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Research Article

Fuzzy Fixed Point Results in $\mathscr{F}$-Metric Spaces with Applications

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In this paper, some concepts of $\mathscr{F}$-metric spaces are used to study a few fuzzy fixed point theorems. Consequently, corresponding fixed point theorems of multivalued and single-valued mappings are discussed. Moreover, one of our obtained results is applied to establish some conditions for existence of solutions of fuzzy Cauchy problems. It is hoped that the established ideas in this work will awake new research directions in fuzzy fixed point theory and related hybrid models in the framework of $\mathscr{F}$-metric spaces.

1. Introduction

One of the challenges in mathematical modeling of practical phenomena relates to the indeterminacy induced by our inability to categorize events with adequate precision. It has been understood that classical mathematics cannot cope effectively with imprecisions. As a result, the concept of fuzzy set was initiated by Zadeh [1] in 1965 as one of the uncertainty approaches to construct mathematical models compatible with real world problems in engineering, life science, economics, medicine, language theory, and so on. The basic ideas of fuzzy set have been extended in different directions. In particular, the notion of fixed point results for fuzzy set-valued mappings and fuzzy contractions was initiated by Heilpern [2] who proved a fixed point theorem parallel to the Banach fixed point theorem (see [3]) in the frame of fuzzy set. Thereafter, several authors have studied and applied fuzzy fixed point results in different settings [4, 5], see, for example [6–14] and the references therein.

Not long ago, Jleli and Samet [9] initiated the concepts of $\mathscr{F}$-metric spaces and obtained a generalization of the Banach fixed point theorem. Meanwhile, researchers have picked keen interests in establishing and improving different results in $\mathscr{F}$-metric spaces, see, for instance, [15–17].

The aim of this paper is twofold. First, we study some common fuzzy fixed point results in the setting of $\mathscr{F}$-complete $\mathscr{F}$-metric spaces. Consequently, corresponding fixed point theorems of multivalued and single-valued mappings are derived. Thereafter, one of our obtained results is applied to discuss some solvability conditions of fuzzy initial value problems. As far as we know, in the setting of $\mathscr{F}$-metric spaces and fuzzy mappings, the results presented herein are new and fundamental. On this note, it can be improved upon when discussed in other generalized hybrid models within the scope of fuzzy mathematics.

2. Preliminaries

In this section, we record some specific definitions and results that will be useful in what follows hereafter.

Definition 1 (see [9]). Let $\mathscr{F}$ be the set of functions $f:(0,+\infty)\longrightarrow\mathbb{R}$ satisfying the following conditions:

($\mathscr{F}_1$) f is nondecreasing, i.e., $0<s<t$ implies $f(s)\leq f(t)$

($\mathscr{F}_2$) for every sequence $\{t_n\}\subset(0,+\infty)$, we have

$$\lim_{n\longrightarrow+\infty} t_n = 0 \text{ implies } \lim_{n\longrightarrow+\infty} f(t_n) = -\infty. \tag{1}$$

Definition 2 (see [9]). Let X be a nonempty set and $D: X\times X\longrightarrow[0,+\infty)$ be a given mapping. Suppose that there exists $(f,\rho)\in\mathscr{F}\times[0,+\infty)$ such that

(D_1) $(x, y) \in X \times X$, $D(x, y) = 0 \Longleftarrow \Rightarrow x = y$

(D_2) $D(x, y) = D(y, x)$, for all $(x, y) \in X \times X$

(D_3) for every $(x, y) \in X \times X$, for every $k \in \mathbb{N}, k \geq 2$, and for every sequence $\{u_i\}_{i=1}^{n} \subset X$ with $(u_1, u_k) = (x, y)$, we have

$$D(x, y) > 0 \Rightarrow f(D(x, y)) \leq f\left(\sum_{i=1}^{k-1} D(u_i, u_{i-1})\right) + \rho. \tag{2}$$

Then, D is said to be an $\mathcal{F}$ -metric and the pair (X, D) is called an $\mathcal{F}$ -metric space.

Example 3 (see [9]). Let $X = \mathbb{R}$. If we define $D : X \times X \longrightarrow [0,+\infty)$ by

$$D(x, y) = \begin{cases} (x-y)^2, & \text{if } (x, y) \in [0, 3] \times [0, 3], \\ |x-y|, & \text{if } (x, y) \notin [0, 3] \times [0, 3], \end{cases} \tag{3}$$

with $f(t) = \ln(t)$ and $\rho = \ln(3)$, then $(X, D)$is an $\mathcal{F}$ -metric space.

Definition 4 (see [9]). Let (X, D) be an $\mathcal{F}$ -metric space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X.

(i) We say that $\{x_n\}_{n\in\mathbb{N}}$ is $\mathcal{F}$ -convergent to x if $\{x_n\}_{n\in\mathbb{N}}$ is convergent to x with respect to the $\mathcal{F}$ -metric D

(ii) $\{x_n\}_{n\in\mathbb{N}}$ is said to be $\mathcal{F}$ -Cauchy, if

$$\lim_{k,n\longrightarrow\infty} D(x_k, x_n) = 0. \tag{4}$$

(iii) (X, D) is $\mathcal{F}$ -complete, if every $\mathcal{F}$ -Cauchy sequence in X is $\mathcal{F}$ -convergent to an element in X

In the sequel, we shall adopt the following notations and definitions in the setting of $\mathcal{F}$-metric space. We shall denote an $\mathcal{F}$-metric by $D_{\mathcal{F}}$ so that $(X, D_{\mathcal{F}})$ represents an $\mathcal{F}$-metric space. Let $C(2^X)$ be the set of all nonempty compact subsets of X and $A, B \in C(2^X)$. Then

$$\begin{aligned} D_{\mathcal{F}}(x, A) &= \inf \{D_{\mathcal{F}}(x, a) : a \in A\}, \\ D_{\mathcal{F}}(A, B) &= \inf \{D_{\mathcal{F}}(a, b) : a \in A, b \in B\}. \end{aligned} \tag{5}$$

Then, the Hausdorff metric $H_{\mathcal{F}}$ on $C(2^X)$ induced by the metric $D_{\mathcal{F}}$ is defined as

$$H_{\mathcal{F}}(A, B) = \begin{cases} \max\{\sup_{a\in A} D_{\mathcal{F}}(a, B), \sup_{b\in B} D_{\mathcal{F}}(A, b)\}, & \text{if it exists,} \\ \infty, & \text{otherwise.} \end{cases} \tag{6}$$

Definition 5 (see [1, 2]). Let X be an arbitrary nonempty set. Then, a fuzzy set in X is a function with domain X and values in $[0, 1] = I$. If B is a fuzzy set in X and $x \in X$, then the function values $B(x)$ is called the grade of membership of $x \in X$. The α -level set of B, denoted by $[B]_\alpha$, is defined as

$$\begin{aligned} [B]_\alpha &= \{x \in X : B(x) \geq \alpha\} \text{ if } x \in (0, 1], \\ [B]_0 &= \overline{\{A(x) > 0\}}. \end{aligned} \tag{7}$$

Here, $\bar{M}$ denotes the closure of the crisp set M. Also, the family of fuzzy sets in a metric space X shall be denoted by I^X.

Definition 6. Let X be an $\mathcal{F}$ -metric space. A subset A of X is called proximal, if for each $x \in X$, there exists $a \in A$ such that $D_{\mathcal{F}}(x, a) = D_{\mathcal{F}}(x, A)$.

Let the set of all nonempty bounded proximal sets in X be denoted by $\mathcal{P}^r(X)$ and the set of all nonempty closed and bounded subsets of X be represented by $CB(X)$. Since every compact set is proximal and any proximal set is closed, we have the inclusions:

$$C(2^X) \subseteq \mathcal{P}^r(X) \subseteq CB(X). \tag{8}$$

Definition 7 (see [2]). Let X be an arbitrary set and Y a metric space. A mapping $B : X \longrightarrow I^X$ is called a fuzzy mapping. A fuzzy mapping B is a fuzzy subset of $X \times Y$ with membership value $B(x)(y)$.

Definition 8 (see [2, 7]). Let A and B be fuzzy mappings from X into I^X. A point $u \in X$ is called fuzzy fixed point of A if $u \in [Au]_\alpha$. The point u is known as a common fuzzy fixed point of A and B if $u \in [Au]_\alpha \cap [Bu]_\alpha$.

Definition 9 (see [18, 19]). A nondecreasing function $\varphi : [0, \infty) \longrightarrow [0,\infty)$ is said to be a comparison function, if $\varphi^n(t) \longrightarrow 0$ as $n \longrightarrow \infty$ for every $t \in [0,\infty)$, where $\varphi^n(t)$ denotes the nth iterate of φ.

Denote by Ω the set of all comparison functions.

Lemma 10 (see [18, 19]). *Let $\varphi \in \Omega$. Then, the following properties hold:*

(i) Each iterate φ^i of φ, for $i \geq 1$ is a comparison function

(ii) φ is continuous at 0

(iii) $\varphi(t) < t$ for all $t > 0$

3. Main Results

First, we present the following auxiliary result.

Lemma 11. *Let A and B be nonempty closed and compact subsets of an $\mathcal{F}$-metric space $(X, D_{\mathcal{F}})$. If $a \in A$, then $D_{\mathcal{F}}(a, B) \leq H_{\mathcal{F}}(A, B)$.*

Proof. The proof is a direct consequence of the definition of $H_{\mathcal{F}}(A, B)$.

Theorem 12. *Let $(X, D_{\mathcal{F}})$ be an $\mathcal{F}$-complete $\mathcal{F}$-metric space and $A, B : X \longrightarrow I^X$ be fuzzy mappings. Assume that for every*

$x \in X$, there exist $\alpha_A(x), \alpha_B(x) \in (0,1]$ such that $[Ax]_{\alpha A(x)}$, $[Bx]_{\alpha B(x)} \in C(2^X)$. Suppose also that the following condition holds:

$$H_{\mathcal{F}}\left([Ax]_{\alpha_A(x)}, [By]_{\alpha_B(x)}\right) \leq \varphi\left(\max\left\{D_{\mathcal{F}}(x,y), D_{\mathcal{F}}\left(x, [A_x]_{\alpha_A(x)}\right), D_{\mathcal{F}}\left(y, [By]_{\alpha_B(y)}\right), \frac{D_{\mathcal{F}}\left(y, [By]_{\alpha_B(y)}\right) + D_{\mathcal{F}}\left(x, [Ax]_{\alpha_A(x)}\right)}{2}\right\}\right), \quad (9)$$

for all $x, y \in X$, where $\phi \in \Omega$. Then, there exists $u \in X$ such that $u \in [Au]_{\alpha A(u)} \cap [Bu]_{\alpha B(u)}$.

Proof. Let $x_0 \in X$ be arbitrary. By hypothesis, there exists $\alpha_A(x_0) \in (0,1]$ such that $[Ax_0]_{\alpha A(x_0)} \in C(2^X)$. Since $[Ax_0]_{\alpha A(x0)}$ is a nonempty compact subset of X, there exists $x_1 \in [Ax_0]_{\alpha A(x_0)}$ such that $D_{\mathcal{F}}(x_0, x_1) = D_{\mathcal{F}}(x_0, [Ax_0]_{\alpha A(x0)})$. Similarly, we can find $\alpha_B(x_1) \in (0,1]$ such that $[Bx_1]_{\alpha B(x1)} \in C(2^X)$ and by compactness of $[Bx_1]_{\alpha B(x1)}$, we can choose $x_2 \in [Bx_1]_{\alpha B(x1)}$ such that $D_{\mathcal{F}}(x_1, x_2) = D_{\mathcal{F}}(x_1, [Bx_1]_{\alpha B(x1)})$. For convenience, denote $\alpha_A(x_i)$ and $\alpha_B(x_i)$ byα_{i+1}, where $i = 0, 1, 2, \cdots$.

By Lemma 11, we have

$$D_{\mathcal{F}}(x_1, x_2) = D_{\mathcal{F}}\left(x_1, [Bx_1]_{\alpha 2}\right) \leq H_{\mathcal{F}}\left([Ax_0]_{\alpha 1}, [Bx_1]_{\alpha 2}\right). \quad (10)$$

Therefore, using (10) together with (9), we have

$$\begin{aligned} D_{\mathcal{F}}(x_1, x_2) &\leq \varphi\left(\max\left\{D_{\mathcal{F}}(x_0, x_1), D_{\mathcal{F}}\left(x_0, [Ax_1]_{\alpha_1}\right), D_{\mathcal{F}}\left(x_1, [Bx_1]_{\alpha_2}\right), \frac{D_{\mathcal{F}}\left(x_0, [Bx_1]_{\alpha_2}\right) + D_{\mathcal{F}}\left(x_1, [Ax_0]_{\alpha_1}\right)}{2}\right\}\right) \\ &\leq \varphi(\max\{\{D_{\mathcal{F}}(x_0, x_1), D_{\mathcal{F}}(x_0, x_1), D_{\mathcal{F}}(x_1, x_2), \frac{D_{\mathcal{F}}(x_0, x_2) + D_{\mathcal{F}}(x_1, x_1)}{2}\}\right) \\ &\leq \varphi(\max\{\{D_{\mathcal{F}}(x_0, x_1), D_{\mathcal{F}}(x_1, x_2), \frac{D_{\mathcal{F}}(x_0, x_1) + D_{\mathcal{F}}(x_1, x_2)}{2}\}\right) \\ &\leq \varphi(\max\{\{D_{\mathcal{F}}(x_0, x_1), D_{\mathcal{F}}(x_1, x_2)\}). \end{aligned} \quad (11)$$

If $\max\{D_{\mathcal{F}}(x_0, x_1), D_{\mathcal{F}}(x_1, x_2)\} = D_F(x_1, x_2)$, then (11) becomes

$$D_{\mathcal{F}}(x_1, x_2) \leq \varphi(D_{\mathcal{F}}(x_0, x_1)) < D_{\mathcal{F}}(x_1, x_2), \quad (12)$$

which is a contradiction. It follows that $\max\{D(x_0, x_1), D_{\mathcal{F}}(x_1, x_2)\} = D(x_0, x_1)$. Therefore, we have

$$D_{\mathcal{F}}(x_1, x_2) \leq \varphi(D_{\mathcal{F}}(x_0, x_1)). \quad (13)$$

By continuous repetition of the above steps, we generate a sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of X with

$$\begin{aligned} x_{2n-1} &\in [Ax_{2n-2}]\alpha_{A(x_{2n-1})}, \\ x_{2n} &\in [Bx_{2n-1}]_{\alpha B(x_{2n-1})}, \end{aligned} \quad (14)$$

such that

$$\begin{aligned} D_{\mathcal{F}}(x_{2n}, x_{2n+1}) &\leq \varphi(D_{\mathcal{F}}(x_{2n-1}, x_{2n})), \\ D_{\mathcal{F}}(x_{2n-1}, x_{2n}) &\leq \varphi(D_{\mathcal{F}}(x_{2n-2}, x_{2n-1})). \end{aligned} \quad (15)$$

Consequently, by induction, for all $n \in \mathbb{N}$, we have

$$D_{\mathcal{F}}(x_n, x_{n+1}) \leq \varphi(D_{\mathcal{F}}(x_{n-1}, x_n)) \leq \cdots \leq \varphi^n(D_{\mathcal{F}}(x_0, x_1)). \quad (16)$$

Let $\eta > 0$ be a given positive number and $(f, \rho) \in \mathcal{F} \times [0,\infty)$ such that condition (D_3) is satisfied. By $(\mathcal{F}_2)$, there exists $\lambda > 0$ such that

$$0 < t < \lambda \text{ implies } f(t) \leq f(\eta) - \rho. \quad (17)$$

Let $n(\eta) \in \mathbb{N}$ such that $0 < \sum_{n\geq n(\eta)} \varphi^n(D_{\mathcal{F}}(x_0, x_1)) < \lambda$. Hence, by (17) and $(\mathcal{F}_1)$, we get

$$f\left(\sum_{i=n}^{k-1} \varphi^i(D_{\mathcal{F}}(x_0, x_1))\right) \leq f\left(\sum_{n\geq n(\eta)} \varphi^n(D_{\mathcal{F}}(x_0, x_1))\right) \leq f(\eta) - \rho. \quad (18)$$

Now, for $D_{\mathcal{F}}(x_k, x_n) > 0$, by (D_3), (16), and (18), we obtain

$$\begin{aligned} f(D_{\mathcal{F}}(x_k, x_n)) &\leq f\left(\sum_{i=n}^{k-1} D_{\mathcal{F}}(x_i, x_i + 1)\right) + \rho \\ &\leq f\left(\sum_{i=n}^{k-1} \varphi^i(D_{\mathcal{F}}(x_0, x_1))\right) + \rho \\ &\leq f\left(\sum_{n\geq n(\eta)} \varphi^n(D_{\mathcal{F}}(x_0, x_1))\right) \leq f(\eta). \end{aligned} \quad (19)$$

It follows from $(\mathcal{F}_1)$ that

$$D_{\mathcal{F}}(x_k, x_n) < \eta, k > n \geq n(\eta). \quad (20)$$

This shows that $\{x_n\}n \in \mathbb{N}$ is $\mathcal{F}$-Cauchy. Hence, $\mathcal{F}$-completeness of $(X, D_{\mathcal{F}})$ implies that there exists $u \in X$ such that $x_n \longrightarrow u$ as $n \longrightarrow \infty$. Now, to prove that $u \in [Au]_{\alpha A(u)}$, assume that $D_{\mathcal{F}}(u, [Au]_{\alpha A(u)}) > 0$. Then by (D_3), we get

$$f\left(D_{\mathcal{F}}\left(u,[Au]_{\alpha_{A(u)}}\right)\right)\le f(D_{\mathcal{F}}(u,x_{2n}) + H_{\mathcal{F}}\left([Bx_{2n-1}]_{\alpha_B(x_{2n-1})},[Au]_{\alpha_{A(u)}}\right)\right)+\rho\le f(D_{\mathcal{F}}(u,x_{2n}) + \varphi\Big(\max\Big\{D_{\mathcal{F}}(u,x_{2n-1}),D_{\mathcal{F}}\left(u,[Au]_{\alpha_{A(u)}}\right), D_{\mathcal{F}}\left(x_{2n-1},[Bx_{2n-1}]_{\alpha_{Bx_{2n-1}}}\right),\frac{D_{\mathcal{F}}\left(u,[Bx_{2n-1}]_{\alpha_{B(x_{2n-1})}}\right)+D_{\mathcal{F}}\left(x_{2n-1},[Au]_{\alpha_{A(u)}}\right)}{2}\Big\}\Big)\Big) + \rho\le f\Big(D_{\mathcal{F}}(u,x_{2n})+\varphi\Big(\max\Big\{D_{\mathcal{F}}(u,x_{2n-1}),D_{\mathcal{F}}\left(u,[Au]_{\alpha_{A(u)}}\right), D_{\mathcal{F}}(x_{2n-1},x_{2n}),\frac{D_{\mathcal{F}}(u,x_{2n})+D_{\mathcal{F}}\left(x_{2n-1},[Au]_{\alpha_{A(u)}}\right)}{2}\Big\}\Big)\Big)+\rho. \tag{21}$$

Now, we analyze (21) under the following cases:

Case (i). If

$$\max\Big\{D_{\mathcal{F}}(u,x_{2n-1}),D_{\mathcal{F}}\left(u,[Au]_{\alpha_A(u)}\right), D_{\mathcal{F}}(x_{2n-1},x_{2n}),\frac{D_{\mathcal{F}}(u,x_{2n})+D_{\mathcal{F}}\left(x_{2n-1},[Au]_{\alpha_A(u)}\right)}{2}\Big\} = D_{\mathcal{F}}(u,x_{2n-1}), \tag{22}$$

then (21) becomes

$$f\left(D_{\mathcal{F}}\left(u,[Au]_{\alpha_A(u)}\right)\right)\le f(D_{\mathcal{F}}(u,x_{2n})+\varphi(D_{\mathcal{F}}(u,x_{2n-1}))+\rho. \tag{23}$$

Since $x_n\longrightarrow u$ as $n\longrightarrow\infty$, then by $(\mathcal{F}_2)$ and the properties of $\phi\in\Omega$,

$$\lim_{n\to\infty} f(D_{\mathcal{F}}(u,x_{2n})+D_{\mathcal{F}}(u,x_{2n-1}))+\rho=-\infty, \tag{24}$$

which is a contradiction.

Case (ii). If

$$\max\Big\{D_{\mathcal{F}}(u,x_{2n-1}),D_{\mathcal{F}}\left(u,[Au]_{\alpha_A(u)}\right), D_{\mathcal{F}}(x_{2n-1},x_{2n}),\frac{D_{\mathcal{F}}(u,x_{2n})+D_{\mathcal{F}}\left(x_{2n-1},[Au]_{\alpha_A(u)}\right)}{2}\Big\} = D_{\mathcal{F}}(x_{2n-1},x_{2n}), \tag{25}$$

then,

$$f\left(D_{\mathcal{F}}\left(u,[Au]_{\alpha_{A(u)}}\right)\right)\le f(D_{\mathcal{F}}(u,x_{2n})+\varphi(D_{\mathcal{F}}(x_{2n-1},x_{2n}))+\rho. \tag{26}$$

Hence, by $(\mathcal{F}_2)$ and the properties of $\phi\in\Omega$, $\lim_{n\to\infty}f(D_{\mathcal{F}}(u,x_{2n})+D_{\mathcal{F}}(x_{2n-1},x_{2n})+\rho=-\infty$, a contradiction.

Case (iii). If

$$\max\Big\{D_{\mathcal{F}}(u,x_{2n-1}),D_{\mathcal{F}}\left(u,[Au]_{\alpha_A(u)}\right), D_{\mathcal{F}}(x_{2n-1},x_{2n}),\frac{D_{\mathcal{F}}(u,x_{2n})+D_{\mathcal{F}}\left(x_{2n-1},[Au]_{\alpha_A(u)}\right)}{2}\Big\} = \frac{D_{\mathcal{F}}(u,x_{2n})+D_{\mathcal{F}}\left(x_{2n-1},[Au]_{\alpha_A(u)}\right)}{2}, \tag{27}$$

then,

$$f\left(D_{\mathcal{F}}\left(u,[Au]_{\alpha_A(u)}\right)\right)\le f\left(D_{\mathcal{F}}(u,x_{2n})+\frac{D_{\mathcal{F}}(u,x_{2n})+D_{\mathcal{F}}\left(x_{2n-1},[Au]_{\alpha_A(u)}\right)}{2}\right)+\rho. \tag{28}$$

By condition $(\mathcal{F}_1)$, from (28), for $\rho=0$, we have

$$D_{\mathcal{F}}\left(u,[Au]_{\alpha_A(u)}\right)<D_{\mathcal{F}}(u,x_{2n})+\frac{D_{\mathcal{F}}(u,x_{2n})+D_{\mathcal{F}}\left(x_{2n-1},[Au]_{\alpha_A(u)}\right)}{2}. \tag{29}$$

As $n\longrightarrow\infty$ in (29), we obtain

$$D_{\mathcal{F}}\left(u,[Au]_{\alpha_{A(u)}}\right)<\frac{D_{\mathcal{F}}\left(u,[Au]_{\alpha_{A(u)}}\right)}{2}<D_{\mathcal{F}}\left(u,[Au]_{\alpha_{A(u)}}\right), \tag{30}$$

which is a contradiction.

Case (iv). If

$$\max\Big\{D_{\mathcal{F}}(u,x_{2n-1}),D_{\mathcal{F}}\left(u,[Au]_{\alpha_A(u)}\right), D_{\mathcal{F}}(x_{2n-1},x_{2n}),\frac{D_{\mathcal{F}}(u,x_{2n})+D_{\mathcal{F}}\left(x_{2n-1},[Au]_{\alpha_A(u)}\right)}{2}\Big\} = D_{\mathcal{F}}\left(u,[Au]_{\alpha_A(u)}\right), \tag{31}$$

then,

$$f\left(D_{\mathcal{F}}\left(u,[Au]_{\alpha_{A(u)}}\right)\right)\le f\left(D_{\mathcal{F}}(u,x_{2n})+\varphi\left(D_{\mathcal{F}}\left(u,[Au]_{\alpha_{A(u)}}\right)\right)\right)+\rho. \tag{32}$$

By $(\mathcal{F}_1)$, from (32), for $\rho=0$, and applying $\phi(t)<t$ for all $t>0$, we get

$$D_{\mathcal{F}}\left(u,[Au]_{\alpha_{A(u)}}\right)<D_{\mathcal{F}}(u,x_{2n})+D_{\mathcal{F}}\left(u,[Au]_{\alpha_{A(u)}}\right)<D_{\mathcal{F}}\left(u,[Au]_{\alpha_{A(u)}}\right),\ \text{as } n\longrightarrow\infty, \tag{33}$$

which is a contradiction. It follows that $D_{\mathcal{F}}(u, [Au]_{\alpha_{A(u)}}) = 0$. On the same steps, one can show that $D_{\mathcal{F}}(u, [Bu]_{\alpha_{B(u)}}) = 0$. Consequently, $u \in [Au]_{\alpha_{A(u)}} \cap [Bu]_{\alpha_{B(u)}}$.

Next, we give an example to support the validity of the hypotheses of Theorem 12.

Example 17. Let $X = \mathbb{R}^+ = [0,\infty)$ and $D_{\mathcal{F}} : X \times X \to [0,\infty)$ be defined by

$$D_{\mathcal{F}}(x,y) = \begin{cases} (x-y)^2, & \text{if } (x,y) \in [0,5] \times [0,5], \\ |x-y|, & \text{if } (x,y) \notin [0,5] \times [0,5], \end{cases} \tag{34}$$

for all $x, y \in X$. It can be seen that $D_{\mathcal{F}}$ satisfies (D_1), (D_2), and (D_3); hence, $(X, D_{\mathcal{F}})$ is an $\mathcal{F}$-metric space with $f(t) = \ln(t)$, $t > 0$, and $\rho = \ln(5)$. Notice that $D_{\mathcal{F}}$ does not satisfy the triangle inequality, since

$$d(1,5) = 16 > d(1,4) + d(4,5) = 10. \tag{35}$$

Moreover, let $\alpha \in (0,1]$ and consider two fuzzy mappings $A, B : X \longrightarrow I^X$ defined as follows:

(i) If $x = 0$

$$A(x)(t) = B(x)(t) = \begin{cases} 1, & \text{if } t = 0, \\ 0, & \text{if } t \neq 0, \end{cases} \tag{36}$$

(ii) If $0 < x < \infty$

$$A(x)(t) = \begin{cases} \alpha, & \text{if } 0 \le t < \dfrac{x^2}{60} \\ \dfrac{\alpha}{4}, & \text{if } \dfrac{x^2}{60} \le t < \dfrac{x^2}{40}, \\ \dfrac{\alpha}{8}, & \text{if } \dfrac{x^2}{40} \le t < x^2, \\ 0, & \text{if } x^2 \le t < \infty, \end{cases}$$
$$B(x)(t) = \begin{cases} \beta, & \text{if } 0 \le t < \dfrac{x^2}{50}, \\ \dfrac{\beta}{4}, & \text{if } \dfrac{x^2}{50} \le t < \dfrac{x^2}{40}, \\ \dfrac{\beta}{32}, & \text{if } \dfrac{x^2}{40} \le t < x^2, \\ 0, & \text{if } x^2 \le t < \infty. \end{cases} \tag{37}$$

Now, define $\varphi : [0,\infty) \longrightarrow [0,\infty)$ by $\varphi(t) = t/16$, $t \ge 0$. Clearly, $\varphi \in \Omega$.

Now, for $x \in X$, there exist $\alpha_A(x) = (\alpha/4) \in (0,1]$ and $\alpha_B(x) = (\beta/8) \in (0,1]$ such that $[Ax]_{(\alpha/4)}, [Bx]_{(\beta/8)} \in C(2^X)$. Specifically, if $x = y = 0$, then $[Ax]_{(\alpha/4)}, [Bx]_{(\beta/8)} = \{0\}$, and hence

$$\begin{aligned} &H_{\mathcal{F}}\left([Ax]_{(\alpha/4)}, [By]_{(\beta/8)}\right) = 0 \\ &\quad \le \varphi\left(\max\left\{D_{\mathcal{F}}(x,y), D_{\mathcal{F}}\left(x, [Ax]_{(\alpha/4)}\right),\right.\right. \\ &\left.\left. D_{\mathcal{F}}\left(y, [By]_{(\beta/8)}\right), \frac{D_{\mathcal{F}}\left(x, [By]_{(\beta/8)}\right) + D_{\mathcal{F}}\left(y, [Ax]_{(\alpha/4)}\right)}{2}\right\}\right). \end{aligned} \tag{38}$$

If $x, y \in (0,\infty)$, then

$$[Ax]_{(\alpha/4)} = \left\{t \in X : A(x)(t) \ge \frac{\alpha}{4}\right\} = \left[0, \frac{x^2}{40}\right]. \tag{39}$$

Similarly, $[By]_{(\beta/8)} = [0, (y^2/40)]$. Therefore, for $x \neq y$, by the definition of $D_{\mathcal{F}}$, we get

$$\begin{aligned} &H_{\mathcal{F}}\left([Ax]_{(\alpha/4)}, [By]_{(\beta/8)}\right) = \left(\frac{x^2}{40} - \frac{y^2}{40}\right)^2 \\ &\quad \le \left|\left(\frac{x+y}{40}(x-y)\right)\right|^2 \le \frac{1}{16}|x-y|^2 \le \frac{1}{16} D_{\mathcal{F}}(x,y) \\ &\quad \le \varphi\left(\max\left\{D_{\mathcal{F}}(x,y), D_{\mathcal{F}}\left(x, [Ax]_{(\alpha/4)}\right),\right.\right. \\ &\left.\left. D_{\mathcal{F}}\left(y, [By]_{(\beta/8)}\right), \frac{D_{\mathcal{F}}\left(x, [By]_{(\beta/8)}\right) + D_{\mathcal{F}}\left(y, [Ax]_{(\alpha/4)}\right)}{2}\right\}\right). \end{aligned} \tag{40}$$

Consequently, all the conditions of Theorem 12 are satisfied to find $u = 0 \in [A0]_{(\alpha/4)} \cap [B0]_{(\beta/8)}$.

By imposing continuity condition on the function $f \in F$, we have the following modification of Theorem 12.

Theorem 18. *Let $(X, D_{\mathcal{F}})$ be an $\mathcal{F}$ -complete $\mathcal{F}$ -metric space and $A, B : X \longrightarrow I^X$ be fuzzy mappings. Assume that for every $x \in X$, there exist $\alpha_A(x), \alpha_B(x) \in (0,1]$ such that $[Ax]_{\alpha A(x)}$, $[Bx]_{\alpha B(x)} \in C(2^X)$. Suppose also that the following conditions hold:*

(i) The function $f \in \mathcal{F}$ is assumed to be continuous. In addition, suppose $\varphi \in \Omega$ satisfies $f(t) > f(\varphi(t)) + \rho$ for all $t \in (0,\infty)$

(ii) And for all x,y $\in X$, we have

$$\begin{aligned} &H_{\mathcal{F}}\left([Ax]_{\alpha_A(x)}, [By]_{\alpha_B(y)}\right) \\ &\quad \le \varphi\left(\max\left\{D_{\mathcal{F}}(x,y), D_{\mathcal{F}}\left(x, [Ax]_{\alpha_A(x)}\right),\right.\right. \\ &\left.\left. D_{\mathcal{F}}\left(y, [By]_{\alpha_B(y)}\right), \frac{D_{\mathcal{F}}\left(x, [By]_{\alpha_B(y)}\right) + D_{\mathcal{F}}\left(y, [Ax]_{\alpha_A(x)}\right)}{2}\right\}\right). \end{aligned} \tag{41}$$

Then, there exists $u \in X$ such that $u \in [Au]_{\alpha_A(u)} \cap [Bu]_{\alpha_B(u)}$.

Proof. Following the proof of Theorem 12, we obtain that $\{x_n\}_{n\in\mathbb{N}}$ is an $\mathcal{F}$-Cauchy sequence in the $\mathcal{F}$-complete metric space $(X, D_{\mathcal{F}})$. Therefore, there exists $u \in X$ such that

$$\lim_{n\to\infty} D_{\mathcal{F}}(x_n, u) = 0. \tag{42}$$

Now, to prove that $u \in [Au]_{\alpha_A(u)}$, we argue by contradiction. So assume $D_F(u, [Au]_{\alpha A(u)}) > 0$. Then by (D_3), and inequation (41), we get

$$\begin{aligned}
& f\left(D_{\mathcal{F}}\left(u, [Au]_{\alpha_{A(u)}}\right)\right) \\
& \quad \le f\left(D_{\mathcal{F}}(u, x_{2n}) + H_{\mathcal{F}}\left([Bx_{2n-1}]_{\alpha_{B(x_{2n-1})}}, [Au]_{\alpha_{A(u)}}\right)\right) \\
& \quad + \rho \le f(D_{\mathcal{F}}(u, x_{2n}) + \varphi(\max\{D_{\mathcal{F}}(u, x_{2n-1}), \\
& D_{\mathcal{F}}\left(u, [Au]_{\alpha_{A(u)}}\right), D_{\mathcal{F}}\left(x_{2n-1}, [Bx_{2n-1}]_{\alpha_{Bx(2n-1)}}\right), \\
& \frac{D_{\mathcal{F}}\left(u, [Bx_{2n-1}]_{\alpha_{B(x_{2n-1})}}\right) + D_{\mathcal{F}}\left(x_{2n-1}, [Au]_{\alpha_{A(u)}}\right)}{2}\Big\}\Big) \\
& \quad + \rho \le f(D_{\mathcal{F}}(u, x_{2n}) + \varphi(\max\{D_{\mathcal{F}}(u, x_{2n-1}), \\
& D_{\mathcal{F}}\left(u, [Au]_{\alpha_{A(u)}}\right), D_{\mathcal{F}}(x_{2n-1}, x_{2n}), \\
& \frac{D_{\mathcal{F}}(u, x_{2n}) + D_{\mathcal{F}}\left(x_{2n-1}, [Au]_{\alpha_{A(u)}}\right)}{2}\Big\}\Big) \\
& \quad + \rho \le f(D_{\mathcal{F}}(u, x_{2n}) + \varphi(\max\{D_{\mathcal{F}}(u, x_{2n-1}), \\
& D_{\mathcal{F}}\left(u, [Au]_{\alpha_{A(u)}}\right), D_{\mathcal{F}}(x_{2n-1}, x_{2n}), \\
& \frac{D_{\mathcal{F}}(u, x_{2n}) + D_{\mathcal{F}}\left(x_{2n-1}, [Au]_{\alpha_{A(u)}}\right)}{2}\Big\}\Big) \\
& \quad + \rho \le f(D_{\mathcal{F}}(u, x_{2n}) + \varphi(\max\{D_{\mathcal{F}}(u, x_{2n-1}), \\
& D_{\mathcal{F}}\left(u, [Au]_{\alpha_{A(u)}}\right), D_{\mathcal{F}}(x_{2n-1}, x_{2n}), D_{\mathcal{F}}(u, x_{2n}) \\
& \quad + D_{\mathcal{F}}\left(x_{2n-1}, [Au]_{\alpha_{A(u)}}\right)\}) + \rho.
\end{aligned} \tag{43}$$

Taking the limit in (43) as $n \to \infty$ and using (42) together with the continuity of f and φ, we have

$$f\left(D_{\mathcal{F}}\left(u, [Au]_{\alpha_{A(u)}}\right)\right) \le f\left(\varphi\left(D_{\mathcal{F}}\left(u, [Au]_{\alpha_{A(u)}}\right)\right)\right) + \rho, \tag{44}$$

which is a contradiction to the condition on f. It follows that $D_{\mathcal{F}}(u, [Au]_{\alpha A(u)}) = 0$. On similar steps, we can show that $D_{\mathcal{F}}(u, [Bu]_{\alpha B(u)}) = 0$. Consequently, we have

$$u \in [Au]_{\alpha_A(u)} \cap [Bu]_{\alpha_B(u)}. \tag{45}$$

Example 19. In line with Example 17, take $f(t) = \ln(t)$, $t > 0$, $\varphi(t) = (t/16)$ and $\rho = \ln(5)$; then, f is continuous on $(0, \infty)$ and φ is continuous for all t. Notice that the condition $f(t) > f(\varphi(t)) + \rho$ for $t > 0$ becomes $\ln(t) > \ln(t/16) + \ln(5)$. Therefore, following the remaining constructions of Example 17, one can easily verify that all the hypotheses of Theorem 18 are satisfied to find some $u \in X$ such that $u \in [Au]_{\alpha A(u)} \cap [Bu]_{\alpha B(u)}$.

4. Consequences

In this section, we apply Theorems 12 and 18 to deduce some fixed point results of multivalued and single-valued mappings in the context of $\mathcal{F}$-metric spaces. To this end, recall that a point $u \in X$ is called a fixed point of a multivalued (single-valued) mapping T on X, if $u \in Tu$ $(u = Tu)$.

Corollary 20. *Let $(X, D_{\mathcal{F}})$ be an $\mathcal{F}$-complete $\mathcal{F}$-metric space and $S, T : X \longrightarrow C(2^X)$ be multivalued mappings. Suppose that the following condition holds:*

$$\begin{aligned}
H_{\mathcal{F}}(Sx, Ty) \le \varphi(\max\{& D_{\mathcal{F}}(x, y), D_{\mathcal{F}}(x, Sx), D_{\mathcal{F}}(y, Ty), \\
& \frac{D_{\mathcal{F}}(x, Ty) + D_{\mathcal{F}}(y, Sx)}{2}\}),
\end{aligned} \tag{46}$$

for all $x, y \in X$, where $\varphi \in \Omega$. Then, there exists $u \in X$ such that $u \in Su \cap Tu$.

Proof. Let $\alpha_A, \alpha_B : X \longrightarrow (0, 1]$ be any two arbitrary mappings, and consider two fuzzy set-valued maps $A, B : X \longrightarrow I^X$ defined as follows:

$$\begin{aligned}
A(x)(t) &= \begin{cases} \alpha_A(x), & \text{if } t \in Sx, \\ 0, & \text{if } t \notin Sx, \end{cases} \\
B(x)(t) &= \begin{cases} \alpha_B(x), & \text{if } t \in Tx, \\ 0, & \text{if } t \notin Tx. \end{cases}
\end{aligned} \tag{47}$$

Then, for all $x \in X$, we have

$$[Ax]_{\alpha_{A(x)}} = \{t \in X : A(x)(t) \ge \alpha_A(x)\} = Sx. \tag{48}$$

Similarly, $[Bx]_{\alpha_{B(x)}} = Tx$. Consequently, Theorem 12 can be applied to find $u \in X$ such that $u \in [Au]_{\alpha_A(u)} \cap [Bu]_{\alpha_B(u)} = Su \cap Tu$.

Following the proof of Corollary 20, we can also apply Theorem 18 to establish the following result.

Corollary 21. *Let $(X, D_{\mathcal{F}})$ be an $\mathcal{F}$-complete $\mathcal{F}$-metric space and $S, T : X \longrightarrow C(2^X)$ be multivalued mappings. Suppose that the following conditions hold:*

(i) *The function $f \in \mathcal{F}$ is assumed to be continuous. In addition, suppose $\varphi \in \Omega$ satisfies $f(t) > f(\varphi(t)) + \rho$ for all $t \in (0,\infty)$*

(ii) *And for all $x, y \in X$, we have*

$$H_{\mathcal{F}}(Sx,Ty)\leq\varphi(\max\{D_{\mathcal{F}}(x,y),D_{\mathcal{F}}(x,Sx),D_{\mathcal{F}}(y,Ty),\cdot\frac{D_{\mathcal{F}}(x,Ty)+D_{\mathcal{F}}(y,Sx)}{2}\}) \tag{49}$$

Then, there exists $u\in X$ such that $u\in Su\cap Tu$.

Corollary 22. *Let $(X,D_{\mathcal{F}})$ be an $\mathcal{F}$ -complete $\mathcal{F}$ -metric space and $g,h:X\longrightarrow X$ be single-valued mappings. Suppose that the following condition holds:*

$$D_{\mathcal{F}}(g(x),h(y))\leq\varphi(\max,\{D_{\mathcal{F}}(x,y),D_{\mathcal{F}}(x,g(x)),D_{\mathcal{F}}(y,h(y)),\cdot\frac{D_{\mathcal{F}}(x,h(y))+D_{\mathcal{F}}(y,g(x))}{2}\}), \tag{50}$$

for all $x,y\in X$, where $\varphi\in\Omega$. Then, there exists $u\in X$ such that $u=g(u)=h(u)$.

Proof. Let $\alpha_A(x),\alpha_B(x)\in(0,1]$ for all $x\in X$. Then, define two fuzzy set-valued maps $A,B:X\longrightarrow I^X$ as follows:

$$A(x)(t)=\begin{cases}\alpha_A(x), & \text{if } t=g(x),\\ 0, & \text{if } t\neq g(x),\end{cases}\quad B(x)(t)=\begin{cases}\alpha_B(x), & \text{if } t=h(x),\\ 0, & \text{if } t\neq h(x).\end{cases} \tag{51}$$

Then,

$$[Ax]_{\alpha A(x)}=\{t\in X: A(x)(t)\geq\alpha_A(x)\}=\{g(x)\}. \tag{52}$$

Similarly, $[Bx]_{\alpha_{B(x)}}=\{h(x)\}$. Obviously, $\{g(x)\},\{h(x)\}\in C(2^X)$, for all $x\in X$. Notice that in this case, $H_{\mathcal{F}}([Ax]\alpha_{A(x)},[By]\alpha_{B(y)})=D_{\mathcal{F}}(g(x),h(y))$. Therefore, Theorem 12 can be applied to find $u\in X$ such that $u\in[Au]_{\alpha_{A(u)}}=\{g(u)\}$ and $[Bu]_{\alpha_{B(u)}}=\{h(u)\}$, which further implies that $u=g(u)=h(u)$.

Following the proof of Corollary 22, one can also employ Theorem 18 to establish the following result.

Corollary 23. *Let $(X,D_{\mathcal{F}})$ be an $\mathcal{F}$ -complete $\mathcal{F}$ -metric space and $g,h:X\longrightarrow X$ be single-valued mappings. Suppose that the following conditions hold:*

(i) The function $f\in\mathcal{F}$ is assumed to be continuous. In addition, suppose that $\varphi\in\Omega$ satisfies $f(t)>f(\varphi(t)(t))+\rho$ for all $t\in(0,\infty)$ and for all $x,y\in X$

(ii) And for all $x,y\in X$, we have

$$D_{\mathcal{F}}(g(x),h(y))\leq\varphi(\max\{D_{\mathcal{F}}(x,y),D_{\mathcal{F}}(x,g(x)),D_{\mathcal{F}}(y,h(y)),\frac{D_{\mathcal{F}}(x,h(y))+D_{\mathcal{F}}(y,g(x))}{2}\}) \tag{53}$$

Then, there exists $u\in X$ such that $u=g(u)=h(u)$.

In the following, we apply Corollary 22 to deduce the main result of Jlei and Samet [9].

Corollary 24 (see [9]). *Let $(X,D_{\mathcal{F}})$ be an $\mathcal{F}$ -complete $\mathcal{F}$ -metric space and $h:X\longrightarrow X$ be a single-valued mapping. If for all $x,y\in X$, there exists $\lambda\in(0,1)$ such that*

$$D_{\mathcal{F}}(h(x),h(y))\leq\lambda D_{\mathcal{F}}(x,y), \tag{54}$$

then there exists $u\in X$ such that $h(u)=u$.

Proof. Consider Corollary 22. Define the function $\varphi:[0,\infty)\longrightarrow[0,\infty)$ by $\varphi(t)=\lambda t$, for all $t\geq0$ and $\lambda\in(0,1)$. Then, by taking $g=h$, all the hypotheses of Corollary 22 coincide with that of Corollary 24; and so, we can find $u\in X$ such that $h(u)=u$.

Remark 25. It is obvious that more consequences of Theorems 12 and 18 can be obtained, but we skip them due to the length of the paper.

5. Application to Fuzzy Initial Value Problems

Fuzzy differential equations (FDEs) and fuzzy integral equations (FIEs) play significant roles in modeling dynamic systems in which uncertainties or vague notions flourish. These concepts have been established in different theoretical directions, and a large number of applications in practical problems have been studied (see, for example, [20–22]). Several techniques for studying FDEs have been presented. The first most popular is using the Hukuhara differentiability (H-differentiability) for fuzzy valued functions (see [20, 23, 24]). On the other hand, the concept of FIEs was initiated by Kaleva [22] and Seikkala [25]. In the study of existence and uniqueness conditions for solutions of FDEs and FIEs, many authors have applied different fixed point theorems. By using the classical Banach fixed point theorem, Subrahmanyam and Sudarsanam [26] proved an existence and uniqueness result for some Volterra integral equations involving fuzzy set-valued mappings. With the help of Shaulder's fixed point theorem and Arzela-Ascoli's theorem, Allahviranloo et al. [27] studied the existence and uniqueness conditions of solutions of some nonlinear fuzzy Volterra integral equations. In [28], the authors discussed some existence results for a fuzzy initial value problem (FIVP) by employing some contractive-like mapping techniques. Congxin and Shiji [29] studied a Cauchy problem of fuzzy differential equation on the basis of the definition of H-differentiability for fuzzy set-valued mappings. They obtained the existence and uniqueness theorem for the Cauchy problem under some generalized Lipschitz condition. Similarly, Villamizar-Roa et al. [30] studied the existence and uniqueness of solution of FIVP in the setting of generalized Hukuhara derivatives. For some intricacies involved in the theory of fuzzy differential equations, the interested reader may consult [22, 24, 31].

In this section, using the ideas of fuzzy mappings in an $\mathcal{F}$-complete $\mathcal{F}$-metric space, we provide some conditions for the existence of solutions of a FIVP. In line with the existence methods, our technique is connected with studying the existence of solutions of the equivalent Volterra integral reformulation of the FIVP.

First, in what follows, we recall a few known results that are needed in the sequel. For most of these basic concepts, we follow [30, 32]. Let $P_\kappa(\mathbb{R})$ denote the family of nonempty compact subsets of $\mathbb{R}$. Define addition and multiplication in $P_\kappa(\mathbb{R})$ as usual, that is, for $A, B \in P_\kappa(\mathbb{R})$ and $\eta \in \mathbb{R}$, we have

$$A + B = \{a + b : a \in A, b \in B\}, \quad \eta A = \{\eta a : a \in A\}. \tag{55}$$

The Hausdorff metric H in $P_\kappa(\mathbb{R})$ is defined as

$$H(A, B) = \max\left\{\sup_{a\in A, b\in B} \inf \|a - b\|_{\mathbb{R}}, \sup_{b\in B, a\in A} \inf \|a - b\|_{\mathbb{R}}\right\}. \tag{56}$$

It is well known that the couple $(P_\kappa(\mathbb{R}), H)$ is a complete metric space. Moreover, the metric H satisfies the following properties for all $A, B, C, D \in P_\kappa(\mathbb{R})$:

$$\begin{gathered} H(\eta A, \eta B) = \eta H(A, B), \\ H(A + B, C + D) \le H(A, C) + H(B, D), \\ H(A + C, B + C) = H(A, B). \end{gathered} \tag{57}$$

In general, $A + (-A) \neq \{0\}$, where $(-1)A = \{-a : a \in A\}$, and hence $P_\kappa(\mathbb{R})$ is not a linear space (cf. [30]).

Definition 26. A fuzzy number in $\mathbb{R}$ is a function $x : \mathbb{R} \longrightarrow [0, 1]$ having the following properties:

(i) x is normal, that is, there exists $t_0 \in \mathbb{R}$ such that $x(t_0) = 1$

(ii) x is fuzzy convex, that is

$$x(\eta t_1 + (1 - \eta)t_2) \ge \min\{x(t_1), x(t_2)\}, \quad \forall t_1, t_2 \in \mathbb{R}, \eta \in [0, 1] \tag{58}$$

(iii) x is upper semicontinuous, that is, $[x]_\alpha$ is closed for all $\alpha \in [0, 1]$

(iv) $[x]_0 = \overline{\{t \in R : x(t) > 0\}}$ is compact

Throughout this section, we shall denote the set of all fuzzy numbers in $\mathbb{R}$ by I^1. The set $[x]_\alpha = \{t \in \mathbb{R} : x(t) \ge \alpha\} = [x^l_\alpha, x^r_\alpha]$ denotes the α-level set of $x \in I^1$. It follows from (i) to (iv) that $[x]_\alpha \in P_\kappa(\mathbb{R})$.

The supremum on I^1 is defined as

$$D_\infty(x, y) = \sup_{0\le\alpha\le1} \max\left\{\left|x^{1,l}_\alpha - x^{2,l}_\alpha\right|, \left|x^{1,r}_\alpha - x^{2,r}_\alpha\right|\right\}, \tag{59}$$

for every $x, y \in I^1$, where $x^r_\alpha - x^l_\alpha = \mathrm{diam}([x]_\alpha)$ is called the diameter of $[x]_\alpha$.

We shall call $C([a, b], I^1)$ the set of all continuous fuzzy functions defined on $[a, b]$. It is verifiable that $C([a, b], I^1)$ is an $\mathcal{F}$-complete $\mathcal{F}$-metric space with respect to the $\mathcal{F}$-metric:

$$D_{\mathcal{F}}(x, y) = \sup_{t\in J} D_\infty(x(t), y(t)), \quad x, y \in C([a, b], I^1). \tag{60}$$

The following lemma summarizes some basic properties of the integral of fuzzy functions.

Lemma 27 (see [22]). *Let $x, y : [a, b] \longrightarrow I^1$ be fuzzy functions and $\eta \in \mathbb{R}$. Then,*

$$\begin{gathered} \int_a^b (x + y)(t)dt = \int_a^b x(t)dt + \int_a^b y(t)dt, \\ \int_a^b \eta x(t)dt = \eta \int_a^b x(t)dt, \\ D_\infty(x(t), y(t)) \text{ is integrable}, \\ D_\infty\left(\int_a^b x(t)dt, \int_a^b y(t)dt\right) \le \int_a^b D_\infty(x(t), y(t))dt. \end{gathered} \tag{61}$$

Definition 28 (see [30]). Let I^n denote the set of all fuzzy numbers in $\mathbb{R}^n$ and $x, y, z \in I^n$. An element z is called the Hukuhara difference (or H-difference) of x and y, if it satisfies the equation $x = y + z$. If the H-difference of x and y exists, it is denoted by $x \ominus_H y$ (or $x - y$). It is easy to see that $x \ominus_H x = \{0\}$, and if $x \ominus_H y$ exists, it is unique.

Definition 29 (see [30]). Let $g : (a, b) \longrightarrow I^n$. The function g is said to be strongly generalized differentiable (or GH-differentiable) at $t_0 \in (a, b)$, if there exists an element $g'_G(t_0) \in I^n$ such that there exists the Hukuhara differences:

$$\begin{gathered} g(t_0 + \delta) \ominus_H g(t_0), \\ g(t_0) \ominus_H g(t_0 - \delta), \\ \lim_{\delta\to0^+} \frac{(g(t_0 + \delta) \ominus_H g(t_0))}{\delta} = \lim_{\delta\to0^+} \frac{(g(t_0) \ominus_H g(t_0 - \delta))}{\delta} = g'_G(t_0). \end{gathered} \tag{62}$$

Here, the limit is taken in the metric space (I^n, D), and at the end points of (a, b), only one-sided derivatives are considered.

Consider the following fuzzy initial value (FIVP):

$$\begin{cases} x'(t) = g(t, x(t)), & t \in J = [a, \rho], \\ x(a) = \rho, \end{cases} \tag{63}$$

where the derivative is considered in the sense of GH-differentiability, the fuzzy function $g : J \times I^1 \longrightarrow I^1$ is continuous, and the initial condition $x(a)$ is a fuzzy number. We denote by $C'(J, I^1)$ the set of all continuous functions $g : J \times I^1 \longrightarrow I^1$ with a continuous derivative.

Lemma 30 (see [20]). *A function $x \in C'(J, I^1)$ is a solution of the FIPV (63) if and only if it satisfies the fuzzy Volterra integral equation:*

$$x(t) = \rho \ominus_H (-1) \int_a^t g(u, x(u))du, \quad t \in J = [a, \rho]. \tag{64}$$

With the above preliminaries, we apply Theorem 12 together with the following result to study the existence conditions of solutions of the FIVP (63)

Theorem 31. *Let $g : J \times I^1 \longrightarrow I^1$ be continuous and assume that the following conditions hold:*

(i) The function g is strictly increasing in the second variable, that is, if $x < y$, then $g(t, x) < g(t, y)$

(ii) There exist some constants $\tau > 0$ and $\lambda \in (0, (1/(2(\rho - a)))]$ such that

$$\|g(t, x(t)) - g(t, y(t))\|\mathbb{R} \leq \lambda \max_{t \in J} \left\{ D_\infty(x, y) e^{-\tau(t-a)} \right\} \tag{65}$$

if $x < y$ for each $t \in J$ and $x, y \in I^1$, where $D_\infty(x, y)$ is the supremum on I^1.

Then, the FIVP (63) has a fuzzy solution in $C(J, I^1)$.

Proof. Let $\tau > 0$ be a constant. We consider the space $C(J, I^1)$ endowed with the weighted $\mathcal{F}$-metric:

$$D_\tau(x, y) = \sup_{t \in J} \left\{ D_\infty(x(t), y(t)) e^{-\tau(t-a)} \right\}, \quad x, y \in C(J, I^1). \tag{66}$$

Notice that $D_\tau(x, y)$ is equivalent to the $\mathcal{F}$-metric $D_{\mathcal{F}}(x, y)$, because

$$e^{(\varrho - a)\tau} D_{\mathcal{F}}(x, y) \leq D_\tau(x, y) \leq D_{\mathcal{F}}(x, y), \tag{67}$$

for all $x, y \in C(J, I^1)$.

Let $M, Q : X \longrightarrow (0, 1]$ be any two mappings. For $x \in X$, take

$$F_\tau(t) = \rho \ominus H(-1) \int_a^t g(u, x(u))du. \tag{68}$$

Assume $x < y$. Then, by hypothesis (i), we have

$$\begin{aligned} F_x(t) &= \rho \ominus H(-1) \int_a^t g(u, x(u))du \\ &< \rho \ominus H(-1) \int_a^t g(u, y(u))du = \theta_y(t). \end{aligned} \tag{69}$$

Hence, $F_x(t) \neq \theta y(t)$. Consider two fuzzy mappings $A, B : X \longrightarrow I^X$ defined by

$$\begin{aligned} \mu_{Ax}(r) &= \begin{cases} M(x), & \text{if } r(t) = F_x(t), \\ 0, & \text{otherwise}, \end{cases} \\ \mu_{By}(r) &= \begin{cases} Q(y), & \text{if } r(t) = \theta_y(t), \\ 0, & \text{otherwise}. \end{cases} \end{aligned} \tag{70}$$

By taking $\alpha_A(x) = M(x)$ and $\alpha_B(y) = Q(y)$, we have

$$[Ax]_{\alpha A(x)} = \{ r \in X : (Ax)(t) \geq M(x) \} = \{ Fx(t) \}, \tag{71}$$

and similarly, $[By]_{\alpha B}(y) = \{\theta_y(t)\}$. Therefore, we have

$$\begin{aligned} & H_{\mathcal{F}}\left([Ax]_{\alpha_A(x)}, [By]_{\alpha_B(y)}\right) \\ &= \max \left\{ \sup_{x \in [Ax]_{\alpha_A(x)}, y \in [By]_{\alpha_B(y)}} \inf \|x - y\|\mathbb{R}, \sup_{y \in [By]_{\alpha_B(y)}, x \in [Ax]_{\alpha_x(y)}} \inf \|x - y\|\mathbb{R}, \right\} \\ &\leq \max \left\{ \sup_{t \in J} \left\| F_x(t) - \theta_y(t) \right\|_{\mathbb{R}} \right\} = \sup_{t \in J} \left\| F_x(t) - \theta_y(t) \right\|_{\mathbb{R}} \\ &= \sup_{t \in J} \left\| \int_a^t g(u, x(u))du - \int_a^t g(u, y(u))du \right\|_{\mathbb{R}} \\ &\leq \sup_{t \in J} \left\{ \int_a^t \|g(u, x(u)) - g(u, y(u))\|_{\mathbb{R}} du \right\} \\ &\leq \sup_{t \in J} \left\{ \int_a^t du \lambda \max \left\{ D\infty(x, y) e^{-\tau(t-a)} \right\} \right\} \\ &\leq \lambda \sup_{t \in J} \left\{ (t - a) \max \left\{ D\infty(x, y) e^{-\tau(t-a)} \right\} \right\} \\ &\leq \lambda(\varrho - a) D_\tau(x, y) \frac{1}{2} D_{\mathcal{F}}(x, y) \varphi(A(x, y)), \end{aligned} \tag{72}$$

where

$$\begin{aligned} A(x, y) = \max \Bigg\{ & D_{\mathcal{F}}(x, y), D_{\mathcal{F}}\left(x, [Ax]_{\alpha_A(x)}\right) \Big|, D_{\mathcal{F}}\left(y, [By]_{\alpha_B(y)}\right), \\ & \frac{D_{\mathcal{F}}\left(x, [By]_{\alpha_B(y)}\right) + D_{\mathcal{F}}\left(y, [Ax]_{\alpha_A(x)}\right)}{2} \Bigg\}. \end{aligned} \tag{73}$$

Hence, all the conditions of Theorem 12 are satisfied with $\varphi(t) = (t/2)$, for all $t \geq 0$. It follows that there exists $x^* \in X$ such that $x^* \in [Ax^*]_{\alpha A(x^*)} \cap [Bx^*]_{\alpha B(x^*)}$. Consequently, x^* is a solution of the FIVP (63).

6. Conclusion

Two problems are addressed in this article. First, two fuzzy fixed point theorems in the context of $\mathcal{F}$-metric spaces are established. Consequently, corresponding fixed point theorems of multivalued and single-valued mappings are deduced. From an application perspective, in the second direction, one of our results is employed to provide some existence conditions for solutions of fuzzy initial value problems. As far as we know, in the setting of $\mathscr{F}$-metric spaces and fuzzy set, the results presented herein are new and fundamental. Consequently, it can be improved upon when discussed in the setting of L-fuzzy mappings, intuitionistic fuzzy mappings, soft set-valued maps, and other generalized hybrid models within the scope of fuzzy mathematics.

Data Availability

No data were used to support this study

Conflicts of Interest

The authors declare that they have no competing interests.

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