Results of Positive Solutions for the Fractional Differential System on an Infinite Interval

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1. Introduction

In this paper, we investigate the following fractional differential system on an infinite interval:

\[
\begin{align*}
D_0^\alpha u(t) + f_1(t, u(t), v(t)) &= 0, \\
D_0^\beta v(t) + f_2(t, u(t), v(t)) &= 0, \\
u(0) &= u'(0) = u''(0) = \cdots = u^{(n-2)}(0) = 0, \\
v(0) &= v'(0) = v''(0) = \cdots = v^{(m-2)}(0) = 0, \\
\end{align*}
\]

where \(1 \leq n - 1 < \alpha_i \leq n\), \(1 \leq m - 1 < \alpha_2 \leq m\), and \(n, m \geq 2\), \(D_0^\alpha\) is the Rieman–Liouville derivative operator. \(\mu_i > 0\) is a constant, \(\int_0^\infty a_i(t)u(t)\,dA_i(t)\) and \(\int_0^\infty a_i(t)v(t)\,dA_i(t)\) denote the Riemann–Stieltjes integral, and \(A_i\) is a function of bounded positive variation. \(a_i \in L[0, +\infty]\), \(\mu_i \int_0^\infty a_i(t)\,dA_i(t) < +\infty\), and \(f_i: [0, +\infty) \times [0, +\infty) \times [0, +\infty) \rightarrow [0, +\infty)\) is continuous, \(i = 1, 2\).

Numerous models in physics, chemistry, biology, medicine, and other fields have promoted the research of differential equations, for instance, evaluation of water quality on receiving

water [1], and the advection-dispersion equation can be formulated as shown for the case of one-dimensional flow:

\[
\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} = D_L \frac{\partial^2 C}{\partial x^2} - f(C),
\]

where \(C\) is the concentration of a generic pollutant, \(t\) is the time, \(x\) is the longitudinal displacement, \(u\) is the velocity, \(D_L\) is the diffusion coefficient, and \(f(C)\) is a generic term for reactions involving the pollutant \(C\). Westerlund [2] established a one-dimensional model to describe the transmission of the electromagnetic wave:

\[
\frac{\partial^2 C}{\partial x^2} = \frac{D_L}{\mu} \frac{\partial C}{\partial t} - f(C).
\]
existence of positive solutions but also the uniqueness of Schauder fixed-point theorem and the Banach contraction value condition. At last, we use two different techniques: the conditions on infinite intervals, which are more general than the infinite interval, many authors put their interest in it [7–16]. Liang and Zhang [17] applied the fixed-point theorem to obtain the existence of positive solutions for the following fractional differential equation:

\begin{equation}
\begin{aligned}
D_0^\alpha u(t) + a(t)f(u(t)) &= 0, \quad 0 < t < +\infty, \\
\lim_{t\to\infty} D_0^{\alpha-1} u(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) \, ds,
\end{aligned}
\end{equation}

where \(2 < \alpha \leq 3\), \(D_0^\alpha\) is the Riemann–Liouville fractional derivative, \(0 < \xi_1 < \xi_2 < \ldots < \xi_{m-2} < +\infty\), \(\beta_i > 0\), \(0 < \sum_{i=1}^{m-1} \beta_i u(\xi_i) < \Gamma(\alpha)\), and \(f: [0, +\infty) \to [0, +\infty)\) is continuous.

For all we know, there are few studies on fractional differential systems of infinite intervals, although it is necessary to do so. In this paper, we aimed at getting the existence and uniqueness of positive solutions for system (1) on infinite interval. Compared with the existing literature, the innovations of this paper are as follows. Firstly, the paper which we discuss is the system rather than a single equation. Secondly, we study the system with integral boundary value conditions on infinite intervals, which are more general than those of two-point, three-point, and multipoint boundary value condition. At last, we use two different techniques: the Schauder fixed-point theorem and the Banach contraction mapping principle, for system (1), not only to obtain the existence of positive solutions but also the uniqueness of positive solutions.

2. Preliminaries and Lemmas

**Definition 1** (see [18, 19]). Let \(\alpha > 0\) and \(u\) be piecewise continuous on \((0, +\infty)\) and integrable on any finite sub-interval of \([0, +\infty)\). Then, for \(t > 0\), we call

\begin{equation}
\begin{aligned}
G_i(t, s) &= G_{i1}(t, s) + G_{i2}(t, s), \quad i = 1, 2, \\
G_{i1}(t, s) &= \frac{1}{\Gamma(\alpha_i)} \left\{ \begin{array}{ll}
t^{\alpha_i-1} - (t-s)^{\alpha_i-1}, & 0 \leq s \leq t \leq +\infty, \\
t^{\alpha_i-1}, & 0 \leq t \leq s \leq +\infty,
\end{array} \right. \\
G_{i2}(t, s) &= \frac{\mu t^{\alpha_i-1}}{\Gamma(\alpha_i) - \mu} \int_0^\infty a_i(t) t^{\alpha_i-1} \, dA_i(t)
\end{aligned}
\end{equation}

the Riemann–Liouville fractional integral of \(u\) of order \(\alpha\).

**Definition 2** (see [18, 19]). The Riemann–Liouville fractional derivative of order \(\alpha > 0\), \(n - 1 \leq \alpha < n\), \(n \in \mathbb{N}\), is defined as

\begin{equation}
\begin{aligned}
D_0^\alpha u(t) &= \frac{1}{\Gamma(n-\alpha)} \left( \frac{d^n}{dt^n} \right)^n \int_0^t (t-s)^{n-\alpha-1} u(s) \, ds,
\end{aligned}
\end{equation}

where \(\mathbb{N}\) denotes the natural number set and the function \(u(t)\) is \(n\) times continuously differentiable on \([0, +\infty)\).

**Lemma 1** (see [18, 19]). Let \(\alpha > 0\), and if the fractional derivative \(D_0^\alpha u(t)\) and \(D_0^\alpha u(t)\) are continuous on \([0, +\infty)\), then

\begin{equation}
\begin{aligned}
D_0^\alpha u(t) &= u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},
\end{aligned}
\end{equation}

where \(c_1, c_2, \ldots, c_n \in (-\infty, +\infty)\), \(n\) is the smallest integer greater than or equal to \(\alpha\).

**Lemma 2.** Let \(y_i \in C(0, +\infty) \cap L(0, +\infty)\); then, the fractional system

\begin{equation}
\begin{aligned}
D_0^\alpha u(t) &= y_1(t) = 0, \\
D_0^\alpha v(t) &= y_2(t) = 0, \\
0 < t < +\infty, \\
D_0^\alpha u(0) &= \mu_1 \int_0^\infty a_1(t) u(t) \, dA_1(t), \\
D_0^\alpha v(0) &= \mu_2 \int_0^\infty a_2(t) v(t) \, dA_2(t),
\end{aligned}
\end{equation}

has a unique integral representation

\begin{equation}
\begin{aligned}
u(t) &= \int_0^\infty G_1(t, s) y_1(s) \, ds, \\
w(t) &= \int_0^\infty G_2(t, s) y_2(s) \, ds,
\end{aligned}
\end{equation}

where

\begin{equation}
\begin{aligned}
G_i(t, s) &= G_{i1}(t, s) + G_{i2}(t, s), \\
G_{i1}(t, s) &= \frac{1}{\Gamma(\alpha_i)} \left\{ \begin{array}{ll}
t^{\alpha_i-1} - (t-s)^{\alpha_i-1}, & 0 \leq s \leq t \leq +\infty, \\
t^{\alpha_i-1}, & 0 \leq t \leq s \leq +\infty,
\end{array} \right. \\
G_{i2}(t, s) &= \frac{\mu t^{\alpha_i-1}}{\Gamma(\alpha_i) - \mu} \int_0^\infty a_i(t) t^{\alpha_i-1} \, dA_i(t).
\end{aligned}
\end{equation}
Proof. By Lemma 1, the equations in system (8) can be transformed into the equivalent integral equations

\[
\begin{align*}
     u(t) &= -I_{0^+}^{a_{j_1}}y_1(t) + c_1 t^{a_{j_1}-1} + c_2 t^{a_{j_2}-2} + \cdots + c_n t^{a_{j_n}-n}, \quad c_j \in (-\infty, +\infty), \quad j = 1, 2, \ldots, n, \\
     v(t) &= -I_{0^+}^{a_{j_1}}y_2(t) + \bar{c}_1 t^{a_{j_1}-1} + \bar{c}_2 t^{a_{j_2}-2} + \cdots + \bar{c}_m t^{a_{j_m}-m}, \quad \bar{c}_j \in (-\infty, +\infty), \quad j = 1, 2, \ldots, m,
\end{align*}
\]  

that is,

\[
\begin{align*}
     u(t) &= -\frac{1}{\Gamma(a_1)} \int_0^t (t-s)^{a_1-1} y_1(s)ds + c_1 t^{a_1-1} + c_2 t^{a_2-2} + \cdots + c_n t^{a_n-n}, \quad c_j \in (-\infty, +\infty), \quad j = 1, 2, \ldots, n, \\
     v(t) &= -\frac{1}{\Gamma(a_2)} \int_0^t (t-s)^{a_2-1} y_2(s)ds + \bar{c}_1 t^{a_2-1} + \bar{c}_2 t^{a_3-2} + \cdots + \bar{c}_m t^{a_m-m}, \quad \bar{c}_j \in (-\infty, +\infty), \quad j = 1, 2, \ldots, m.
\end{align*}
\]  

Since

\[
\begin{align*}
     u(0) = u'(0) = \cdots = u^{(m-2)}(0) = 0, \\
     v(0) = v'(0) = \cdots = v^{(m-2)}(0) = 0,
\end{align*}
\]  

we have

\[
\begin{align*}
     c_2 = c_3 = \cdots = c_n = 0, \\
     \bar{c}_2 = \bar{c}_3 = \cdots = \bar{c}_m = 0.
\end{align*}
\]  

So,

\[
\begin{align*}
     u(t) &= -\frac{1}{\Gamma(a_1)} \int_0^t (t-s)^{a_1-1} y_1(s)ds + c_1 t^{a_1-1}, \\
     v(t) &= -\frac{1}{\Gamma(a_2)} \int_0^t (t-s)^{a_2-1} y_2(s)ds + \bar{c}_1 t^{a_2-1}.
\end{align*}
\]  

We also have

\[
\begin{align*}
     D_{0^+}^{a_1-1}u(t) &= c_1 \Gamma(a_1) \int_0^t y_1(s)ds, \\
     D_{0^+}^{a_2-1}v(t) &= \bar{c}_1 \Gamma(a_2) \int_0^t y_2(s)ds.
\end{align*}
\]  

For

\[
\begin{align*}
     D_{0^+}^{a_1-1}u(+\infty) &= \mu_1 \int_0^{+\infty} a_1(s) u(s)dA_1(s), \\
     D_{0^+}^{a_2-1}v(+\infty) &= \mu_2 \int_0^{+\infty} a_2(s) v(s)dA_2(s),
\end{align*}
\]  

we obtain

\[
\begin{align*}
     c_1 &= \frac{1}{\Gamma(a_1)} \int_0^{+\infty} a_1(s) u(s)dA_1(s) + \int_0^{+\infty} y_1(s)ds, \\
     \bar{c}_1 &= \frac{1}{\Gamma(a_2)} \int_0^{+\infty} a_2(s) v(s)dA_2(s) + \int_0^{+\infty} y_2(s)ds.
\end{align*}
\]  

Combining (15) and (18), we have

\[
\begin{align*}
     u(t) &= -\frac{1}{\Gamma(a_1)} \int_0^t (t-s)^{a_1-1} y_1(s)ds + c_1 t^{a_1-1} \\
     &= \int_0^{+\infty} G_{11}(t,s) y_1(s)ds + \frac{\mu_1 t^{a_1-1}}{\Gamma(a_1)} \int_0^{+\infty} a_1(t) u(t)dA_1(t), \\
     v(t) &= -\frac{1}{\Gamma(a_2)} \int_0^t (t-s)^{a_2-1} y_2(s)ds + \bar{c}_1 t^{a_2-1} \\
     &= \int_0^{+\infty} G_{21}(t,s) y_2(s)ds + \frac{\mu_2 t^{a_2-1}}{\Gamma(a_2)} \int_0^{+\infty} a_2(t) v(t)dA_2(t).
\end{align*}
\]  

(19) and (20) are multiplied by \(a_1(t)\) and \(a_2(t)\), respectively, and then solved the integral from 0 to +\(\infty\) with respect to \(A_1(t)\) and \(A_2(t)\); then, we have

\[
\begin{align*}
     &\int_0^{+\infty} a_1(t) u(t)dA_1(t) = \frac{\Gamma(a_1)}{\Gamma(a_1) - \mu_1} \int_0^{+\infty} a_1(t) dA_1(t) \int_0^{+\infty} a_1(t) G_{11}(t,s) y_1(s)ds dA_1(t), \\
     &\int_0^{+\infty} a_2(t) v(t)dA_2(t) = \frac{\Gamma(a_2)}{\Gamma(a_2) - \mu_2} \int_0^{+\infty} a_2(t) dA_2(t) \int_0^{+\infty} a_2(t) G_{21}(t,s) y_2(s)ds dA_2(t).
\end{align*}
\]  

(21)
Therefore,
\[ u(t) = \int_{0}^{\infty} G_{11}(t, s) y_1(s) ds + \frac{\mu_1 t^{\alpha_1 - 1}}{\Gamma(\alpha_1)} \int_{0}^{\infty} a_1(t) u(t) da_1(t) = \int_{0}^{\infty} G_{11}(t, s) y_1(s) ds + \frac{\mu_1 t^{\alpha_1 - 1}}{\Gamma(\alpha_1) - \mu_1} \int_{0}^{\infty} a_1(t) da_1(t) \]
\[ v(t) = \int_{0}^{\infty} G_{21}(t, s) y_2(s) ds + \frac{\mu_2 t^{\alpha_2 - 1}}{\Gamma(\alpha_2)} \int_{0}^{\infty} a_2(t) v(t) da_2(t) \]
\[ = \int_{0}^{\infty} G_{21}(t, s) y_2(s) ds + \frac{\mu_2 t^{\alpha_2 - 1}}{\Gamma(\alpha_2) - \mu_2} \int_{0}^{\infty} a_2(t) da_2(t) \]
\[ v(t) = \int_{0}^{\infty} G_{21}(t, s) y_2(s) ds + \frac{\mu_2 t^{\alpha_2 - 1}}{\Gamma(\alpha_2) - \mu_2} \int_{0}^{\infty} a_2(t) da_2(t) \]
\[ + \left( \frac{1}{\Gamma(\alpha_1)} + \frac{\mu_1}{\Gamma(\alpha_1) - \mu_1} \right) \int_{0}^{\infty} a_1(t) da_1(t) \]
\[ + \left( \frac{1}{\Gamma(\alpha_2)} + \frac{\mu_2}{\Gamma(\alpha_2) - \mu_2} \right) \int_{0}^{\infty} a_2(t) da_2(t) \]
\[ \omega = \max \left\{ \frac{1}{\Gamma(\alpha_1)} + \frac{\mu_1}{\Gamma(\alpha_1) - \mu_1} \int_{0}^{\infty} a_1(t) da_1(t) \right\} \]  

The space \( X = E_1 \times E_2 \) will be used in the study of system (1), where
\[
E_1 = \left\{ u \in C[0, +\infty) : \sup_{t \in [0, \infty)]} \frac{u(t)}{1 + t^{\alpha_1 - 1}} < +\infty \right\},
\]
\[
E_2 = \left\{ v \in C[0, +\infty) : \sup_{t \in [0, \infty)]} \frac{v(t)}{1 + t^{\alpha_2 - 1}} < +\infty \right\}.
\]

Then, \( (E_1, \|\cdot\|) \) and \( (E_2, \|\cdot\|) \) are the Banach space with the norm
\[
\|u\| = \sup_{t \in [0, \infty]} \frac{u(t)}{1 + t^{\alpha_1 - 1}},
\]
\[
\|v\| = \sup_{t \in [0, \infty]} \frac{v(t)}{1 + t^{\alpha_2 - 1}}.
\]

Clearly, \( (X, \|\cdot\|) \) is a Banach space with the norm \( \|u, v\| = \|u\| + \|v\| \). Define nonlinear integral operators \( T_i : X \rightarrow E_i \) and \( T : X \rightarrow X \) by
\[
T_i(u, v)(t) = \int_{0}^{\infty} G_i(t, s) f_i(s, u(s), v(s)) ds, \quad i = 1, 2,
\]
\[
T(u, v) = (T_1(u, v), T_2(u, v)).
\]

Thus, the existence of solutions to system (1) is equivalent to the existence of solutions in \( X \) for operator equation \( T(u, v) = (u, v) \) defined by (27).

**Lemma 4** (see [20, 21]). Let \( E \) be defined as (24) and \( M \) be any bounded subset of \( E \). Then, \( M \) is relatively compact in \( E \) if \( \{x(t)/1 + t^{\alpha_1 - 1} : x \in M \} \) is equicontinuous on any finite subinterval of \( I \), and for any given \( \epsilon > 0 \), there exists \( N > 0 \) such that \( \|x(t_1)/1 + t_1^{\alpha_1 - 1} - (x(t_2)/1 + t_2^{\alpha_1 - 1})\| < \epsilon \) uniformly with respect to all \( x \in M \), and \( t_1, t_2 > N \).

**3. Main Results**

We list the conditions to be used later: (H1) there exist nonnegative functions \( p_i(t), g_i(t), h_i(t) \in L^1[0, +\infty) \) and \( t^{\alpha_i - 1} g_i(t), t^{\alpha_i - 1} h_i(t) \in L^1[0, +\infty) \) such that
\[
|f_i(t, u, v)| \leq p_i(t) + g_i(t)|u| + h_i(t)|v|, \quad (t, u, v) \in [0, +\infty) \times [0, +\infty) \times [0, +\infty).
\]

\[
(H_2) |f_i(t, 0, 0)| \in L^1[0, +\infty), \quad \text{there exist nonnegative functions} \quad g_i(t), \quad h_i(t) \in L^1[0, +\infty), \quad \text{and} \quad t^{\alpha_i - 1} g_i(t), \quad t^{\alpha_i - 1} h_i(t) \in L^1[0, +\infty), \quad \text{such that}
\]
\[
|f_i(t, u_1, v_1) - f_i(t, u_2, v_2)| \leq g_i(t)|u_1 - u_2| + h_i(t)|v_1 - v_2|, \quad (t, u_1, v_1) \in [0, +\infty) \times [0, +\infty) \times [0, +\infty).
\]

**Remark 1.** If (H1) holds, then
\[
\int_{0}^{\infty} |f_i(t, u, v)| dt \leq p_i^* + (g_i^* + h_i^*)\|u, v\|, \quad (u, v) \in X,
\]

where
\[
p_i^* = \int_{0}^{\infty} p_i(t) dt,
\]
\[
g_i^* = \int_{0}^{\infty} \left(1 + t^{\alpha_i - 1}\right) g_i(t) dt,
\]
\[
h_i^* = \int_{0}^{\infty} \left(1 + t^{\alpha_i - 1}\right) h_i(t) dt.
\]
In fact, by \((H_1)\), for any \((u, v) \in X\), we have
\[
\int_0^\infty |f(t, u(t), v(t))| \, dt
\]
\[
\leq \int_0^\infty (p_t(t) + g_t(t)|u(t)| + h_t(t)|v(t)|) \, dt
\]
\[
= \int_0^\infty \left( p_t(t) + (1 + t^{\alpha - 1}) g_t(t) \frac{|u(t)|}{1 + t^{\beta - 1}} + (1 + t^{\alpha - 1}) h_t(t) \frac{|v(t)|}{1 + t^{\beta - 1}} \right) \, dt
\]
\[
\leq \int_0^\infty (p_t(t) + (1 + t^{\alpha - 1}) g_t(t)\|u\| + (1 + t^{\alpha - 1}) h_t(t)\|v\|) \, dt
\]
\[
= \int_0^\infty p_t(t) \, dt + \int_0^\infty (1 + t^{\alpha - 1}) g_t(t) \, dt \|u\| + \int_0^\infty (1 + t^{\alpha - 1}) h_t(t) \, dt \|v\|
\]
\[
= p_t^* + g_t^* \|u\| + h_t^* \|v\| \leq p_t^* + (g_t^* + h_t^*) \|(u, v)\|.
\]

**Theorem 1.** Assume that \((H_1)\) holds; then, \(T: X \longrightarrow X\) is a completely continuous operator.

**Proof.** First, we show that \(T: X \longrightarrow X\) is continuous. Suppose \(\{(u_n, v_n)\} \subset X\), \((u, v) \in X\) with \(\|(u_n, v_n) - (u, v)\| \longrightarrow 0\) \((n \rightarrow +\infty)\), and there exists a constant \(r > 0\) such that \(\|(u_n, v_n)\| \leq r\) and \(\|(u, v)\| \leq r\). By \((H_1)\) and (30), we have

\[
\left| \int_0^\infty G_1(t, s) f_1(s, u_n(s), v_n(s)) \, ds - \int_0^\infty G_1(t, s) f_1(s, u(s), v(s)) \, ds \right|
\]
\[
\leq \int_0^\infty G_1(t, s) f_1(s, u_n(s), v_n(s)) \, ds + \int_0^\infty G_1(t, s) f_1(s, u(s), v(s)) \, ds
\]
\[
\leq \omega \int_0^\infty f_1(s, u_n(s), v_n(s)) \, ds + \omega \int_0^\infty f_1(s, u(s), v(s)) \, ds
\]
\[
\leq 2\omega (p_t^* + g_t^* \|u\| + h_t^* \|v\|) \leq 2\omega (p_t^* + (g_t^* + h_t^*) \|(u, v)\|) < +\infty.
\]

From \((H_1)\) and (33), for any \(\varepsilon > 0\), there exists sufficiently large \(M_0\) such that
\[
\int_{M_0}^\infty G_1(t, s) f_1(s, u_n(s), v_n(s)) \, ds + \int_{M_0}^\infty G_1(t, s) f_1(s, u(s), v(s)) \, ds
\]
\[
\cdot (s, u(s), v(s)) \, ds < \frac{\varepsilon}{2}.
\]

On the contrary, by the continuity of \(f_1(t, u, v)\) on \([0, M_0] \times [0, 1 + M_0^{\alpha - 1}) \times [0, 1 + M_0^{\beta - 1})\), there exists \(N > 0\) such that when \(n > N\) and \(t \in [0, M_0]\), we have
\[
\left| f_1(s, u_n(s), v_n(s)) - f_1(s, u(s), v(s)) \right| < \frac{\varepsilon}{2\omega M_0}
\]
\[
\text{Hence, for any } t \in [0, +\infty) \text{ and } n > N, \text{ we obtain}
\]
\[
\left| \frac{T_1(u_n, v_n)(t) - T_1(u, v)(t)}{1 + t^{\alpha - 1}} \right|
\]
\[
= \left| \int_0^\infty G_1(t, s) f_1(s, u_n(s), v_n(s)) \, ds - \int_0^\infty G_1(t, s) f_1(s, u(s), v(s)) \, ds \right|
\]
\[
\leq \int_{M_0}^\infty G_1(t, s) \left| f_1(s, u_n(s), v_n(s)) - f_1(s, u(s), v(s)) \right| \, ds
\]
\[
+ \int_{M_0}^\infty G_1(t, s) f_1(s, u_n(s), v_n(s)) \, ds + \int_{M_0}^\infty G_1(t, s) f_1(s, u(s), v(s)) \, ds
\]
Thus, we know that \( \|T_1(u_n, v_n) - T_1(u, v)\| \to 0 \) (\( n \to +\infty \)). By the similar proof as (33)–(36), we know \( \|T_2(u_n, v_n) - T_2(u, v)\| \to 0 \) (\( n \to +\infty \)). So, \( T: X \to X \) is continuous.

Next, we show that \( T: X \to X \) is relatively compact. Let \( \Omega \) be a bounded subset of \( X \); then, there exists constant \( M > 0 \) such that \( \| (u, v) \| \leq M, \ (u, v) \in \Omega \). For any \( (u, v) \in \Omega, \ t \in [0, +\infty) \), and by (32), we obtain

\[
\frac{T_1(t)(u_n, v_n)(t)}{1 + t_n^{a-1}} = \int_0^\infty G_1(t, s) f_1(s, u_n(s), v_n(s))ds
\]

\[
\leq \omega \int_0^\infty |f_1(s, u_n(s), v_n(s))|ds
\]

\[
\leq \omega (p^*_1 + (g_1^* + h_1^*)\| (u, v) \|)
\]

\[
\leq \omega (p^*_1 + (g_1^* + h_1^*)M) < +\infty.
\]

Similarly, we have

\[
\frac{T_2(t)(u_n, v_n)(t)}{1 + t_n^{a-1}} \leq \omega (p^*_1 + (g_1^* + h_1^*)\| (u, v) \|)
\]

\[
\leq \omega (p^*_1 + (g_1^* + h_1^*)M) < +\infty.
\]

Consequently, \( T(\Omega) \) is uniformly bounded.

Given \( l \in [0, +\infty) \) be any compact interval. For any \( t_1, t_2 \in I, t_1 < t_2 \) and \( (u, v) \in \Omega \), we deduce

\[
\frac{T_1(t_2)(u_n, v_n)(t_2)}{1 + t_n^{a-1}} - \frac{T_1(t_1)(u_n, v_n)(t_1)}{1 + t_n^{a-1}}
\]

\[
= \int_0^\infty G_1(t_2, s) f_1(s, u(s), v(s))ds - \int_0^\infty G_1(t_1, s) f_1(s, u(s), v(s))ds
\]

\[
\leq \int_0^\infty \left| G_1(t_2, s) f_1(s, u(s), v(s)) - G_1(t_1, s) f_1(s, u(s), v(s)) \right| ds
\]

\[
+ \int_0^\infty \frac{\mu_1}{\Gamma(a_1)} \int_0^\infty a_1(t) G_1(t, s) dA_1(t)
\]

\[
\leq \int_0^\infty \frac{\mu_1}{\Gamma(a_1)} \int_0^\infty a_1(t) G_1(t, s) dA_1(t)
\]

\[
+ \int_0^\infty \frac{\mu_1}{\Gamma(a_1)} \int_0^\infty a_1(t) G_1(t, s) dA_1(t)
\]

\[
+ \int_0^\infty \frac{\mu_1}{\Gamma(a_1)} \int_0^\infty a_1(t) G_1(t, s) dA_1(t)
\]
Since
\[ \int_0^\infty \left| \frac{G_{11}(t_2, s)}{1 + t_2^{a_i-1}} - \frac{G_{11}(t_1, s)}{1 + t_2^{a_i-1}} \right| f_1(s, u(s), v(s)) ds \]
\[ \leq \int_0^{t_1} \left| \frac{G_{11}(t_2, s)}{1 + t_2^{a_i-1}} - \frac{G_{11}(t_1, s)}{1 + t_2^{a_i-1}} \right| f_1(s, u(s), v(s)) ds \]
\[ + \int_{t_1}^{t_2} \left| \frac{G_{11}(t_2, s)}{1 + t_2^{a_i-1}} - \frac{G_{11}(t_1, s)}{1 + t_2^{a_i-1}} \right| f_1(s, u(s), v(s)) ds \]
\[ + \int_{t_2}^{\infty} \left| \frac{G_{11}(t_2, s)}{1 + t_2^{a_i-1}} - \frac{G_{11}(t_1, s)}{1 + t_2^{a_i-1}} \right| f_1(s, u(s), v(s)) ds \]
\[ \leq \frac{1}{\Gamma(a_i)} \int_0^{t_1} \frac{t_1^{a_i-1} - t_2^{a_i-1} + (t_2 - s)^{a_i-1} - (t_1 - s)^{a_i-1}}{1 + t_2^{a_i-1}} f_1(s, u(s), v(s)) ds \]
\[ + \frac{1}{\Gamma(a_i)} \int_{t_1}^{t_2} \frac{t_2^{a_i-1} - t_1^{a_i-1} + (t_2 - s)^{a_i-1} - (t_1 - s)^{a_i-1}}{1 + t_2^{a_i-1}} f_1(s, u(s), v(s)) ds \]
\[ + \frac{1}{\Gamma(a_i)} \int_{t_2}^{\infty} \frac{t_2^{a_i-1} - t_1^{a_i-1}}{1 + t_2^{a_i-1}} f_1(s, u(s), v(s)) ds \]
\[ \longrightarrow 0, \quad \text{as } t_1 \longrightarrow t_2. \] (40)

In the same way, we can know
\[ \int_0^\infty \left| \frac{G_{11}(t_2, s)}{1 + t_2^{a_i-1}} - \frac{G_{11}(t_1, s)}{1 + t_2^{a_i-1}} \right| f_1(s, u(s), v(s)) ds \]
\[ \longrightarrow 0, \quad \text{as } t_1 \longrightarrow t_2. \] (41)

So,
\[ \frac{T_1(u, v)(t_2) - T_1(u, v)(t_1)}{1 + t_2^{a_i-1}} \longrightarrow 0, \quad \text{as } t_1 \longrightarrow t_2. \] (42)

Similar to (39)–(41), we have
\[ \frac{T_2(u, v)(t_2) - T_2(u, v)(t_1)}{1 + t_2^{a_i-1}} \longrightarrow 0, \quad \text{as } t_1 \longrightarrow t_2. \] (43)

Therefore, \( T(\Omega) \) is equicontinuous.

By \((H_1)\) and (30), for any \( \varepsilon > 0 \), there exists \( \kappa > 0 \) such that
\[ \int_\kappa^{\infty} f(s, u_n(s), v_n(s)) ds < \varepsilon. \] (44)

Due to \( \lim_{t \to \infty} \left( t \varepsilon / (a_i - 1) \right) = 1 \), there exists sufficiently large \( N_1 > 0 \) such that, for any \( t_1, t_2 > N_1 \), we have
\[ \frac{t_2^{a_i-1}}{1 + t_2^{a_i-1}} - \frac{t_1^{a_i-1}}{1 + t_1^{a_i-1}} < \varepsilon. \] (45)

Also because of \( \lim_{t \to \infty} \left( (t - \kappa)^{a_i-1} / (a_i - 1) \right) = 1 \), there exists sufficiently large \( N_2 > \kappa \) such that, for any \( t_1, t_2 > N_2 \) and \( 0 \leq s \leq \kappa \), we have
\[ \frac{(t_2 - s)^{a_i-1}}{1 + t_2^{a_i-1}} - \frac{(t_1 - s)^{a_i-1}}{1 + t_1^{a_i-1}} < \varepsilon. \] (46)

Choose \( N > \max\{N_1, N_2\} \); for any \( t_1, t_2 > N \), we get
In (47),
\[
\begin{align*}
\int_0^{\infty} & \left| \frac{G(t_2, s)}{1 + t_2^{\alpha_1 - 1}} - \frac{G(t_1, s)}{1 + t_1^{\alpha_1 - 1}} \right| f_1(s, u, v(s)) \, ds \\
\leq & \int_0^t \left| \frac{G(t_2, s)}{1 + t_2^{\alpha_1 - 1}} - \frac{G(t_1, s)}{1 + t_1^{\alpha_1 - 1}} \right| f_1(s, u(s), v(s)) \, ds \\
+ & \int_t^\infty \left| \frac{G(t_2, s)}{1 + t_2^{\alpha_1 - 1}} - \frac{G(t_1, s)}{1 + t_1^{\alpha_1 - 1}} \right| f_1(s, u(s), v(s)) \, ds \\
+ & \int_0^t \left| \frac{G(t_2, s)}{1 + t_2^{\alpha_1 - 1}} - \frac{G(t_1, s)}{1 + t_1^{\alpha_1 - 1}} \right| f_1(s, u, v(s)) \, ds \\
\leq & \frac{1}{\Gamma(\alpha_1)} \int_0^t \left( \left| \frac{t_2^{\alpha_1 - 1} - t_1^{\alpha_1 - 1}}{1 + t_2^{\alpha_1 - 1}} \right| + \left| \frac{(t_2 - s)^{\alpha_1 - 1} - (t_1 - s)^{\alpha_1 - 1}}{1 + t_2^{\alpha_1 - 1}} \right| \right) f_1(s, u(s), v(s)) \, ds \\
+ & \frac{1}{\Gamma(\alpha_1)} \int_t^\infty \left( \left| \frac{t_2^{\alpha_1 - 1} - t_1^{\alpha_1 - 1}}{1 + t_2^{\alpha_1 - 1}} \right| + \left| \frac{(t_2 - s)^{\alpha_1 - 1} - (t_1 - s)^{\alpha_1 - 1}}{1 + t_2^{\alpha_1 - 1}} \right| \right) f_1(s, u(s), v(s)) \, ds \\
+ & \frac{1}{\Gamma(\alpha_1)} \int_0^t \left( \left| \frac{t_2^{\alpha_1 - 1} - t_1^{\alpha_1 - 1}}{1 + t_2^{\alpha_1 - 1}} \right| + \left| \frac{(t_2 - s)^{\alpha_1 - 1} - (t_1 - s)^{\alpha_1 - 1}}{1 + t_2^{\alpha_1 - 1}} \right| \right) f_1(s, u(s), v(s)) \, ds \\
\leq & \frac{1}{\Gamma(\alpha_1)} \int_0^t \left( \left| \frac{t_2^{\alpha_1 - 1} - t_1^{\alpha_1 - 1}}{1 + t_2^{\alpha_1 - 1}} \right| + \left| \frac{(t_2 - s)^{\alpha_1 - 1} - (t_1 - s)^{\alpha_1 - 1}}{1 + t_2^{\alpha_1 - 1}} \right| \right) f_1(s, u(s), v(s)) \, ds \\
+ & \frac{1}{\Gamma(\alpha_1)} \int_0^t \left( \left| \frac{t_2^{\alpha_1 - 1} - t_1^{\alpha_1 - 1}}{1 + t_2^{\alpha_1 - 1}} \right| + \left| \frac{(t_2 - s)^{\alpha_1 - 1} - (t_1 - s)^{\alpha_1 - 1}}{1 + t_2^{\alpha_1 - 1}} \right| \right) f_1(s, u(s), v(s)) \, ds \\
+ & \frac{1}{\Gamma(\alpha_1)} \int_0^t \left( \left| \frac{t_2^{\alpha_1 - 1} - t_1^{\alpha_1 - 1}}{1 + t_2^{\alpha_1 - 1}} \right| + \left| \frac{(t_2 - s)^{\alpha_1 - 1} - (t_1 - s)^{\alpha_1 - 1}}{1 + t_2^{\alpha_1 - 1}} \right| \right) f_1(s, u(s), v(s)) \, ds \\
\leq & \frac{1}{\Gamma(\alpha_1)} \int_0^\infty \left( \left| \frac{t_2^{\alpha_1 - 1} - t_1^{\alpha_1 - 1}}{1 + t_2^{\alpha_1 - 1}} \right| + \left| \frac{(t_2 - s)^{\alpha_1 - 1} - (t_1 - s)^{\alpha_1 - 1}}{1 + t_2^{\alpha_1 - 1}} \right| \right) f_1(s, u(s), v(s)) \, ds \\
+ & \frac{9}{\Gamma(\alpha_1)} \int_0^\infty f_1(s, u(s), v(s)) \, ds \\
\leq & \frac{1}{\Gamma(\alpha_1)} \int_0^\infty \frac{9}{\Gamma(\alpha_1)} f_1(s, u(s), v(s)) \, ds \\
\leq & \frac{1}{\Gamma(\alpha_1)} \frac{\max_{s \in [0, \kappa]} f_1(s, u(s), v(s))}{2 \kappa \epsilon + \frac{9}{\Gamma(\alpha_1)} \epsilon}.
\end{align*}
\]
By (47) and (48), we have
\[
\left| T_1(u, v)(t_2) - T_1(u, v)(t_1) \right| \leq \max_{s \in [u, v]} \left| f_1(s, u(s), v(s)) \right| \frac{2\alpha + 9}{\Gamma(\alpha_1)} \rho \varepsilon + \epsilon 
\]
(49)

So, \( T_1(u, v) \) is equicontinuous on \(+\infty\), proof similar to (49), and we know \( T_2(u, v) \) is equicontinuous on \(+\infty\); thus, \( T(u, v) \) is equicontinuous on \(+\infty\). It follows from Lemma 4 that \( T: X \longrightarrow X \) is relatively compact. Therefore, \( T: X \longrightarrow X \) is completely continuous. The proof is completed. \( \square \)

**Theorem 2.** Assume that \( (H_2) \) holds; then, system (1) has at least one positive solution if \( \omega(g_1^* + g_2^* + h_1^* + h_2^*) < 1 \).

*Proof.* Let
\[
r \geq \frac{\omega(p_1^* + p_2^*)}{1 - \omega(g_1^* + g_2^* + h_1^* + h_2^*)}
\]
(50)

\[K = \{(u, v) \in X, \| (u, v) \| \leq r \}.
\]

Now, we illustrate \( T(K) \subset K \) for any \((u, v) \in K\) and \( t \in [0, +\infty)\); by Lemma 3 and Remark 1, we have

\[
\int_0^\infty |f_i(t, u(t), v(t))|dt 
\leq \int_0^\infty \left( (g_1(t)|u(t)| + h_1(t)|v(t)| + |f_i(t, 0, 0)|) \right)dt 
\leq \int_0^\infty \left( (1 + t^{n-1})g_1(t)\frac{|u(t)|}{1 + t^{n-1}} + (1 + t^{n-1})h_1(t)\frac{|v(t)|}{1 + t^{n-1}} + |f_i(t, 0, 0)| \right)dt 
\leq \int_0^\infty \left( (1 + t^{n-1})g_1(t)||u(t)|| + (1 + t^{n-1})h_1(t)||v(t)|| + |f_i(t, 0, 0)| \right)dt 
\leq \int_0^\infty (1 + t^{n-1})g_1(t)||u(t)|| + \int_0^\infty (1 + t^{n-1})h_1(t)||v(t)|| + \int_0^\infty |f_i(t, 0, 0)|dt 
= g_1^* ||u|| + h_1^* ||v|| + \int_0^\infty |f_i(t, 0, 0)|dt 
\leq (g_1^* + h_1^*)((u, v)) + \int_0^\infty |f_i(t, 0, 0)|dt.
\]
(55)
For any \((u_1, v_1), (u_2, v_2) \in X\) and \(t \in [0, +\infty)\), by Lemma 3, we have

\[
T_1(u_1, v_1)(t) - T_1(u_2, v_2)(t) \leq \frac{1}{1 + t^{\alpha_1 - 1}} \int_0^\infty G_1(t, s) \left| f_1(s, u_1(s), v_1(s)) - f_1(s, u_2(s), v_2(s)) \right| ds
\]

\[
\leq \bar{\omega} \int_0^\infty \left| g_1(s)(u_1(s) - u_2(s)) \right| + h_1(s)(|v_1(s) - v_2(s)|) ds
\]

\[
\leq \bar{\omega} \left\| (g_1^* u_1 - u_2) + h_1^* v_1 - v_2 \right\|.
\]

(56)

By the similar proof, we have

\[
T_2(u_1, v_1)(t) - T_2(u_2, v_2)(t) \leq \bar{\omega} \left\| (g_2^* u_1 - u_2) + h_2^* v_1 - v_2 \right\|.
\]

(57)

Now, inequalities (56) and (57) can show that

\[
\|T(u, v)\| \leq \bar{\omega} \left( g_1^* + g_2^* + h_1^* + h_2^* \right) \left\| (u_1 - u_2) + (v_1 - v_2) \right\|
\]

(58)

Thus, by the Banach contraction mapping theorem that

\( T \) has a unique fixed point in \( X \), system (1) has a unique positive solution. The proof is completed.

4. An Example

Consider the following fractional differential system:

\[
\begin{cases}
D_{0^+}^{(5/2)}u(t) + f_1(t, u(t), v(t)) = 0,

D_{0^+}^{(7/2)}v(t) + f_2(t, u(t), v(t)) = 0, & 0 < t < +\infty,

u(0) = u'(0) = 0,

D_{0^+}^{(3/2)}u(+) = \frac{1}{5} \int_0^\infty e^{-t}u(t) dt,

v(0) = v'(0) = 0,

D_{0^+}^{(5/2)}v(+) = \frac{1}{4} \int_0^\infty e^{-t}v(t) dt,
\end{cases}
\]

(59)

where \( \alpha_1 = (5/2), \alpha_2 = (7/2), \mu_1 = (1/5), \mu_2 = (1/4), A_1(t) = A_2(t) = t, a_1(t) = e^{-t} = a_2(t) = e^{-t} \). Then, we have

\[
\int_0^\infty a_1(t) dA_1(t) = \int_0^\infty a_2(t) dA_2(t) = \int_0^\infty e^{-t} dt = 1 < +\infty,
\]

\[
\int_0^\infty a_1(t) t^\alpha_1 - 1 dA_1(t) = \int_0^\infty e^{-t} t^{3/2} dt = 2.3562 < +\infty,
\]

\[
\int_0^\infty a_2(t) t^\alpha_2 - 1 dA_2(t) = \int_0^\infty e^{-t} t^{5/2} dt = 3.3233 < +\infty,
\]

\[\bar{\omega} = 2.0929.\]

(60)

Take

\[
f_1(t, u(t), v(t)) = \frac{1}{1 + e^{-t}} \left( u(t) \cos t + v(t) \sin t \right) \frac{1}{7(1 + t^{(5/2)})} e^{3t},
\]

\[
f_2(t, u(t), v(t)) = \frac{1}{2(1 + e^{-t})} + \frac{u(t)}{(1 + t^{(3/2)})} e^{3t} + \frac{v(t)}{3(1 + t^{(5/2)})} e^{3t}.
\]

(61)

Let

\[
g_1(t) = \frac{1}{1 + t^{(3/2)}} e^{3t},
\]

\[
g_2(t) = \frac{1}{1 + t^{(3/2)}} e^{3t},
\]

(62)

\[
h_1(t) = \frac{1}{7(1 + t^{(5/2)})} e^{3t},
\]

\[
h_2(t) = \frac{1}{3(1 + t^{(5/2)})} e^{3t}.
\]

Through calculation, we get

\[
g_1^* = \int_0^\infty \left( 1 + t^{\alpha_1 - 1} \right) g_1(t) dt = \int_0^\infty e^{t^3} dt = \frac{1}{7},
\]

\[
g_2^* = \int_0^\infty \left( 1 + t^{\alpha_2 - 1} \right) g_2(t) dt = \int_0^\infty e^{t^3} dt = \frac{1}{12},
\]

\[
h_1^* = \int_0^\infty \left( 1 + t^{\alpha_1 - 1} \right) h_1(t) dt = \int_0^\infty e^{3t} dt = \frac{1}{21},
\]

\[
h_2^* = \int_0^\infty \left( 1 + t^{\alpha_2 - 1} \right) h_2(t) dt = \frac{1}{3} \int_0^\infty e^{4t} dt = \frac{1}{12},
\]

\[\bar{\omega}(g_1^* + g_2^* + h_1^* + h_2^*) = 2.0929(\frac{1}{7} + \frac{1}{12} + \frac{1}{21} + 1) = 0.9219 < 1.\]

(63)

Then, by Theorem 2, system (59) has at least one positive solution.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.
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