^\alpha$ be the sequence of positive linear operators. Then, for each function $f \in C_\lambda^*[0, \infty)$, we have

$$\lim_{m \rightarrow \infty} \|\mathcal{D}_{m,\sigma}^{\alpha_m}(f) - f\|_\lambda = 0. \quad (14)$$

Proof. From Lemma 1, we obtain

$$\begin{aligned}\|\mathcal{D}_{m,\sigma}^{\alpha_m}(1; x) - 1\|_\lambda &= 0, \\ \|\mathcal{D}_{m,\sigma}^{\alpha_m}(\sigma; x) - \sigma\|_\lambda &\leq \frac{\alpha_m}{1 - \alpha_m}, \\ \|\mathcal{D}_{m,\sigma}^{\alpha_m}(\sigma^2) - \sigma^2\|_\lambda &\leq \frac{m - (m-1)(1 - \alpha_m)^2}{(m-1)(1 - \alpha_m)^2} + \frac{\alpha^2 - 2\alpha_m + 3}{(m-1)(1 - \alpha_m)^3}.\end{aligned}\quad (15)$$

Hence, by Theorem 4, we deduce

$$\lim_{m \rightarrow \infty} \|\mathcal{D}_{m,\sigma}^\alpha(f) - f\|_\lambda = 0. \quad (16)$$

4. Rate of Convergence

In this part, we would like to determine the rate of convergence for $\mathcal{D}_{m,\sigma}^\alpha$ by weighted modulus of continuity $\omega_\sigma(f; \delta)$ which was introduced by Holhos [26] in 2008, as follows:

$$\omega_\sigma(f; \delta) = \sup_{x, \vartheta \in [0, \infty), |\sigma(\vartheta) - \sigma(x)| \leq \delta} \frac{|f(\vartheta) - f(x)|}{\lambda(\vartheta) + \lambda(x)}, \quad \delta > 0, \quad (17)$$

where $f \in \mathcal{C}_\lambda[0, \infty)$, with the following properties:

- (i) $\omega_\sigma(f; 0) = 0$
- (ii) $\omega_\sigma(f; \lambda) \geq 0, \lambda \geq 0$, for $f \in \mathcal{C}_\lambda[0, \infty)$
- (iii) $\lim_{\lambda \rightarrow 0} \omega_\sigma(f; \lambda) = 0$, for each $f \in U_\lambda[0, \infty)$

Theorem 7 (see [26]). Let $\mathcal{T}_m : \mathcal{C}_\lambda[0, \infty) \rightarrow \mathcal{B}_\lambda[0, \infty)$ be a sequence of positive linear operators with

$$\begin{aligned}\|\mathcal{T}_m(\sigma^0) - \sigma^0\|_{\lambda^0} &= a_m, \\ \|\mathcal{T}_m(\sigma) - \sigma\|_{\lambda^{1/2}} &= b_m, \\ \|\mathcal{T}_m(\sigma^2) - \sigma^2\|_\lambda &= c_m, \\ \|\mathcal{T}_m(\sigma^3) - \sigma^3\|_{\lambda^{3/2}} &= d_m,\end{aligned}\quad (18)$$

where the sequences a_m, b_m, c_m , and d_m converge to zero as $m \rightarrow \infty$. Then

$$\|\mathcal{T}_m(f) - f\|_{\lambda^{3/2}} \leq (7 + 4a_m + 2c_m)\omega_\sigma(f; \lambda_m) + \|f\|_\lambda a_m, \quad (19)$$

for all $f \in \mathcal{C}_\lambda[0, \infty)$, where

$$\lambda_m = 2\sqrt{(a_m + 2b_m + c_m)(1 + a_m)} + a_m + 3b_m + 3c_m + d_m. \quad (20)$$

Theorem 8. Let $0 \leq \alpha < 1$ such that $\lim_{m \rightarrow \infty} \alpha_m = 0$, and also, let $\mathcal{D}_{m,\sigma}^\alpha$ be the sequence of positive linear operators. Then for all $f \in C_\lambda[0, \infty)$, we have

$$\begin{aligned}\|\mathcal{D}_{m,\sigma}^{\alpha_m}(f) - f\|_{\lambda^{3/2}} &\leq \left(7 + \frac{2m - 2(m-1)(1 - \alpha_m)^2}{(m-1)(1 - \alpha_m)^2} \right. \\ &\quad \left. + \frac{2\alpha_m^2 - 4\alpha_m + 6}{(m-1)(1 - \alpha_m)^3} \right) \omega_\sigma(f; \delta_m),\end{aligned}\quad (21)$$

where

$$\begin{aligned} \delta_m = 2 & \sqrt{\frac{2\alpha_m}{1-\alpha_m} + \frac{m-(m-1)(1-\alpha_m)^2}{(m-2)(1-\alpha_m)^2} + \frac{\alpha_m^2 - 2\alpha_m + 3}{(m-1)(1-\alpha_m)^3}} \\ & + \frac{3\alpha_m}{1-\alpha_m} + \frac{3m-3(m-1)(1-\alpha_m)^2}{(m-1)(1-\alpha_m)^2} + \frac{3\alpha_m^2 - 6\alpha_m + 6}{(m-1)(1-\alpha_m)^3} \\ & + \frac{m^2 - (m-1)(m-2)(1-\alpha_m)^3}{(m-1)(m-2)(1-\alpha_m)^3} + \frac{6m^2 - 3m(1-\alpha_m)^2}{(m-1)(m-2)(1-\alpha_m)^4} \\ & + \frac{2(1-\alpha_m)^4 + 6(1-\alpha_m)^2 + 6(\alpha_m + 1)}{(m-1)(m-2)(1-\alpha_m)^5}. \end{aligned} \quad (22)$$

Proof. We should calculate the sequences (a_m) , (b_m) , (c_m) , and (d_m) , in order to apply Theorem 7. In light of Lemma 1, clearly, we have

$$\begin{aligned} a_m &= \|\mathcal{D}_{m,\sigma}^{\alpha_m}(\sigma^0) - \sigma^0\|_{\lambda^0} = 0, \\ b_m &= \|\mathcal{D}_{m,\sigma}^{\alpha_m}(\sigma) - \sigma\|_{\lambda^{1/2}} \leq \frac{\alpha_m}{1-\alpha_m}, \\ c_m &= \|\mathcal{D}_{m,\sigma}^{\alpha_m}(\sigma^2) - \sigma^2\|_{\lambda} \leq \frac{m-(m-1)(1-\alpha_m)^2}{(m-1)(1-\alpha_m)^2} + \frac{\alpha_m^2 - 2\alpha_m + 3}{(m-1)(1-\alpha_m)^3}. \end{aligned} \quad (23)$$

Finally,

$$\begin{aligned} d_m &= \|\mathcal{D}_{m,\sigma}^{\alpha_m}(\sigma^3) - \sigma^3\|_{\lambda^{3/2}} \leq \frac{m^2 - (m-1)(m-2)(1-\alpha_m)^3}{(m-1)(m-2)(1-\alpha_m)^3} \\ &+ \frac{6m - 3m(1-\alpha_m)^2}{(m-1)(m-2)(1-\alpha_m)^4} \\ &+ \frac{2(1-\alpha_m)^4 + 6(1-\alpha_m)^2 + 6(\alpha_m + 1)}{(m-1)(m-2)(1-\alpha_m)^5}. \end{aligned} \quad (24)$$

Thus, Theorem 7 is satisfied. Hence, we have the desired result.

Remark 9. For $\lim_{\lambda \rightarrow 0} \omega_\sigma(f; \lambda) = 0$ in Theorem 8, we obtain

$$\lim_{m \rightarrow \infty} \|\mathcal{D}_{m,\sigma}^{\alpha_m}(f) - f\|_{\lambda^{3/2}} = 0, \quad \text{for } f \in U_\lambda[0, \infty). \quad (25)$$

5. Pointwise Convergence of $\mathcal{D}_{m,\sigma}^\alpha$

In this section, we shall analyze pointwise convergence of $\mathcal{D}_{m,\sigma}^\alpha$ by obtaining the Voronovskaya theorem in a quantitative form by using the same technique in [7].

Theorem 10. Let $0 \leq \alpha < 1$ such that $\lim_{m \rightarrow \infty} \alpha_m = 0$, and also, let $f \in \mathcal{E}_\lambda[0, \infty)$, $x \in [0, \infty)$ and suppose that $(f \circ \sigma^{-1})'$ and $(f \circ \sigma^{-1})''$ exist at $\sigma(x)$. If $(f \circ \sigma^{-1})''$ is bounded on $[0, \infty)$, then, we have

$$\lim_{m \rightarrow \infty} m[\mathcal{D}_{m,\sigma}^\alpha(f; x) - f(x)] = \frac{3\sigma(x)(f \circ \sigma^{-1})''(\sigma(x))}{2}. \quad (26)$$

Proof. By using Taylor expansion of $(f \circ \sigma^{-1})$ at $\sigma(x) \in [0, \infty)$, we have

$$\begin{aligned} f(\vartheta) &= (f \circ \sigma^{-1})(\sigma(\vartheta)) \\ &= (f \circ \sigma^{-1})(\sigma(x)) + (f \circ \sigma^{-1})'(\sigma(x))(\sigma(\vartheta) - \sigma(x)) \\ &\quad + \frac{(f \circ \sigma^{-1})''(\sigma(x))(\sigma(\vartheta) - \sigma(x))^2}{2} \\ &\quad + \lambda_x(\vartheta)(\sigma(\vartheta) - \sigma(x))^2, \end{aligned} \quad (27)$$

where

$$\lambda_x(\vartheta) = \frac{(f \circ \sigma^{-1})''(\sigma(\vartheta)) - (f \circ \sigma^{-1})''(\sigma(x))}{2}. \quad (28)$$

Therefore, (28) together with the assumption on f ensures that

$$|\lambda_x(\vartheta)| \leq \mathcal{K}, \quad \text{for all } \vartheta \in [0, \infty) \quad (29)$$

and is convergent to zero as $\vartheta \rightarrow x$. Now applying the operators (6) to the equality (27), we obtain

$$\begin{aligned} [\mathcal{D}_{m,\sigma}^\alpha(f; x) - f(x)] &= (f \circ \sigma^{-1})'(\sigma(x))\mathcal{D}_{m,\sigma}^\alpha((\sigma(\vartheta) - \sigma(x)); x) \\ &\quad + \frac{(f \circ \sigma^{-1})''(\sigma(x))\mathcal{D}_{m,\sigma}^\alpha((\sigma(\vartheta) - \sigma(x))^2; x)}{2} \\ &\quad + \mathcal{D}_{m,\sigma}^\alpha(\lambda_x(\vartheta)((\sigma(\vartheta) - \sigma(x))^2; x)). \end{aligned} \quad (30)$$

From Lemma 2, we get

$$\lim_{m \rightarrow \infty} m\mathcal{D}_{m,\sigma}^\alpha((\sigma(\vartheta) - \sigma(x)); x) = 0, \quad (31)$$

$$\lim_{m \rightarrow \infty} m\mathcal{D}_{m,\sigma}^\alpha((\sigma(\vartheta) - \sigma(x))^2; x) \leq 3\sigma(x). \quad (32)$$

By estimating the equality (30), we will get the proof.

Since from (28), for every $\varepsilon > 0$, $\lim_{\vartheta \rightarrow x} \lambda_x(\vartheta) = 0$. Let $\delta > 0$ such that $|\lambda_x(\vartheta)| < \varepsilon$ for every $\vartheta \geq 0$. By Cauchy-Schwartz inequality, we get

$$\begin{aligned} \lim_{m \rightarrow \infty} m\mathcal{D}_{m,\sigma}^\alpha(|\lambda_x(\vartheta)|(\sigma(\vartheta) - \sigma(x))^2; x) \\ \leq \varepsilon \lim_{m \rightarrow \infty} m\mathcal{D}_{m,\sigma}^\alpha((\sigma(\vartheta) - \sigma(x))^2; x) \\ + \frac{\mathcal{K}}{\delta^2} \lim_{m \rightarrow \infty} \mathcal{D}_{m,\sigma}^\alpha((\sigma(\vartheta) - \sigma(x))^4; x). \end{aligned} \quad (33)$$

Since

$$\lim_{m \rightarrow \infty} m\mathcal{D}_{m,\sigma}^\alpha((\sigma(\vartheta) - \sigma(x))^4; x) = 0, \quad (34)$$

we obtain

$$\lim_{m \rightarrow \infty} m \mathcal{D}_{m,\sigma}^\alpha (|\lambda_x(\vartheta)| (\sigma(\vartheta) - \sigma(x))^2; x) = 0. \quad (35)$$

Thus, by taking into account equations (31), (32), and (35) to equation (30), the proof is completed.

Remark 11. If we choose $\alpha_n = 1/((m+1)^a)$ with $a > 1$, then, one can easily see that $\lim_{m \rightarrow \infty} \alpha_m = 0$ and $m \lim_{m \rightarrow \infty} \alpha_m = 0$.

6. Local Approximation

In this section, for the operators $\mathcal{D}_{m,\sigma}^\alpha$, we shall present local approximation theorems. Let $\mathcal{C}_B[0, \infty)$ denote the space of real-valued continuous and bounded functions f defined on the interval $[0, \infty)$. The norm $\|\cdot\|$ on the space $\mathcal{C}_B[0, \infty)$ is defined by

$$\|f\| = \sup_{0 \leq x < \infty} |f(x)|. \quad (36)$$

$\mathcal{K}$ -functional is defined as

$$\mathcal{K}_2(f, \delta) = \inf_{s \in W^2} \left\{ \|f - s\| + \delta \|f''\| \right\}, \quad (37)$$

where $\delta > 0$ and $W^2 = \{s \in \mathcal{C}_B[0, \infty): s', s'' \in \mathcal{C}_B[0, \infty)\}$. By Devore and Lorentz ([27], p. 177, Theorem 6.4), there exists an absolute constant $\mathcal{C} > 0$ such that

$$\mathcal{K}(f, \delta) \leq \mathcal{C} \omega_2(f, \sqrt{\delta}). \quad (38)$$

The second order modulus of smoothness is as follows:

$$\omega_2(f, \sqrt{\delta}) = \sup_{0 < h \leq \sqrt{\delta}} \sup_{x \in [0, \infty)} |f(x+2h) - 2f(x+h) + f(x)|, \quad (39)$$

where $f \in C_B[0, \infty)$. The usual modulus of continuity of $f \in C_B[0, \infty)$ is defined by

$$\omega(f, \delta) = \sup_{0 < h \leq \delta} \sup_{x \in [0, \infty)} |f(x+h) - f(x)|. \quad (40)$$

Theorem 12. Let $0 \leq \alpha < 1$ such that $\lim_{m \rightarrow \infty} \alpha_m = 0$ and for all $f \in \mathcal{C}_B[0, \infty)$. Also, let σ be a function satisfying the conditions (σ_1) and (σ_2) , and $\|\sigma''\|$ is finite. Then, there exists an absolute constant $\mathcal{C} > 0$ such that

$$|\mathcal{D}_{m,\sigma}^\alpha(f; x) - f(x)| \leq \mathcal{C} \mathcal{K}(f, \delta_m(x)), \quad (41)$$

where

$$\delta_m(x) = \left(\frac{m}{(m-1)(1-\alpha)^2} - \frac{1+\alpha}{1-\alpha} \right) \sigma^2(x) + \frac{\alpha^2 - 2\alpha + 3}{(m-1)(1-\alpha)^3} \sigma(x). \quad (42)$$

Proof. Let $s \in W^2$ and $x, \vartheta \in [0, \infty)$. By Taylor's formula, we have

$$\begin{aligned} s(\vartheta) &= s(x) + (s\sigma^{-1})'(\sigma(x))(\sigma(\vartheta) - \sigma(x)) \\ &\quad + \int_{\sigma(x)}^{\sigma(\vartheta)} (\sigma(\vartheta) - \nu) (s\sigma^{-1})''(\nu) d\nu. \end{aligned} \quad (43)$$

By using the equality,

$$(s\sigma^{-1})''(\sigma(x)) = \frac{s''(x)}{(\sigma'(x))^2} - s''(x) \frac{\sigma''(x)}{(\sigma'(x))^3}. \quad (44)$$

Now, putting $\nu = \sigma(x)$ in the last term in equality (43), we get

$$\begin{aligned} &\int_{\sigma(x)}^{\sigma(\vartheta)} (\sigma(\vartheta) - \nu) (s\sigma^{-1})''(\nu) d\nu \\ &= \int_x^\vartheta (\sigma(\vartheta) - \sigma(x)) \left[\frac{s''(x)\sigma'(x) - s'(x)\sigma''(x)}{(\sigma'(x))^2} \right] dx \\ &= \int_{\sigma(x)}^{\sigma(\vartheta)} (\sigma(\vartheta) - \nu) \frac{s''(\sigma^{-1}(\nu))}{(\sigma'(\sigma^{-1}(\nu)))^2} d\nu \\ &\quad - \int_{\sigma(x)}^{\sigma(\vartheta)} (\sigma(\vartheta) - \nu) \frac{s'(\sigma^{-1}(\nu))\sigma''(\sigma^{-1}(\nu))}{(\sigma'(\sigma^{-1}(\nu)))^3} d\nu. \end{aligned} \quad (45)$$

By applying operator (6) to the both sides of equality (43) and from Lemma 1, we deduce

$$\begin{aligned} \mathcal{D}_{m,\sigma}^\alpha(s; x) &= s(x) \\ &\quad + \mathcal{D}_{m,\sigma}^\alpha \left(\int_{\sigma(x)}^{\sigma(\vartheta)} (\sigma(\vartheta) - \nu) \frac{s''(\sigma^{-1}(\nu))}{(\sigma'(\sigma^{-1}(\nu)))^2} d\nu; x \right) \\ &\quad - \mathcal{D}_{m,\sigma}^\alpha \left(\int_{\sigma(x)}^{\sigma(\vartheta)} (\sigma(\vartheta) - \nu) \frac{s'(\sigma^{-1}(\nu))\sigma''(\sigma^{-1}(\nu))}{(\sigma'(\sigma^{-1}(\nu)))^3} d\nu; x \right). \end{aligned} \quad (46)$$

As we know, σ is strictly increasing on $[0, \infty)$, and with condition (σ_2) , we get

$$|\mathcal{D}_{m,\sigma}^\alpha(s; x) - s(x)| \leq \mathcal{M}_{m,2}^\sigma(x) \left(\|s''\| + \|s'\| \|\sigma''\| \right), \quad (47)$$

where

$$\mathcal{M}_{m,2}^\sigma(x) = \mathcal{D}_{m,\sigma}^\alpha((\sigma(\vartheta) - \sigma(x))^2; x). \quad (48)$$

Also,

$$|\mathcal{D}_{m,\sigma}^\alpha| \leq \|f\|. \quad (49)$$

Hence, we have

$$\begin{aligned} & |\mathcal{D}_{m,\sigma}^\alpha(f; x) - f(x)| \\ & \leq |\mathcal{D}_{m,\sigma}^\alpha(f - s; x)| + |\mathcal{D}_{m,\sigma}^\alpha(s; x) - s(x)| + |s(x) - f(x)| \\ & \leq 2\|f - s\| + \left\{ \left(\frac{m}{(m-1)(1-\alpha)^2} - \frac{1+\alpha}{1-\alpha} \right) \sigma^2(x) \right. \\ & \quad \left. + \frac{\alpha^2 - 2\alpha + 3}{(m-1)(1-\alpha)^3} \sigma(x) \right\} (\|s''\| + \|s'\| \|\sigma''\|), \end{aligned} \quad (50)$$

if $\mathcal{C} = \max \{2, \|\sigma''\|\}$; then

$$\begin{aligned} & |\mathcal{D}_{m,\sigma}^\alpha(f; x) - f(x)| \\ & \leq \mathcal{C} \left(2\|f - s\| + \left\{ \left(\frac{m}{(m-1)(1-\alpha)^2} - \frac{1+\alpha}{1-\alpha} \right) \sigma^2(x) \right. \right. \\ & \quad \left. \left. + \frac{\alpha^2 - 2\alpha + 3}{(m-1)(1-\alpha)^3} \sigma(x) \right\} \|s''\|_{W^2} \right). \end{aligned} \quad (51)$$

Taking infimum over all $s \in W^2$, we obtain

$$|\mathcal{D}_{m,\sigma}^\alpha(f; x) - f(x)| \leq \mathcal{C}\mathcal{H}(f, \delta_m(x)). \quad (52)$$

Theorem 12 is proved.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

We declare that there is no conflict of interest.

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