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Research Article

Estimates of Upper Bound for Differentiable Functions Associated with k-Fractional Integrals and Higher Order Strongly s-Convex Functions

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In this paper, we establish two integral identities associated with differentiable functions and the k-Riemann-Liouville fractional integrals. The results are then used to derive the estimates of upper bound for functions whose first or second derivatives absolute values are higher order strongly s-convex functions.

1. Introduction

Fractional calculus also known as noninteger calculus is a branch of mathematical analysis in which we discuss the integrals and derivatives of arbitrary order. The study of fractional calculus has a very long history, which can be traced back to the end of the 17th century; in 1695, L'Hospital wrote to Leibniz to discuss fractional derivative about a function. For hundreds of years, many mathematicians, such as Euler, Laplace, Fourier, Abel, Liouville, and Riemann, have carried out in-depth research on this subject (see [1]). Especially, in recent decades, the fractional calculus has found numerous applications in various fields such as mechanics, electricity, chemistry, biology, economics, notably control theory, and signal and image processing. Due to the backgrounds in practical applications, the fractional calculus has developed rapidly and has become a hot research topic (see [2–4]).

Among several known forms of fractional integrals, the Riemann-Liouville fractional integral has been investigated extensively, which is defined as follows:

Definition 1 ([3]). Let $f \in L[a, b]$. The Riemann-Liouville integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$
\begin{aligned}
J^{\alpha}_{a^+}f(v) &= \frac{1}{\Gamma(\alpha)}\int_a^v (v-u)^{\alpha-1} f(u)\mathrm{d}u, \quad v > a, \\
J^{\alpha}_{b^-}f(v) &= \frac{1}{\Gamma(\alpha)}\int_v^b (u-v)^{\alpha-1} f(u)\mathrm{d}u, \quad v < b,
\end{aligned}
\tag{1}
$$

where

$$
\Gamma(\alpha) = \int_0^{\infty} e^{-u} u^{\alpha-1}\mathrm{d}u \tag{2}
$$

is the gamma function.

In recent years, several researchers have utilized the concepts of fractional calculus to obtain the fractional analogues of classical inequalities. For example, Sarikaya et al. [5] established a generalized Hermite-Hadamard inequality via the Riemann-Liouville fractional integrals; Set [6] gave some fractional integral inequalities of the Ostrowski type through s-convex functions; Du et al. [7] deduced some variants of the fractional Hermite-Hadamard's inequality using the class of generalized (α,m)-preinvex functions; Noor et al. [8] used the class of the s-Godunova-Levin convex functions to obtain refinements of the fractional Hermite-Hadamard inequalities; Peng et al. [9] obtained the Riemann-Liouville fractional Simpson inequalities through generalized (m,h_1,h_2)-preinvex functions; Wu and Awan [10] used the class of h-convex functions to derive the upper bound estimates of function involving fractional integrals; Wu et al. [11] established some fractional integral inequalities using k-th order differentiable strongly h-preinvex functions; Zhang et al. [12] provided some variations of the fractional Hermite-Hadamard's inequalities. For more results related to this topic, we refer the interested reader to [13–17] and references cited therein.

In [18], Mubeen and Habibullah introduced the k-fractional integral of the Riemann-Liouville type as follows:

Definition 2 ([18]). Let $F \in L[a,b]$. The k-Riemann-Liouville fractional integrals $J^{\alpha}_{a+}F$ and $J^{\alpha}_{b-}F$ of order $\alpha > 0$ with $a \geq 0$, $k > 0$ are defined by

$$\begin{aligned} {}_kJ^{\alpha}_{a^+}F(v) &= \frac{1}{k\Gamma_k(\alpha)}\int_a^v (v-u)^{\alpha/k-1}F(u)\mathrm{d}u, v > a, \\ {}_kJ^{\alpha}_{b^-}F(v) &= \frac{1}{k\Gamma_k(\alpha)}\int_v^b (u-v)^{\alpha/k-1}F(u)\mathrm{d}u, v < b, \end{aligned} \tag{3}$$

where

$$\Gamma_k(\alpha) = \int_0^{\infty} e^{-u^k/k}u^{\alpha-1}\mathrm{d}u \tag{4}$$

is the k-gamma function.

Note that if $k \longrightarrow 1$, then the k-Riemann-Liouville fractional integrals reduces to the classical Riemann-Liouville fractional integral.

Sarikaya et al. [19, 20] generalized the k-Riemann-Liouville fractional integrals and discussed their properties. Moreover, for the k-gamma function, they showed that $\Gamma_k(\alpha) = k^{\alpha/k-1}\Gamma(\alpha/k)$ and $\Gamma_k(\alpha+k) = \alpha\Gamma_k(\alpha)$. For k-beta function, it is defined by

$$B_k(x,y) = \frac{1}{k}\int_0^1 u^{x/k-1}(1-u)^{y/k-1}\mathrm{d}u, x > 0, y > 0, k > 0, \tag{5}$$

which implies that $B_k(x,y) = (1/k)B((x/k),(y/k))$ and $B_k(x,y) = \Gamma_k(x)\Gamma_k(y)/\Gamma_k(x+y)$.

Motivated by the ideas of [19, 20], in this paper, we first establish two identities for the k -Riemann-Liouville fractional integrals associated with differentiable functions. We then apply the results to derive some estimates of the upper bound for differentiable functions involving k-fractional integrals via higher order strongly s-convex functions.

2. Preliminaries and Lemmas

Let us briefly summarize the concepts on generalized convex functions which are related to the contents of this paper.

As a strengthening property of convexity, Polyak [21] introduced the strongly convex functions, as follows:

Definition 3. Let $I \subset \mathbb{R}$ be an interval. A function $f : I \longrightarrow \mathbb{R}$ is said to be strongly convex function with modulus $\mu > 0$, if

$$\begin{aligned} f((1-t)x+ty) &\leq (1-t)f(x) + tf(y) \\ &\quad - \mu t(1-t)|y-x|^2, \forall x,y \in I, t \in [0,1]. \end{aligned} \tag{6}$$

Angulo et al. [22] presented an extension of strongly convex functions which is called strongly s-convex functions, i.e.:

Definition 4. Let $I \subset \mathbb{R}$ be an interval, $s \in (0,1]$. A function $f : I \longrightarrow \mathbb{R}$ is said to be strongly s -convex function with modulus $\mu > 0$, if

$$\begin{aligned} f((1-t)x+ty) &\leq (1-t)^s f(x) + t^s f(y) \\ &\quad - \mu t(1-t)|y-x|^2, \forall x,y \in I, t \in [0,1]. \end{aligned} \tag{7}$$

Here we provide a further extension of strongly s-convex functions, as follows:

Definition 5. Let $I \subset \mathbb{R}$ be an interval, $s \in (0,1]$, $\mu > 0$. A function $f : I \longrightarrow \mathbb{R}$ is said to be strongly s-convex functions of order $\sigma > 0$, if

$$\begin{aligned} f((1-t)x+ty) &\leq (1-t)^s f(x) + t^s f(y) - \mu(t^{\sigma}(1-t) \\ &\quad + t(1-t)^{\sigma})|y-x|^{\sigma}, \forall x,y \in I, t \in [0,1]. \end{aligned} \tag{8}$$

Remark 6. If $\sigma = 2$, then the class of strongly s-convex functions of order $\sigma > 0$ reduces to the class of strongly s-convex functions. For $\sigma = 2$ and $s = 1$, we have the class of classical strongly convex functions.

In the following, we establish two integral identities associated with differentiable functions and the k-Riemann-Liouville fractional integrals. These integral identities play important role in dealing with subsequent results.

Lemma 7. *Let $f : [a,b] \longrightarrow \mathbb{R}$ be a differentiable function and $f' \in L[a,b]$. Then for any $\alpha > 0$, $k > 0$, and $x \in (a,b)$, we have*

$$\begin{aligned} &\frac{(x-a)[(x+a)f(x) - xf(a)] + (b-x)[(b+x)f(x) - xf(b)]}{(b-a)^2} \\ &- \frac{\Gamma_k(\alpha+k)}{(b-a)^2}\left[\frac{a}{(x-a)^{\alpha/k-1}}\,{}_kJ^{\alpha}_{x^-}f(a) + \frac{b}{(b-x)^{\alpha/k-1}}\,{}_kJ^{\alpha}_{x^+}f(b)\right] \\ &= \frac{(x-a)^2}{(b-a)^2}\int_0^1 \left(at^{\alpha/k} + x\right)f'((1-t)a+tx)dt \\ &\quad - \frac{(b-x)^2}{(b-a)^2}\int_0^1 \left(b(1-t)^{\alpha/k} + x\right)f'((1-t)x+tb)dt. \end{aligned} \tag{9}$$

Proof. Let

$$\begin{aligned} I &= \frac{(x-a)^2}{(b-a)^2}\int_0^1\left(at^{\alpha/k}+x\right)f'((1-t)a+tx)\mathrm{d}t \\ &\quad -\frac{(b-x)^2}{(b-a)^2}\int_0^1\left(b(1-t)^{\alpha/k}+x\right)f'((1-t)x+tb)\mathrm{d}t \\ &\triangleq I_1-I_2. \end{aligned} \tag{10}$$

Integrating by parts gives

$$\begin{aligned} I_1 &= \frac{(x-a)^2}{(b-a)^2}\int_0^1\left(at^{\alpha/k}+x\right)f'((1-t)a+tx)\mathrm{d}t \\ &= \frac{(x-a)[(x+a)f(x)-xf(a)]}{(b-a)^2} \\ &\quad -\frac{a\alpha(x-a)}{k(b-a)^2}\int_0^1 t^{\alpha/k-1}f((1-t)a+tx)\mathrm{d}t \\ &= \frac{(x-a)[(x+a)f(x)-xf(a)]}{(b-a)^2} \\ &\quad -\frac{a\Gamma_k(\alpha+k)}{(x-a)^{\alpha/k-1}(b-a)^2k\Gamma_k(\alpha)}\int_a^x(u-a)^{\alpha/k-1}f(u)\mathrm{d}u \\ &= \frac{(x-a)[(x+a)f(x)-xf(a)]}{(b-a)^2} \\ &\quad -\frac{a\Gamma_k(\alpha+k)}{(x-a)^{\alpha/k-1}(b-a)^2}\,{}_kJ^{\alpha}_{x-}f(a). \end{aligned} \tag{11}$$

Similarly, we have

$$\begin{aligned} I_2 &= \frac{(b-x)^2}{(b-a)^2}\int_0^1\left(b(1-t)^{\alpha/k}+x\right)f'((1-t)x+tb)\mathrm{d}t \\ &= \frac{(b-x)[xf(b)-(b+x)f(x)]}{(b-a)^2} \\ &\quad -\frac{b\alpha(b-x)}{k(b-a)^2}\int_0^1(1-t)^{\alpha/k-1}f((1-t)x+tb)\mathrm{d}t \\ &= \frac{(b-x)[xf(b)-(b+x)f(x)]}{(b-a)^2} \\ &\quad +\frac{b\Gamma_k(\alpha+k)}{(b-x)^{\alpha/k-1}(b-a)^2k\Gamma_k(\alpha)}\int_x^b(b-u)^{\alpha/k-1}f(u)\mathrm{d}u \\ &= \frac{(b-x)[xf(b)-(b+x)f(x)]}{(b-a)^2} \\ &\quad +\frac{b\Gamma_k(\alpha+k)}{(b-x)^{\alpha/k-1}(b-a)^2}\,{}_kJ^{\alpha}_{x+}f(b). \end{aligned} \tag{12}$$

Using (11) and (12) in (10) leads to (9). This completes the proof of Lemma 7.

Lemma 8. *Let $f:[a,b]\longrightarrow\mathbb{R}$ be twice differentiable function and $f''\in L[a,b]$. Then for any $\alpha>0$, $k>0$ and $x\in(a,b)$, we have*

$$\begin{aligned} &\frac{1}{(b-a)^2}\left[\frac{(x(\alpha+k)+ka)f(x)-kaf(a)}{k(a-x)}\right. \\ &\quad\left.+\frac{(a(\alpha+k)+kx)f(x)-kxf(b)}{k(x-b)}\right] \\ &\quad+\frac{(\alpha+k)\Gamma_k(\alpha+k)}{k(b-a)^2}\left[\frac{x}{(x-a)^{\alpha/k+1}}\,{}_kJ^{\alpha}_{x-}f(a)\right. \\ &\quad\left.+\frac{a}{(b-x)^{\alpha/k+1}}\,{}_kJ^{\alpha}_{x+}f(b)\right] \\ &\quad=\frac{(x-a)}{(b-a)^2}\int_0^1\left(t^{\alpha/k+1}x+ta\right)f''(tx+(1-t)a)dt \\ &\quad\quad+\frac{(b-x)}{(b-a)^2}\int_0^1\left((1-t)^{\alpha/k+1}a+(1-t)x\right)f'' \\ &\quad\quad\cdot(tb+(1-t)x)dt. \end{aligned} \tag{13}$$

Proof. Let

$$\begin{aligned} I &= \frac{(x-a)}{(b-a)^2}\int_0^1\left(t^{\alpha/k+1}x+ta\right)f''(tx+(1-t)a)\mathrm{d}t \\ &\quad+\frac{(b-x)}{(b-a)^2}\int_0^1\left((1-t)t^{\alpha/k+1}a+(1-t)x\right)f'' \\ &\quad\cdot(tb+(1-t)x)\mathrm{d}t\triangleq I_1+I_2. \end{aligned} \tag{14}$$

Integrating by parts, we obtain

$$\begin{aligned} I_1 &= \frac{(x-a)}{(b-a)^2}\int_0^1\left(t^{a/k+1}x+ta\right)f''(tx+(1-t)a)\mathrm{d}t \\ &= \frac{(x+a)f'(x)}{(b-a)^2}-\frac{1}{(b-a)^2}\int_0^1 \\ &\quad\cdot\left(\left(\frac{\alpha}{k}+1\right)t^{\alpha/k}x+a\right)f'(tx+(1-t)a)\mathrm{d}t \\ &= \frac{(x+a)f'(x)}{(b-a)^2}+\frac{((\alpha+k)x+ka)f(x)-kaf(a)}{k(a-x)(b-a)^2} \\ &\quad+\frac{x\alpha(\alpha+k)}{k^2(x-a)(b-a)^2}\int_0^1 t^{\alpha/k-1}f(tx+(1-t)a)\mathrm{d}t \\ &= \frac{(x+a)f'(x)}{(b-a)^2}+\frac{((\alpha+k)x+ka)f(x)-kaf(a)}{k(a-x)(b-a)^2} \\ &\quad+\frac{x(\alpha+k)\Gamma_k(\alpha+k)}{\Gamma_k(\alpha)k^2(x-a)^{\alpha/k+1}(b-a)^2}\int_a^x(u-a)^{\alpha/k-1}f(u)\mathrm{d}u \\ &= \frac{(x+a)f'(x)}{(b-a)^2}+\frac{(x(\alpha+k)+ka)f(x)-kaf(a)}{k(a-x)(b-a)^2} \\ &\quad+\frac{x(\alpha+k)\Gamma_k(\alpha+k)}{k(x-a)^{\alpha/k+1}(b-a)^2}\,{}_kJ^{\alpha}_{x-}f(a). \end{aligned} \tag{15}$$

Similarly, we have

$$\begin{aligned}
I_2 &= \frac{(b-x)}{(b-a)^2}\int_0^1\left((1-t)^{a/k+1}a+(1-t)x\right)f''(tb+(1-t)x)\mathrm{d}t\\
&= -\frac{(x+a)f'(x)}{(b-a)^2}+\frac{1}{(b-a)^2}\int_0^1\\
&\quad\cdot\left(\left(\frac{\alpha}{k}+1\right)(1-t)^{\alpha/k}a+x\right)f'(tb+(1-t)x)\mathrm{d}t\\
&= -\frac{(x+a)f'(x)}{(b-a)^2}+\frac{((\alpha+k)a+kx)f(x)-kxf(b)}{k(x-b)(b-a)^2}\\
&\quad+\frac{a\alpha(\alpha+k)}{k^2(b-x)(b-a)^2}\int_0^1(1-t)^{\alpha/k-1}f(tb+(1-t)x)\mathrm{d}t\\
&= -\frac{(x+a)f'(x)}{(b-a)^2}+\frac{((\alpha+k)a+kx)f(x)-kxf(b)}{k(x-b)(b-a)^2}\\
&\quad+\frac{a(\alpha+k)\Gamma_k(\alpha+k)}{\Gamma_k(\alpha)k^2(b-x)^{\alpha/k+1}(b-a)^2}\int_x^b(b-u)^{\alpha/k-1}f(u)\mathrm{d}u\\
&= -\frac{(x+a)f'(x)}{(b-a)^2}+\frac{(a(\alpha+k)+kx)f(x)-kxf(b)}{k(x-b)(b-a)^2}\\
&\quad+\frac{a(\alpha+k)\Gamma_k(\alpha+k)}{k(b-x)^{\alpha/k+1}(b-a)^2}\,{}_kJ^{\alpha}_{x+}f(b).
\end{aligned} \tag{16}$$

Combining (14), (15), and (16) leads to (13). Lemma 8 is proved.

3. Main Results

3.1. Estimates of the Upper Bound for Functions Whose First Derivatives Absolute Values Are Higher Order Strongly s-Convex Functions

Theorem 9. *Let $f:[a,b]\subset(0,\infty)\longrightarrow\mathbb{R}$ be a differentiable function and $f'\in L[a,b]$. If $|f'|$ is strongly s-convex functions of order $\sigma>0$, then for any $\alpha>0$, $k>0$, $\mu>0$, $s\in(0,1]$, and $x\in(a,b)$, we have*

$$\begin{aligned}
&\left|\frac{(x-a)[(x+a)f(x)-xf(a)]+(b-x)[(b+x)f(x)-xf(b)]}{(b-a)^2}\right.\\
&\left.-\frac{\Gamma_k(\alpha+k)}{(b-a)^2}\left[\frac{a}{(x-a)^{\alpha/k-1}}\,{}_kJ^{\alpha}_{x-}f(a)+\frac{b}{(b-x)^{\alpha/k-1}}\,{}_kJ^{\alpha}_{x+}f(b)\right]\right|\\
&\quad\le\frac{(x-a)^2}{(b-a)^2}\left(L_1|f'(a)|+L_2|f'(x)|-\mu L_3|x-a|^{\sigma}\right)\\
&\quad+\frac{(b-x)^2}{(b-a)^2}\left(L_4|f'(x)|+L_5|f'(b)|-\mu L_6|b-x|^{\sigma}\right),
\end{aligned} \tag{17}$$

where

$$\begin{aligned}
L_1&=akB_k(\alpha+k,sk+k)+\frac{x}{s+1},\\
L_2&=\frac{ak}{\alpha+ks+k}+\frac{x}{s+1},\\
L_3&=\frac{ak^2}{(\alpha+k\sigma+k)(\alpha+k\sigma+2k)}+akB_k(\alpha+2k,k\sigma+k)\\
&\quad+\frac{x}{(\sigma+1)(\sigma+2)}+xkB_k(2k,k\sigma+k),\\
L_4&=\frac{bk}{\alpha+ks+k}+\frac{x}{s+1},\\
L_5&=bkB_k(ks+k,\alpha+k)+\frac{x}{s+1},\\
L_6&=bkB_k(k\sigma+k,\alpha+2k)-bkB_k(k\sigma+2k,\alpha+k)\\
&\quad+bkB_k(2k,\alpha+k\sigma+k)+\frac{x}{(\sigma+1)(\sigma+2)}\\
&\quad+xkB_k(2k,k\sigma+k).
\end{aligned} \tag{18}$$

Proof. By Lemma 7 and the property of absolute value, we have

$$\begin{aligned}
&\left|\frac{(x-a)[(x+a)f(x)-xf(a)]+(b-x)[(b+x)f(x)-xf(b)]}{(b-a)^2}\right.\\
&\left.-\frac{\Gamma_k(\alpha+k)}{(b-a)^2}\left[\frac{a}{(x-a)^{\alpha/k-1}}\,{}_kJ^{\alpha}_{x-}f(a)+\frac{b}{(b-x)^{\alpha/k-1}}\,{}_kJ^{\alpha}_{x+}f(b)\right]\right|\\
&\quad\le\frac{(x-a)^2}{(b-a)^2}\int_0^1\left(at^{\alpha/k}+x\right)\left|f'((1-t)a+tx)\right|\mathrm{d}t\\
&\quad+\frac{(b-x)^2}{(b-a)^2}\int_0^1\left(b(1-t)^{\alpha/k}+x\right)\left|f'((1-t)x+tb)\right|\mathrm{d}t.
\end{aligned} \tag{19}$$

Utilizing the fact that $|f'|$ is strongly s-convex functions of the order $\sigma>0$, we obtain

$$\begin{aligned}
&\frac{(x-a)^2}{(b-a)^2}\int_0^1\left(at^{\alpha/k}+x\right)\left|f'((1-t)a+tx)\right|\mathrm{d}t\\
&\quad+\frac{(b-x)^2}{(b-a)^2}\int_0^1\left(b(1-t)^{\alpha/k}+x\right)\left|f'((1-t)x+tb)\right|\mathrm{d}t\\
&\quad\le\frac{(x-a)^2}{(b-a)^2}\int_0^1\left(at^{\alpha/k}+x\right)\left[(1-t)^s|f'(a)|+t^s|f'(x)|\right.\\
&\quad\left.-\mu\{t^{\sigma}(1-t)+t(1-t)^{\sigma}\}|x-a|^{\sigma}\right]\mathrm{d}t\\
&\quad+\frac{(b-x)^2}{(b-a)^2}\int_0^1\left(b(1-t)^{\alpha/k}+x\right)\left[(1-t)^s|f'(x)|\right.\\
&\quad\left.+t^s|f'(b)|-\mu\{t^{\sigma}(1-t)+t(1-t)^{\sigma}\}|b-x|^{\sigma}\right]\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(x-a)^2}{(b-a)^2}\left[\left(akB_k(\alpha+k, sk+k) + \frac{x}{s+1}\right)\left|f'(a)\right|\right. \\
&\quad + \left(\frac{ak}{\alpha+ks+k} + \frac{x}{s+1}\right)\left|f'(x)\right| \\
&\quad - \mu\left(\frac{ak^2}{(\alpha+k\sigma+k)(\alpha+k\sigma+2k)}\right. \\
&\quad + akB_k(\alpha+2k, k\sigma+k) + \frac{x}{(\sigma+1)(\sigma+2)} \\
&\quad \left.\left. + xkB_k(2k, k\sigma+k)\right)|x-a|^\sigma\right] + \frac{(b-x)^2}{(b-a)^2} \\
&\quad \cdot \left[\left(\frac{bk}{\alpha+ks+k} + \frac{x}{s+1}\right)\left|f'(x)\right|\right. \\
&\quad + \left(bkB_k(ks+k, \alpha+k) + \frac{x}{s+1}\right)\left|f'(b)\right| \\
&\quad - \mu\left(bkB_k(k\sigma+k, \alpha+2k) - bkB_k(k\sigma+2k, \alpha+k)\right. \\
&\quad + bkB_k(2k, \alpha+k\sigma+k) + \frac{x}{(\sigma+1)(\sigma+2)} \\
&\quad \left.\left. + xkB_k(2k, k\sigma+k)\right)|b-x|^\sigma\right],
\end{aligned} \tag{20}
$$

which implies the desired inequality (17). The proof of Theorem 9 is complete. is complete.

Theorem 10. *Let $f : [a,b] \subset (0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function and $f' \in L[a,b]$, and let $p > 1$, $q > 1$, $1/p + 1/q = 1$. If $|f'|^q$ is strongly s-convex functions of the order $\sigma > 0$, then for any $\alpha > 0$,*
$k > 0$, $\mu > 0$, $s \in (0,1]$, and $x \in (a,b)$, we have

$$
\begin{aligned}
&\left|\frac{(x-a)[(x+a)f(x) - xf(a)] + (b-x)[(b+x)f(x) - xf(b)]}{(b-a)^2}\right. \\
&\quad \left.- \frac{\Gamma_k(\alpha+k)}{(b-a)^2}\left[\frac{a}{(x-a)^{\alpha/k-1}}\, {}_kJ^\alpha_{x-}f(a) + \frac{b}{(b-x)^{\alpha/k-1}}\, {}_kJ^\alpha_{x+}f(b)\right]\right| \\
&\le \frac{(x-a)^2}{(b-a)^2}\left(\frac{ak}{\alpha+k} + x\right)^{1/p}\left(L_1\left|f'(a)\right|^q\right. \\
&\quad \left.+ L_2\left|f'(x)\right|^q - \mu L_3|x-a|^\sigma\right)^{1/q} + \frac{(b-x)^2}{(b-a)^2} \\
&\quad \cdot \left(\frac{bk}{\alpha+k} + x\right)^{1/p}\left(L_4\left|f'(x)\right|^q + L_5\left|f'(b)\right|^q\right. \\
&\quad \left.- \mu L_6|b-x|^\sigma\right)^{1/q},
\end{aligned} \tag{21}
$$

where L_1, L_2, L_3, L_4, L_5, and L_6 are given by the same expressions as described in Theorem 9.

Proof. Using Lemma 7, Hölder's inequality, and the fact that $|f'|^q$ is strongly s-convex functions of the order $\sigma > 0$, it follows that

$$
\begin{aligned}
&\left|\frac{(x-a)[(x+a)f(x) - xf(a)] + (b-x)[(b+x)f(x) - xf(b)]}{(b-a)^2}\right. \\
&\quad \left.- \frac{\Gamma_k(\alpha+k)}{(b-a)^2}\left[\frac{a}{(x-a)^{\alpha/k-1}}\, {}_kJ^\alpha_{x-}f(a) + \frac{b}{(b-x)^{\alpha/k-1}}\, {}_kJ^\alpha_{x+}f(b)\right]\right| \\
&\le \frac{(x-a)^2}{(b-a)^2}\int_0^1\left(at^{\alpha/k} + x\right)\left|f'((1-t)a + tx)\right|\mathrm{d}t \\
&\quad + \frac{(b-x)^2}{(b-a)^2}\int_0^1\left(b(1-t)^{\alpha/k} + x\right)\left|f'((1-t)x + tb)\right|\mathrm{d}t \\
&\le \frac{(x-a)^2}{(b-a)^2}\left(\int_0^1\left(at^{\alpha/k} + x\right)\mathrm{d}t\right)^{1/p} \\
&\quad \cdot \left(\int_0^1\left(at^{\alpha/k} + x\right)\left|f'((1-t)a + tx)\right|^q\mathrm{d}t\right)^{1/q} \\
&\quad + \frac{(b-x)^2}{(b-a)^2}\left(\int_0^1\left(b(1-t)^{\alpha/k} + x\right)\mathrm{d}t\right)^{1/p} \\
&\quad \cdot \left(\int_0^1\left(b(1-t)^{\alpha/k} + x\right)\left|f'((1-t)x + tb)\right|^q\mathrm{d}t\right)^{1/q} \\
&\le \frac{(x-a)^2}{(b-a)^2}\left(\int_0^1\left(at^{\alpha/k} + x\right)\mathrm{d}t\right)^{1/p} \\
&\quad \times \left(\int_0^1\left(at^{\alpha/k} + x\right)\left[(1-t)^s\left|f'(a)\right|^q + t^s\left|f'(x)\right|^q\right.\right. \\
&\quad \left.\left.- \mu(t^\sigma(1-t) + t(1-t)^\sigma)|x-a|^\sigma\right]\mathrm{d}t\right)^{1/q} \\
&\quad + \frac{(b-x)^2}{(b-a)^2}\left(\int_0^1\left(b(1-t)^{\alpha/k} + x\right)\mathrm{d}t\right)^{1/p} \\
&\quad \times \left(\int_0^1\left(b(1-t)^{\alpha/k} + x\right)\left[(1-t)^s\left|f'(x)\right|^q\right.\right. \\
&\quad \left.\left.+ t^s\left|f'(b)\right|^q - \mu(t^\sigma(1-t) + t(1-t)^\sigma)|b-x|^\sigma\right]\mathrm{d}t\right)^{1/q} \\
&= \frac{(x-a)^2}{(b-a)^2}\left(\frac{ak}{\alpha+k} + x\right)^{1/p} \times \left[\left(akB_k(\alpha+k, sk+k)\right.\right. \\
&\quad \left.+ \frac{x}{s+1}\right)\left|f'(a)\right|^q + \left(\frac{ak}{\alpha+ks+k} + \frac{x}{s+1}\right)\left|f'(x)\right|^q \\
&\quad - \mu\left(\frac{ak^2}{(\alpha+k\sigma+k)(\alpha+k\sigma+2k)} + akB_k(\alpha+2k, k\sigma+k)\right. \\
&\quad \left.\left.+ \frac{x}{(\sigma+1)(\sigma+2)} + xkB_k(2k, k\sigma+k)\right)|x-a|^\sigma\right]^{1/q} \\
&\quad + \frac{(b-x)^2}{(b-a)^2}\left(\frac{bk}{\alpha+k} + x\right)^{1/p} \times \left[\left(\frac{bk}{\alpha+ks+k} + \frac{x}{s+1}\right)\right. \\
&\quad \cdot \left|f'(x)\right|^q + \left(bkB_k(ks+k, \alpha+k) + \frac{x}{s+1}\right)\left|f'(b)\right|^q
\end{aligned}
$$

$$
\begin{aligned}
&-\mu\bigg(bkB_k(k\sigma+k,\alpha+2k)-bkB_k(k\sigma+2k,\alpha+k)\\
&+bkB_k(2k,\alpha+k\sigma+k)+\frac{x}{(\sigma+1)(\sigma+2)}\\
&+xkB_k(2k,k\sigma+k)\bigg)|b-x|^{\sigma}\bigg]^{1/q}=\frac{(x-a)^2}{(b-a)^2}\\
&\cdot\left(\frac{ak}{\alpha+k}+x\right)^{1/p}\left(L_1\left|f'(a)\right|^q+L_2\left|f'(x)\right|^q-\mu L_3|x-a|^{\sigma}\right)^{1/q}\\
&+\frac{(b-x)^2}{(b-a)^2}\left(\frac{bk}{\alpha+k}+x\right)^{1/p}\left(L_4\left|f'(x)\right|^q+L_5\left|f'(b)\right|^q\right.\\
&\left.-\mu L_6|b-x|^{\sigma}\right)^{1/q}.
\end{aligned}
\tag{22}
$$

This completes the proof of Theorem 10.

3.2. Estimates of the Upper Bound for Functions Whose Second Derivatives Absolute Values Are Higher Order Strongly s-Convex Functions

Theorem 11. *Let $f:[a,b]\subset(0,\infty)\longrightarrow\mathbb{R}$ be a differentiable function and $f''\in L[a,b]$. If $|f''|$ is strongly s -convex functions of the order $\sigma>0$, then for any $\alpha>0$, $k>0$, $\mu>0$, $s\in($ $0,1]$, and $x\in(a,b)$, we have*

$$
\begin{aligned}
&\left|\frac{1}{(b-a)^2}\left[\frac{(x(\alpha+k)+ka)f(x)-kaf(a)}{k(a-x)}\right.\right.\\
&\left.+\frac{(a(\alpha+k)+kx)f(x)-kxf(b)}{k(x-b)}\right]\\
&+\frac{(\alpha+k)\Gamma_k(\alpha+k)}{k(b-a)^2}\left[\frac{x}{(x-a)^{\alpha/k+1}}\,{}_kJ^{\alpha}_{x-}f(a)\right.\\
&\left.\left.+\frac{a}{(b-x)^{\alpha/k+1}}\,{}_kJ^{\alpha}_{x+}f(b)\right]\right|\\
&\quad\le\frac{(x-a)}{(b-a)^2}\left(M_1\left|f''(x)\right|+M_2\left|f''(a)\right|-\mu M_3|x-a|^{\sigma}\right)\\
&\quad+\frac{(b-x)}{(b-a)^2}\left(M_4\left|f''(b)\right|+M_5\left|f''(x)\right|-\mu M_6|b-x|^{\sigma}\right),
\end{aligned}
\tag{23}
$$

where

$$
\begin{aligned}
M_1&=\frac{xk}{\alpha+ks+2k}+\frac{a}{s+2},\\
M_2&=xkB_k(\alpha+2k,ks+k)+akB_k(2k,ks+k),\\
M_3&=\frac{xk^2}{(\alpha+k\sigma+2k)(\alpha+k\sigma+3k)}\\
&\quad+xkB_k(\alpha+3k,k\sigma+k)+\frac{a}{(\sigma+2)(\sigma+3)}\\
&\quad+akB_k(3k,k\sigma+k),\\
M_4&=akB_k(ks+k,\alpha+2k)+\frac{x}{(s+1)(s+2)},\\
M_5&=\frac{ka}{\alpha+ks+2k}+\frac{x}{s+2},\\
M_6&=kaB_k(k\sigma+k,\alpha+3k)+kaB_k(2k,\alpha+k\sigma+2k)\\
&\quad+xkB_k(k\sigma+k,3k)+xkB_k(2k,k\sigma+2k).
\end{aligned}
\tag{24}
$$

Proof. By Lemma 8 and the property of absolute value, we have

$$
\begin{aligned}
&\left|\frac{1}{(b-a)^2}\left[\frac{(x(\alpha+k)+ka)f(x)-kaf(a)}{k(a-x)}\right.\right.\\
&\left.+\frac{(a(\alpha+k)+kx)f(x)-kxf(b)}{k(x-b)}\right]+\frac{(\alpha+k)\Gamma_k(\alpha+k)}{k(b-a)^2}\\
&\left.\cdot\left[\frac{x}{(x-a)^{\alpha/k+1}}\,{}_kJ^{\alpha}_{x-}f(a)+\frac{a}{(b-x)^{\alpha/k+1}}\,{}_kJ^{\alpha}_{x+}f(b)\right]\right|\\
&=\left|\frac{(x-a)}{(b-a)^2}\int_0^1\left(t^{\alpha/k+1}x+ta\right)f''(tx+(1-t)a)\mathrm{d}t\right.\\
&+\frac{(b-x)}{(b-a)^2}\int_0^1\left((1-t)^{\alpha/k+1}a+(1-t)x\right)f''\\
&\left.\cdot(tb+(1-t)x)\mathrm{d}t\right|\le\frac{(x-a)}{(b-a)^2}\int_0^1\left(t^{\alpha/k+1}x+ta\right)\\
&\cdot\left|f''(tx+(1-t)a)\right|\mathrm{d}t+\frac{(b-x)}{(b-a)^2}\int_0^1\\
&\cdot\left((1-t)^{\alpha/k+1}a+(1-t)x\right)\left|f''(tb+(1-t)x)\right|\mathrm{d}t.
\end{aligned}
\tag{25}
$$

Utilizing the fact that $|f''|$ is strongly s-convex functions of the order $\sigma>0$, we obtain

$$
\begin{aligned}
&\frac{(x-a)}{(b-a)^2}\int_0^1\left(t^{\alpha/k+1}x+ta\right)\left|f''(tx+(1-t)a)\right|\mathrm{d}t\\
&+\frac{(b-x)}{(b-a)^2}\int_0^1\left((1-t)^{\alpha/k+1}a+(1-t)x\right)\left|f''(tb+(1-t)x)\right|\mathrm{d}t\\
&\le\frac{(x-a)}{(b-a)^2}\int_0^1\left(t^{\alpha/k+1}x+ta\right)\left[\left(t^s\left|f''(x)\right|+(1-t)^s\left|f''(a)\right|\right)\right.\\
&\left.-\mu\left(t^{\sigma}(1-t)+t(1-t)^{\sigma}\right)|x-a|^{\sigma}\right]\mathrm{d}t\\
&+\frac{(b-x)}{(b-a)^2}\int_0^1\left((1-t)^{\alpha/k+1}a+(1-t)x\right)\left[\left(t^s\left|f''(b)\right|\right.\right.\\
&\left.\left.+(1-t)^s\left|f''(x)\right|\right)-\mu\left(t^{\sigma}(1-t)+t(1-t)^{\sigma}\right)|b-x|^{\sigma}\right]\mathrm{d}t
\end{aligned}
$$

$$= \frac{(x-a)}{(b-a)^2}\left[\left(\frac{xk}{\alpha+ks+2k}+\frac{a}{s+2}\right)\left|f''(x)\right| + \left(xkB_k(\alpha+2k,ks+k)+akB_k(2k,ks+k)\right)\left|f''(a)\right| - \mu\left(\frac{xk^2}{(\alpha+k\sigma+2k)(\alpha+k\sigma+3k)} + xkB_k(\alpha+3k,k\sigma+k) + \frac{a}{(\sigma+2)(\sigma+3)} + akB_k(3k,k\sigma+k)\right)|x-a|^\sigma\right] + \frac{(b-x)}{(b-a)^2}\left[\left(akB_k(ks+k,\alpha+2k)+\frac{x}{(s+1)(s+2)}\right)\cdot\left|f''(b)\right| + \left(\frac{ka}{\alpha+ks+2k}+\frac{x}{s+2}\right)\left|f''(x)\right| - \mu\left(kaB_k(k\sigma+k,\alpha+3k)+kaB_k(2k,\alpha+k\sigma+2k) + xkB_k(k\sigma+k,3k)+xkB_k(2k,k\sigma+2k)\right)|b-x|^\sigma\right], \tag{26}$$

which implies the desired inequality (23). This completes the proof of Theorem 11 .

Theorem 12. *Let $f:[a,b]\subset(0,\infty)\longrightarrow\mathbb{R}$ be twice differentiable function and $f''\in L[a,b]$, and let $p>1, q>1, 1/p+1/q=1$. If $|f''|^q$ is strongly s-convex functions of the order $\sigma>0$, then for any $\alpha>0$, $k>0$, $\mu>0$, $s\in(0,1]$, and $x\in(a,b)$, we have*

$$\left|\frac{1}{(b-a)^2}\left[\frac{(x(\alpha+k)+ka)f(x)-kaf(a)}{k(a-x)} + \frac{(a(\alpha+k)+kx)f(x)-kxf(b)}{k(x-b)}\right] + \frac{(\alpha+k)\Gamma_k(\alpha+k)}{k(b-a)^2}\cdot\left[\frac{x}{(x-a)^{\alpha/k+1}}\,{}_kJ^{\alpha}_{x-}f(a) + \frac{a}{(b-x)^{\alpha/k+1}}\,{}_kJ^{\alpha}_{x+}f(b)\right]\right| \le \frac{(x-a)}{(b-a)^2}\left(\frac{xk}{\alpha+2k}+\frac{a}{2}\right)^{1/p}\left(M_1\left|f''(x)\right|^q + M_2\left|f''(a)\right|^q - \mu M_3|x-a|^\sigma\right)^{1/q} + \frac{(b-x)}{(b-a)^2}\cdot\left(\frac{ak}{\alpha+2k}+\frac{x}{2}\right)^{1/p}\left(M_4\left|f''(b)\right|^q + M_5\left|f''(x)\right|^q - \mu M_6|b-x|^\sigma\right)^{1/q}, \tag{27}$$

where M_1, M_2, M_3, M_4, M_5, and M_6 are given by the same expressions as described in Theorem 11.

Proof. Using Lemma 8, Hölder's inequality, and the fact that $|f''|^q$ is strongly s-convex functions of the order $\sigma>0$, we obtain

$$\left|\frac{1}{(b-a)^2}\left[\frac{(x(\alpha+k)+ka)f(x)-kaf(a)}{k(a-x)} + \frac{(a(\alpha+k)+kx)f(x)-kxf(b)}{k(x-b)}\right] + \frac{(\alpha+k)\Gamma_k(\alpha+k)}{k(b-a)^2}\cdot\left[\frac{x}{(x-a)^{\alpha/k+1}}\,{}_kJ^{\alpha}_{x-}f(a) + \frac{a}{(b-x)^{\alpha/k+1}}\,{}_kJ^{\alpha}_{x+}f(b)\right]\right|$$
$$\le \frac{(x-a)}{(b-a)^2}\int_0^1\left(t^{\alpha/k+1}x+ta\right)\left|f''(tx+(1-t)a)\right|\mathrm{d}t + \frac{(b-x)}{(b-a)^2}\int_0^1\left((1-t)^{\alpha/k+1}a+(1-t)x\right)\cdot\left|f''(tb+(1-t)x)\right|\mathrm{d}t$$
$$\le \frac{(x-a)}{(b-a)^2}\left(\int_0^1\left(t^{\alpha/k+1}x+ta\right)\mathrm{d}t\right)^{1/p}\times\left(\int_0^1\left(t^{\alpha/k+1}x+ta\right)\left|f''(tx+(1-t)a)\right|^q\mathrm{d}t\right)^{1/q} + \frac{(b-x)}{(b-a)^2}\left(\int_0^1\left((1-t)^{\alpha/k+1}a+(1-t)x\right)\mathrm{d}t\right)^{1/p}\times\left(\int_0^1\left((1-t)^{\alpha/k+1}a+(1-t)x\right)\left|f''(tb+(1-t)x)\right|^q\mathrm{d}t\right)^{1/q}$$
$$\le \frac{(x-a)}{(b-a)^2}\left(\int_0^1\left(t^{\alpha/k+1}x+ta\right)\mathrm{d}t\right)^{1/p}\times\left(\int_0^1\left(t^{\alpha/k+1}x+ta\right)\left[\left(t^s\left|f''(x)\right|^q+(1-t)^s\left|f''(a)\right|^q\right) - \mu\left(t^\sigma(1-t)+t(1-t)^\sigma\right)|x-a|^\sigma\right]\mathrm{d}t\right)^{1/q} + \frac{(b-x)}{(b-a)^2}\left(\int_0^1\left((1-t)^{\alpha/k+1}a+(1-t)x\right)\mathrm{d}t\right)^{1/p}\times\left(\int_0^1\left((1-t)^{\alpha/k+1}a+(1-t)x\right)\left[t^s\left|f''(b)\right|^q + (1-t)^s\left|f''(x)\right|^q - \mu\left(t^\sigma(1-t)+t(1-t)^\sigma\right)\cdot|b-x|^\sigma\right]\mathrm{d}t\right)^{1/q}$$
$$= \frac{(x-a)}{(b-a)^2}\left(\frac{xk}{\alpha+2k}+\frac{a}{2}\right)^{1/p}\cdot\left(\left(\frac{xk}{\alpha+ks+2k}+\frac{a}{s+2}\right)\left|f''(x)\right|^q + \left(xkB_k(\alpha+2k,ks+k)+akB_k(2k,ks+k)\right)\cdot\left|f''(a)\right|^q - \mu\left(\frac{xk^2}{(\alpha+k\sigma+2k)(\alpha+k\sigma+3k)} + xkB_k(\alpha+3k,k\sigma+k) + \frac{a}{(\sigma+2)(\sigma+3)} + akB_k(3k,k\sigma+k)\right)|x-a|^\sigma\right)^{1/q} + \frac{(b-x)}{(b-a)^2}\left(\frac{ak}{\alpha+2k}+\frac{x}{2}\right)^{1/p}\cdot\left(\left(akB_k(ks+k,\alpha+2k)+\frac{x}{(s+1)(s+2)}\right)\right.$$

$$\cdot \left|f''(b)\right|^q + \left(\frac{ka}{\alpha+ks+2k} + \frac{x}{s+2}\right)\left|f''(x)\right|^q - \mu(kaB_k(k\sigma+k,\alpha+3k) + kaB_k(2k,\alpha+k\sigma+2k) + xkB_k(k\sigma+k,3k) + xkB_k(2k,k\sigma+2k))|b-x|^\sigma\Bigg)^{1/q}. \tag{28}$$

The proof of Theorem 12 is complete.

4. Conclusion

In this paper, we establish two identities involving differentiable functions and the k-Riemann-Liouville fractional integrals. Utilizing the identities, we obtain the estimates of the upper bound for functions whose first or second derivatives absolute values are higher order strongly s-convex functions. It is worth mentioning that our results contain, as a special case $(k, s, \sigma) = (1,1,2)$, the estimates of the upper bound of functions for the classical Riemann-Liouville fractional integrals and strongly convex functions.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All authors contributed equally and significantly in this paper. All authors read and approved the final manuscript.

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