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Research Article

Results of Positive Solutions for the Fractional Differential System on an Infinite Interval

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The chief topic of this paper is to investigate the fractional differential system on an infinite interval. By introducing an appropriate compactness criterion in a special function space and applying the Schauder fixed-point theorem and the Banach contraction mapping principle, we established the results for the existence and uniqueness of positive solutions. An example is then given to show the utilization of the main results.

1. Introduction

In this paper, we investigate the following fractional differential system on an infinite interval:

$$\begin{cases} D_{0^+}^{\alpha_1} u(t) + f_1(t, u(t), v(t)) = 0, & \\ D_{0^+}^{\alpha_2} v(t) + f_2(t, u(t), v(t)) = 0, & 0 < t < +\infty, \\ u(0) = u'(0) = u''(0) = \cdots = u^{(n-2)}(0) = 0, & D_{0^+}^{\alpha_1 - 1} u(+\infty) = \mu_1 \int_0^\infty a_1(t) u(t) \mathrm{d}A_1(t), \\ v(0) = v'(0) = v''(0) = \cdots = v^{(m-2)}(0) = 0, & D_{0^+}^{\alpha_2 - 1} v(+\infty) = \mu_2 \int_0^\infty a_2(t) v(t) \mathrm{d}A_2(t), \end{cases} \tag{1}$$

where $1 \le n-1 < \alpha_1 \le n$, $1 \le m-1 < \alpha_2 \le m$, and n, $m \ge 2$, $D_{0^+}^{\alpha_i}$ is the Riemann–Liouville derivative operator. $\mu_i > 0$ is a constant, $\int_0^\infty a_1(t) u(t) \mathrm{d}A_1(t)$ and $\int_0^\infty a_2(t) v(t) \mathrm{d}A_2(t)$ denote the Riemann–Stieltjes integral, and A_i is a function of bounded positive variation. $a_i \in L[0, +\infty)$, $\mu_i \int_0^\infty a_i(t) t^{\alpha_i - 1} \mathrm{d}A_i(t) < \Gamma(\alpha_i)$, $\int_0^\infty a_i(t) \mathrm{d}A_i(t) < +\infty$, and f_i: $[0, +\infty) \times [0, +\infty) \times [0, +\infty) \longrightarrow [0, +\infty)$ is continuous, $i = 1, 2$.

Numerous models in physics, chemistry, biology, medicine, and other fields have promoted the research of differential equations, for instance, evaluation of water quality on receiving water [1], and the advection-dispersion equation can be formulated as shown for the case of one-dimensional flow:

$$\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} = D_L \frac{\partial^2 C}{\partial t^2} - f(C), \tag{2}$$

where C is the concentration of a generic pollutant, t is the time, x is the longitudinal displacement, u is the velocity, D_L is the diffusion coefficient, and $f(C)$ is a generic term for reactions involving the pollutant C. Westerlund [2] established a one-dimensional model to describe the transmission of the electromagnetic wave:

$$\mu\varepsilon\frac{\partial^2 E(x,t)}{\partial x^2} + \mu\varepsilon\zeta D_t^{\gamma} E(x,t) + \frac{\partial^2 E(x,t)}{\partial t^2} = 0, \quad (3)$$

where μ, ε, and ζ are constants and $D_t^{\gamma} E(x,t) = (\partial^{\gamma} E(x,t))/\partial t^{\gamma}$ is a fractional derivative. In the process of establishing the model, k-Hessian equations [3], Sobolev equations [4], and Schrödinger elliptic equations [5, 6], there are also huge applications.

Under the proper initial or boundary conditions, to study the positive solution of the above models is very necessary; especially, for the boundary value problems on the infinite interval, many authors put their interest in it [7–16]. Liang and Zhang [17] applied the fixed-point theorem to obtain the existence of positive solutions for the following fractional differential equation:

$$\begin{cases} D_{0^+}^{\alpha} u(t) + a(t) f(u(t)) = 0, \quad 0 < t < +\infty, \\ u(0) = u'(0) = 0, \quad \lim_{t \longrightarrow +\infty} D_{0^+}^{\alpha-1} u(t) = \sum_{i=1}^{m-2} \beta_i u(\xi_i), \end{cases} \quad (4)$$

where $2 < \alpha \le 3$, $D_{0^+}^{\alpha}$ is the Riemann–Liouville fractional derivative, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < +\infty$, $\beta_i > 0$, $0 < \sum_{i=1}^{m-2} \beta_i u(\xi_i) < \Gamma(\alpha)$, and $f: [0, +\infty) \longrightarrow [0, +\infty)$is continuous.

For all we know, there are few studies on fractional differential systems of infinite intervals, although it is necessary to do so. In this paper, we aimed at getting the existence and uniqueness of positive solutions for system (1) on infinite interval. Compared with the existing literature, the innovations of this paper are as follows. Firstly, the paper which we discuss is the system rather than a single equation. Secondly, we study the system with integral boundary value conditions on infinite intervals, which are more general than those of two-point, three-point, and multipoint boundary value condition. At last, we use two different techniques: the Schauder fixed-point theorem and the Banach contraction mapping principle, for system (1), not only to obtain the existence of positive solutions but also the uniqueness of positive solutions.

2. Preliminaries and Lemmas

Definition 1 (see [18, 19]). Let $\alpha > 0$ and u be piecewise continuous on $(0, +\infty)$ and integrable on any finite subinterval of $[0, +\infty)$. Then, for $t > 0$, we call

$$I_{0^+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds \quad (5)$$

the Riemann–Liouville fractional integral of u of order α.

Definition 2 (see [18, 19]). The Riemann–Liouville fractional derivative of order $\alpha > 0$, $n-1 \le \alpha < n$, $n \in \mathbb{N}$, is defined as

$$D_{0^+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} u(s) ds, \quad (6)$$

where $\mathbb{N}$ denotes the natural number set and the function $u(t)$ is n times continuously differentiable on $[0, +\infty)$.

Lemma 1 (see [18, 19]). *Let $\alpha > 0$, and if the fractional derivative $D_{0^+}^{\alpha-1} u(t)$ and $D_{0^+}^{\alpha} u(t)$ are continuous on $[0, +\infty)$, then,*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \quad (7)$$

where $c_1, c_2, \ldots, c_n \in (-\infty, +\infty)$, n is the smallest integer greater than or equal to α.

Lemma 2. *Let $y_i \in C(0, +\infty) \cap L[0, +\infty)$; then, the fractional system*

$$\begin{cases} D_{0+}^{\alpha_1} u(t) + y_1(t) = 0, \\ D_{0+}^{\alpha_1} v(t) + y_2(t) = 0, \\ 0 < t < +\infty, \\ u(0) = u'(0) = u''(0) = \cdots = u^{(n-2)}(0) = 0, \\ D_{0+}^{\alpha_1-1} u(+\infty) = \mu_1 \int_0^{\infty} a_1(t) u(t) dA_1(t), \\ v(0) = v'(0) = v''(0) = \cdots = v^{(m-2)}(0) = 0, \\ D_{0+}^{\alpha_2-1} v(+\infty) = \mu_2 \int_0^{\infty} a_2(t) v(t) dA_2(t), \end{cases} \quad (8)$$

has a unique integral representation

$$\begin{cases} u(t) = \int_0^{\infty} G_1(t,s) y_1(s) ds, \\ v(t) = \int_0^{\infty} G_2(t,s) y_2(s) ds, \end{cases} \quad (9)$$

where

$$G_i(t,s) = G_{i1} = (t,s) + G_{i2}(t,s), \quad i = 1, 2,$$

$$G_{i1}(t,s) = \frac{1}{\Gamma(\alpha_i)} \begin{cases} t^{\alpha_i-1} - (t-s)^{\alpha_i-1}, & 0 \le s \le t \le +\infty, \\ t^{\alpha_i-1}, & 0 \le t \le s \le +\infty, \end{cases} \quad (10)$$

$$G_{i2}(t,s) = \frac{\mu_i t^{\alpha_i-1}}{\Gamma(\alpha_i) - \mu_i \int_0^{\infty} a_i(t) t^{\alpha_i-1} dA_i(t)} \int_0^{\infty} a_i(t) G_{i1}(t,s) dA_i(t).$$

Proof. By Lemma 1, the equations in system (8) can be transformed into the equivalent integral equations

$$u(t) = -I_{0+}^{\alpha_1} y_1(t) + c_1 t^{\alpha_1-1} + c_2 t^{\alpha_1-2} + \cdots + c_n t^{\alpha_1-n}, c_j \in (-\infty, +\infty), \quad j = 1, 2, \ldots, n,$$
$$v(t) = -I_{0+}^{\alpha_2} y_2(t) + \bar{c}_1 t^{\alpha_2-1} + \bar{c}_2 t^{\alpha_2-2} + \cdots + c_m t^{\alpha_2-m}, \bar{c}_j \in (-\infty, +\infty), \quad \bar{j} = 1, 2, \ldots, m, \tag{11}$$

that is,

$$u(t) = -\frac{1}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1-1} y_1(s) ds + c_1 t^{\alpha_1-1} + c_2 t^{\alpha_1-2} + \cdots + c_n t^{\alpha_1-n}, c_j \in (-\infty, +\infty), \quad j = 1, 2, \ldots, n,$$
$$v(t) = -\frac{1}{\Gamma(\alpha_2)} \int_0^t (t-s)^{\alpha_2-1} y_2(s) ds + \bar{c}_1 t^{\alpha_2-1} + \bar{c}_2 t^{\alpha_2-2} + \cdots + \bar{c}_m t^{\alpha_2-m}, \bar{c}_j \in (-\infty, +\infty), \quad j = 1, 2, \ldots, m. \tag{12}$$

Since

$$u(0) = u'(0) = \cdots = u^{(n-2)} = 0,$$
$$v(0) = v'(0) = \cdots = v^{(m-2)}(0) = 0, \tag{13}$$

we have

$$c_2 = c_3 = \cdots = c_n = 0,$$
$$\bar{c}_2 = \bar{c}_3 = \cdots = \bar{c}_m = 0. \tag{14}$$

So,

$$u(t) = -\frac{1}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1-1} y_1(s) ds + c_1 t^{\alpha_1-1},$$
$$v(t) = -\frac{1}{\Gamma(\alpha_2)} \int_0^t (t-s)^{\alpha_2-1} y_2(s) ds + \bar{c}_1 t^{\alpha_2-1}. \tag{15}$$

We also have

$$D_{0^+}^{\alpha_1-1} u(t) = c_1 \Gamma(\alpha_1) - \int_0^t y_1(s) ds,$$
$$D_{0^+}^{\alpha_2-1} v(t) = \bar{c}_1 \Gamma(\alpha_2) - \int_0^t y_2(s) ds. \tag{16}$$

For

$$D_{0^+}^{\alpha_1-1} u(+\infty) = \mu_1 \int_0^\infty a_1(s) u(s) dA_1(s),$$
$$D_{0^+}^{\alpha_2-1} v(+\infty) = \mu_2 \int_0^\infty a_2(s) v(s) dA_2(s), \tag{17}$$

we obtain

$$c_1 = \frac{1}{\Gamma(\alpha_1)} \left(\mu_1 \int_0^\infty a_1(s) u(s) dA_1(s) + \int_0^{+\infty} y_1(s) ds \right),$$
$$\bar{c}_1 = \frac{1}{\Gamma(\alpha_2)} \left(\mu_2 \int_0^\infty a_2(s) v(s) dA_2(s) + \int_0^{+\infty} y_2(s) ds \right). \tag{18}$$

Combining (15) and (18), we have

$$u(t) = -\frac{1}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1-1} y_1(s) ds + c_1 t^{\alpha_1-1}$$
$$= \int_0^\infty G_{11}(t,s) y_1(s) ds + \frac{\mu_1 t^{\alpha_1-1}}{\Gamma(\alpha_1)} \int_0^\infty a_1(t) u(t) dA_1(t), \tag{19}$$

$$v(t) = -\frac{1}{\Gamma(\alpha_2)} \int_0^t (t-s)^{\alpha_2-1} y_2(s) ds + \bar{c}_1 t^{\alpha_2-1}$$
$$= \int_0^\infty G_{21}(t,s) y_2(s) ds + \frac{\mu_2 t^{\alpha_2-1}}{\Gamma(\alpha_2)} \int_0^\infty a_2(t) v(t) dA_2(t). \tag{20}$$

(19) and (20) are multiplied by $a_1(t)$ and $a_2(t)$, respectively, and then solved the integral from 0 to $+\infty$ with respect to $A_1(t)$ and $A_2(t)$; then, we have

$$\int_0^\infty a_1(t) u(t) dA_1(t) = \frac{\Gamma(\alpha_1)}{\Gamma(\alpha_1) - \mu_1 \int_0^\infty a_1(t) dA_1(t)} \int_0^\infty a_1(t) \int_0^\infty G_{11}(t,s) y_1(s) ds \, dA_1(t),$$
$$\int_0^\infty a_2(t) v(t) dA_2(t) = \frac{\Gamma(\alpha_2)}{\Gamma(\alpha_2) - \mu_2 \int_0^\infty a_2(t) dA_2(t)} \int_0^\infty a_2(t) \int_0^\infty G_{21}(t,s) y_2(s) ds \, dA_2(t). \tag{21}$$

Therefore,

$$u(t)=\int_0^\infty G_{11}(t,s)y_1(s)\mathrm{d}s+\frac{\mu_1 t^{\alpha_1-1}}{\Gamma(\alpha_1)}\int_0^\infty a_1(t)u(t)\mathrm{d}A_1(t)$$
$$=\int_0^\infty G_{11}(t,s)y_1(s)\mathrm{d}s+\frac{\mu_1 t^{\alpha_1-1}}{\Gamma(\alpha_1)-\mu_1\int_0^\infty a_1(t)\mathrm{d}A_1(t)}\cdot\int_0^\infty a_1(t)\int_0^\infty G_{11}(t,s)y_1(s)\mathrm{d}s\,\mathrm{d}A_1(t)$$
$$=\int_0^\infty G_{11}(t,s)y_1(s)\mathrm{d}s+\int_0^\infty G_{12}(t,s)y_1(s)\mathrm{d}s,$$
$$v(t)=\int_0^\infty G_{21}(t,s)y_2(s)\mathrm{d}s+\frac{\mu_2 t^{\alpha_2-1}}{\Gamma(\alpha_2)}\int_0^\infty a_2(t)v(t)\mathrm{d}A_2(t)$$
$$=\int_0^\infty G_{21}(t,s)y_2(s)\mathrm{d}s+\frac{\mu_2 t^{\alpha_2-1}}{\Gamma(\alpha_2)-\mu_2\int_0^\infty a_2(t)\mathrm{d}A_2(t)}\cdot\int_0^\infty a_2(t)\int_0^\infty G_{21}(t,s)y_2(s)\mathrm{d}s\,\mathrm{d}A_2(t)$$
$$=\int_0^\infty G_{21}(t,s)y_2(s)\mathrm{d}s+\int_0^\infty G_{22}(t,s)y_2(s)\mathrm{d}s. \tag{22}$$

So, (9) holds. The proof is completed.

Lemma 3. *The Green function in Lemma 2 has the following properties:*

(1) $G_{i1}(t,s)$ is continuous and $G_{i1}(t,s)\geq 0$, $(t,s)\in[0,+\infty)\times[0,+\infty)$.

(2) $(G_{i1}(t,s)/1+t^{\alpha_i-1})\leq(1/\Gamma(\alpha_i))$, $(G_i(t,s)/1+t^{\alpha_i-1})\leq\varpi$, $(t,s)\in[0,+\infty)\times[0,+\infty)$,

where

$$\varpi=\max\left\{\begin{array}{l}\dfrac{1}{\Gamma(\alpha_1)}+\dfrac{\Gamma(\alpha_1)+\mu_1\int_0^\infty a_1(t)\mathrm{d}A_1(t)}{\Gamma(\alpha_1)\left(\Gamma(\alpha_1)-\mu_1\int_0^\infty a_1(t)t^{\alpha_1-1}\mathrm{d}A_1(t)\right)},\\ \\ \dfrac{1}{\Gamma(\alpha_2)}+\dfrac{\Gamma(\alpha_2)+\mu_2\int_0^\infty a_2(t)\mathrm{d}A_2(t)}{\Gamma(\alpha_2)\left(\Gamma(\alpha_2)-\mu_2\int_0^\infty a_2(t)t^{\alpha_2-1}\mathrm{d}A_2(t)\right)}\end{array}\right\}. \tag{23}$$

The space $X=E_1\times E_2$ will be used in the study of system (1), where

$$E_1=\left\{u\in C[0,+\infty):\sup_{t\in[0,+\infty)}\frac{u(t)}{1+t^{\alpha_1-1}}<+\infty\right\},$$
$$E_2=\left\{v\in C[0,+\infty):\sup_{t\in[0,+\infty)}\frac{v(t)}{1+t^{\alpha_2-1}}<+\infty\right\}. \tag{24}$$

Then, $(E_1,\|\cdot\|)$ and $(E_2,\|\cdot\|)$ are the Banach space with the norm

$$\|u\|=\sup_{t\in[0,+\infty)}\frac{u(t)}{1+t^{\alpha_1-1}},$$
$$\|v\|=\sup_{t\in[0,+\infty)}\frac{v(t)}{1+t^{\alpha_2-1}}. \tag{25}$$

Clearly, $(X,\|\cdot\|)$ is a Banach space with the norm $\|u,v\|=\|u\|+\|v\|$. Define nonlinear integral operators T_i: $X\longrightarrow E_i$ and $T:X\longrightarrow X$ by

$$T_i(u,v)(t)=\int_0^\infty G_i(t,s)f_i(s,u(s),v(s))\mathrm{d}s,\quad i=1,2, \tag{26}$$

$$T(u,v)=(T_1(u,v),T_2(u,v)). \tag{27}$$

Thus, the existence of solutions to system (1) is equivalent to the existence of solutions in X for operator equation $T(u,v)=(u,v)$ defined by (27).

Lemma 4 (see [20, 21]). *Let E be defined as (24) and M be any bounded subset of E. Then, M is relatively compact in E if $\{(x(t)/1+t^{\alpha-1}):x\in M\}$ is equicontinuous on any finite subinterval of J, and for any given $\varepsilon>0$, there exists $N>0$ such that $|(x(t_1)/1+t_1^{\alpha-1})-(x(t_2)/1+t_2^{\alpha-1})|<\varepsilon$ uniformly with respect to all $x\in M$, and t_1, $t_2>N$.*

3. Main Results

We list the conditions to be used later: $(\mathbf{H}_1)$ there exist nonnegative functions $p_i(t)$, $g_i(t)$, $h_i(t)\in L^1[0,+\infty)$ and $t^{\alpha_1-1}g_i(t)$, $t^{\alpha_2-1}h_i(t)\in L^1[0,+\infty)$ such that

$$|f_i(t,u,v)|\leq p_i(t)+g_i(t)|u|+h_i(t)|v|,\ (t,u,v)\in[0,+\infty)\times[0,+\infty)\times[0,+\infty). \tag{28}$$

$(\mathbf{H}_2)$ $|f_i(t,0,0)|\in L^1[0,+\infty)$, there exist nonnegative functions $g_i(t)$, $h_i(t)\in L^1[0,+\infty)$ and $t^{\alpha_1-1}g_i(t)$, $t^{\alpha_2-1}h_i(t)\in L^1[0,+\infty)$ such that

$$|f_i(t,u_1,v_1)-f_i(t,u_2,v_2)|\leq g_i(t)|u_1-u_2|+h_i(t)|v_1-v_2|,$$
$$(t,u_1,v_1),(t,u_1,v_1)\in[0,+\infty)\times[0,+\infty)\times[0,+\infty). \tag{29}$$

Remark 1. If $(\mathbf{H}_1)$ holds, then

$$\int_0^\infty|f_i(t,u(t),v(t))|\mathrm{d}t\leq p_i^*+(g_i^*+h_i^*)\|(u,v)\|,\quad(u,v)\in X, \tag{30}$$

where

$$p_i^*=\int_0^\infty p_i(t)\mathrm{d}t,$$
$$g_i^*=\int_0^\infty\left(1+t^{\alpha_1-1}\right)g_i(t)\mathrm{d}t, \tag{31}$$
$$h_i^*=\int_0^\infty\left(1+t^{\alpha_2-1}\right)h_i(t)\mathrm{d}t.$$

In fact, by $(\mathbf{H}_1)$, for any $(u,v)\in X$, we have

$$
\begin{aligned}
&\int_0^\infty \left|f_i(t,u(t),v(t))\right|dt\\
&\le \int_0^\infty \left(p_i(t)+g_i(t)|u(t)|+h_i(t)|v(t)|\right)dt\\
&=\int_0^\infty \left(p_i(t)+\left(1+t^{\alpha_1-1}\right)g_i(t)\frac{|u(t)|}{1+t^{\alpha_1-1}}+\left(1+t^{\alpha_2-1}\right)h_i(t)\frac{|v(t)|}{1+t^{\alpha_2-1}}\right)dt\\
&\le \int_0^\infty \left(p_i(t)+\left(1+t^{\alpha_1-1}\right)g_i(t)\|u\|+\left(1+t^{\alpha_2-1}\right)h_i(t)\|v\|\right)dt\\
&=\int_0^\infty p_i(t)dt+\int_0^\infty\left(1+t^{\alpha_1-1}\right)g_i(t)dt\|u\|+\int_0^\infty\left(1+t^{\alpha_2-1}\right)h_i(t)dt\|v\|\\
&=p_i^*+g_i^*\|u\|+h_i^*\|v\|\le p_i^*+\left(g_i^*+h_i^*\right)\|(u,v)\|.
\end{aligned}
\tag{32}
$$

Theorem 1. *Assume that ($\mathbf{H}_1$) holds; then, $T:X\longrightarrow X$ is a completely continuous operator.*

Proof. First, we show that $T:X\longrightarrow X$ is continuous. Suppose $\{(u_n,v_n)\}\subset X$, $(u,v)\in X$ with $\|(u_n,v_n)-(u,v)\|\longrightarrow 0$ $(n\longrightarrow+\infty)$, and there exists a constant $r>0$ such that $\|(u_n,v_n)\|\le r$ and $\|(u,v)\|\le r$. By $(\mathbf{H}_1)$ and (30), we have

$$
\begin{aligned}
&\left|\int_0^\infty G_1(t,s)f_1(s,u_n(s),v_n(s))ds-\int_0^\infty G_1(t,s)f_1(s,u(s),v(s))ds\right|\\
&\le \int_0^\infty G_1(t,s)f_1(s,u_n(s),v_n(s))ds+\int_0^\infty G_1(t,s)f_1(s,u(s),v(s))ds\\
&\le \varpi\int_0^\infty f_1(s,u_n(s),v_n(s))ds+\varpi\int_0^\infty f_1(s,u(s),v(s))ds\\
&\le 2\varpi\left(p_1^*+g_1^*\|u\|+h_1^*\|v\|\right)\le 2\varpi\left(p_1^*+\left(g_1^*+h_1^*\right)\|(u,v)\|\right)<+\infty.
\end{aligned}
\tag{33}
$$

From $(\mathbf{H}_1)$ and (33), for any $\varepsilon>0$, there exists sufficiently large M_0 such that

$$
\int_{M_0}^\infty G_1(t,s)f_1(s,u_n(s),v_n(s))ds+\int_{M_0}^\infty G_1(t,s)f_1\cdot(s,u(s),v(s))ds<\frac{\varepsilon}{2}.
\tag{34}
$$

On the contrary, by the continuity of $f_1(t,u,v)$ on $[0,M_0]\times[0,(1+M_0^{\alpha_1-1})r]\times[0,(1+M_0^{\alpha_1-1})r]$, there exists $N>0$ such that when $n>N$ and $t\in[0,M_0]$, we have

$$
\left|f_1(s,u_n(s),v_n(s))-f_1(s,u(s),v(s))\right|<\frac{\varepsilon}{2\varpi M_0}.
\tag{35}
$$

Hence, for any $t\in[0,+\infty)$ and $n>N$, we obtain

$$
\begin{aligned}
&\left|\frac{T_1(u_n,v_n)(t)}{1+t^{\alpha_1-1}}-\frac{T_1(u,v)(t)}{1+t^{\alpha_1-1}}\right|\\
&=\left|\int_0^\infty\frac{G_1(t,s)}{1+t^{\alpha_1-1}}f_1(s,u_n(s),v_n(s))ds-\int_0^\infty\frac{G_1(t,s)}{1+t^{\alpha_1-1}}f_1(s,u(s),v(s))ds\right|\\
&\le\int_0^{M_0}\frac{G_1(t,s)}{1+t^{\alpha_1-1}}\left|f_1(s,u_n(s),v_n(s))-f_1(s,u(s),v(s))\right|ds\\
&\quad+\int_{M_0}^\infty\frac{G_1(t,s)}{1+t^{\alpha_1-1}}f_1(s,u_n(s),v_n(s))ds+\int_{M_0}^\infty\frac{G_1(t,s)}{1+t^{\alpha_1-1}}f_1(s,u(s),v(s))ds
\end{aligned}
$$

$$\begin{aligned}
&\leq \int_0^{M_0} \frac{G_1(t,s)}{1+t^{\alpha_1-1}} \left| f_1(s,u_n(s),v_n(s)) - f_1(s,u(s),v(s)) \right| ds \\
&\quad + \int_{M_0}^{\infty} G_1(t,s) f_1(s,u_n(s),v_n(s)) ds + \int_{M_0}^{\infty} G_1(t,s) f_1(s,u(s),v(s)) ds \\
&< \varpi \int_0^{M_0} \left| f_1(s,u_n(s),v_n(s)) ds - f_1(s,u(s),v(s)) \right| ds + \frac{\varepsilon}{2} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned} \tag{36}$$

Thus, we know that $\|T_1(u_n,v_n) - T_1(u,v)\| \longrightarrow 0$ $(n \longrightarrow +\infty)$. By the similar proof as (33)–(36), we know $\|T_2(u_n,v_n) - T_2(u,v)\| \longrightarrow 0$ $(n \longrightarrow +\infty)$. So, $T: X \longrightarrow X$ is continuous.

Next, we show that $T: X \longrightarrow X$ is relatively compact. Let Ω be a bounded subset of X; then, there exists constant $M > 0$ such that $\|(u,v)\| \leq M$, $(u,v) \in \Omega$. For any $(u,v) \in \Omega$, $t \in [0, +\infty)$, and by (32), we obtain

$$\begin{aligned}
\left| \frac{T_1(u_n,v_n)(t)}{1+t^{\alpha_1-1}} \right| &= \left| \int_0^{\infty} \frac{G_1(t,s)}{1+t^{\alpha_1-1}} f_1(s,u_n(s),v_n(s)) ds \right| \\
&\leq \varpi \int_0^{\infty} \left| f_1(s,u_n(s),v_n(s)) \right| ds \\
&\leq \varpi \left(p_1^* + (g_1^* + h_1^*) \|(u,v)\| \right) \\
&\leq \varpi \left(p_1^* + (g_1^* + h_1^*) M \right) < +\infty.
\end{aligned} \tag{37}$$

Similarly, we have

$$\begin{aligned}
\left| \frac{T_2(u_n,v_n)(t)}{1+t^{\alpha_2-1}} \right| &\leq \varpi \left(p_1^* + (g_1^* + h_1^*) \|(u,v)\| \right) \\
&\leq \varpi \left(p_1^* + (g_1^* + h_1^*) M \right) < +\infty.
\end{aligned} \tag{38}$$

Consequently, $T(\Omega)$ is uniformly bounded.

Given $I \subset [0, +\infty)$ be any compact interval. For any $t_1, t_2 \in I$, $t_1 < t_2$ and $(u,v) \in \Omega$, we deduce

$$\begin{aligned}
&\left| \frac{T_1(u,v)(t_2)}{1+t_2^{\alpha_1-1}} - \frac{T_1(u,v)(t_1)}{1+t_1^{\alpha_1-1}} \right| \\
&\quad = \left| \int_0^{\infty} \frac{G_1(t_2,s)}{1+t_2^{\alpha_1-1}} f_1(s,u(s),v(s)) ds - \int_0^{\infty} \frac{G_1(t_1,s)}{1+t_1^{\alpha_1-1}} f_1(s,u(s),v(s)) ds \right| \\
&\quad \leq \int_0^{\infty} \left| \frac{G_{11}(t_2,s)}{1+t_2^{\alpha_1-1}} - \frac{G_{11}(t_1,s)}{1+t_1^{\alpha_1-1}} \right| f_1(s,u(s),v(s)) ds \\
&\qquad + \int_0^{\infty} \left| \frac{G_{12}(t_2,s)}{1+t_2^{\alpha_1-1}} - \frac{G_{12}(t_1,s)}{1+t_1^{\alpha_1-1}} \right| f_1(s,u(s),v(s)) ds \\
&\quad \leq \int_0^{\infty} \left| \frac{G_{11}(t_2,s)}{1+t_2^{\alpha_1-1}} - \frac{G_{11}(t_1,s)}{1+t_1^{\alpha_1-1}} \right| f_1(s,u(s),v(s)) ds \\
&\qquad + \left| \frac{t_2^{\alpha_1-1}}{1+t_2^{\alpha_1-1}} - \frac{t_1^{\alpha_1-1}}{1+t_1^{\alpha_1-1}} \right| \int_0^{\infty} \frac{\mu_1 \int_0^{\infty} a_1(t) G_{11}(t,s) dA_1(t)}{\Gamma(\alpha_1) - \mu_1 \int_0^{\infty} a_1(t) t^{\alpha_1-1} dA_1(t)} f_1(s,u(s),v(s)) ds \\
&\quad \leq \int_0^{\infty} \left| \frac{G_{11}(t_2,s)}{1+t_2^{\alpha_1-1}} - \frac{G_{11}(t_1,s)}{1+t_2^{\alpha_1-1}} \right| f_1(s,u(s),v(s)) ds \\
&\qquad + \int_0^{\infty} \left| \frac{G_{11}(t_1,s)}{1+t_2^{\alpha_1-1}} - \frac{G_{11}(t_1,s)}{1+t_1^{\alpha_1-1}} \right| f_1(s,u(s),v(s)) ds \\
&\qquad + \left| \frac{t_2^{\alpha_1-1}}{1+t_2^{\alpha_1-1}} - \frac{t_1^{\alpha_1-1}}{1+t_1^{\alpha_1-1}} \right| \int_0^{\infty} \frac{\mu_1 \int_0^{\infty} a_1(t) G_{11}(t,s) dA_1(t)}{\Gamma(\alpha_1) - \mu_1 \int_0^{\infty} a_1(t) t^{\alpha_1-1} dA_1(t)} f_1(s,u(s),v(s)) ds.
\end{aligned} \tag{39}$$

Since

$$\begin{aligned}
&\int_0^{\infty}\left|\frac{G_{11}(t_2,s)}{1+t_2^{\alpha_1-1}}-\frac{G_{11}(t_1,s)}{1+t_2^{\alpha_1-1}}\right|f_1(s,u(s),v(s))\mathrm{d}s \\
&\le \int_0^{t_1}\left|\frac{G_{11}(t_2,s)}{1+t_2^{\alpha_1-1}}-\frac{G_{11}(t_1,s)}{1+t_2^{\alpha_1-1}}\right|f_1(s,u(s),v(s))\mathrm{d}s \\
&\quad+\int_{t_1}^{t_2}\left|\frac{G_{11}(t_2,s)}{1+t_2^{\alpha_1-1}}-\frac{G_{11}(t_1,s)}{1+t_2^{\alpha_1-1}}\right|f_1(s,u(s),v(s))\mathrm{d}s \\
&\quad+\int_{t_2}^{+\infty}\left|\frac{G_{11}(t_2,s)}{1+t_2^{\alpha_1-1}}-\frac{G_{11}(t_1,s)}{1+t_2^{\alpha_1-1}}f_1(s,u(s),v(s))\mathrm{d}s\right| \\
&\le\frac{1}{\Gamma(\alpha_1)}\int_0^{t_1}\frac{\left|t_2^{\alpha_1-1}-t_1^{\alpha_1-1}\right|+\left|(t_2-s)^{\alpha_1-1}-(t_1-s)^{\alpha_1-1}\right|}{1+t_2^{\alpha_1-1}}f_1 \\
&\quad\cdot(s,u(s),v(s))\mathrm{d}s \\
&\quad+\frac{1}{\Gamma(\alpha_1)}\int_{t_1}^{t_2}\frac{\left|t_2^{\alpha_1-1}-t_1^{\alpha_1-1}\right|+\left|(t_2-s)^{\alpha_1-1}\right|}{1+t_2^{\alpha_1-1}}f_1(s,u \\
&\quad\cdot(s),v(s))\mathrm{d}s \\
&\quad+\frac{1}{\Gamma(\alpha_1)}\int_{t_2}^{+\infty}\frac{\left|t_2^{\alpha_1-1}-t_1^{\alpha_1-1}\right|}{1+t_2^{\alpha_1-1}}f_1(s,u(s),v(s))\mathrm{d}s \\
&\longrightarrow 0, \qquad \text{as } t_1\longrightarrow t_2.
\end{aligned} \tag{40}$$

In the same way, we can know

$$\int_0^{\infty}\left|\frac{G_{11}(t_1,s)}{1+t_2^{\alpha_1-1}}-\frac{G_{11}(t_1,s)}{1+t_1^{\alpha_1-1}}\right|f_1(s,u(s),v(s))\mathrm{d}s\longrightarrow 0, \quad \text{as } t_1\longrightarrow t_2. \tag{41}$$

So,

$$\left|\frac{T_1(u,v)(t_2)}{1+t_2^{\alpha_1-1}}-\frac{T_1(u,v)(t_1)}{1+t_1^{\alpha_1-1}}\right|\longrightarrow 0, \quad \text{as } t_1\longrightarrow t_2. \tag{42}$$

Similar to (39)–(41), we have

$$\left|\frac{T_2(u,v)(t_2)}{1+t_2^{\alpha_2-1}}-\frac{T_2(u,v)(t_1)}{1+t_1^{\alpha_2-1}}\right|\longrightarrow 0, \quad \text{as } t_1\longrightarrow t_2. \tag{43}$$

Therefore, $T(\Omega)$ is equicontinuous.

By $(\mathbf{H}_1)$ and (30), for any $\varepsilon>0$, there exists $\kappa>0$ such that

$$\int_{\kappa}^{\infty} f(s,u_n(s),v_n(s))\mathrm{d}s_1<\varepsilon. \tag{44}$$

Due to $\lim_{t\longrightarrow+\infty}(t^{\alpha_1-1}/1+t^{\alpha_1-1})=1$, there exists sufficiently large $N_1>0$ such that, for any t_1, $t_2>N_1$, we have

$$\left|\frac{t_2^{\alpha_1-1}}{1+t_2^{\alpha_1-1}}-\frac{t_1^{\alpha_1-1}}{1+t_1^{\alpha_1-1}}\right|<\varepsilon. \tag{45}$$

Also because of $\lim_{t\longrightarrow+\infty}((t-\kappa)^{\alpha_1-1}/1+t^{\alpha_1-1})=1$, there exists sufficiently large $N_2>\kappa$ such that, for any t_1, $t_2>N_2$ and $0\le s\le\kappa$, we have

$$\left|\frac{(t_2-s)^{\alpha_1-1}}{1+t_2^{\alpha_1-1}}-\frac{(t_1-s)^{\alpha_1-1}}{1+t_1^{\alpha_1-1}}\right|<\varepsilon. \tag{46}$$

Choose $N>\max\{N_1, N_2\}$; for any t_1, $t_2>N$, we get

$$\begin{aligned}
&\left|\frac{T_1(u,v)(t_2)}{1+t_2^{\alpha_1-1}}-\frac{T_1(u,v)(t_1)}{1+t_1^{\alpha_1-1}}\right| \\
&\le\int_0^{\infty}\left|\frac{G_{11}(t_2,s)}{1+t_2^{\alpha_1-1}}-\frac{G_{11}(t_1,s)}{1+t_1^{\alpha_1-1}}\right|f_1(s,u(s),v(s))\mathrm{d}s \\
&\quad+\left|\frac{t_2^{\alpha_1-1}}{1+t_2^{\alpha_1-1}}-\frac{t_1^{\alpha_1-1}}{1+t_1^{\alpha_1-1}}\right|\int_0^{\infty}\frac{\mu_1\int_0^{\infty}a_1(t)G_{11}(t,s)\mathrm{d}A_1(t)}{\Gamma(\alpha_1)-\mu_1\int_0^{\infty}a_1(t)t^{\alpha_1-1}\mathrm{d}A_1(t)}f_1(s,u(s),v(s))\mathrm{d}s.
\end{aligned} \tag{47}$$

In (47),

$$
\begin{aligned}
&\int_0^{\infty}\left|\frac{G_{11}(t_2,s)}{1+t_2^{\alpha_1-1}}-\frac{G_{11}(t_1,s)}{1+t_1^{\alpha_1-1}}\right| f_1(s,u(s),v(s))\mathrm{d}s \\
&\quad\le \int_0^{t_1}\left|\frac{G_{11}(t_2,s)}{1+t_2^{\alpha_1-1}}-\frac{G_{11}(t_1,s)}{1+t_1^{\alpha_1-1}}\right| f_1(s,u(s),v(s))\mathrm{d}s \\
&\qquad+\int_{t_1}^{t_2}\left|\frac{G_{11}(t_2,s)}{1+t_2^{\alpha_1-1}}-\frac{G_{11}(t_1,s)}{1+t_1^{\alpha_1-1}}\right| f_1(s,u(s),v(s))\mathrm{d}s \\
&\qquad+\int_{t_2}^{\infty}\left|\frac{G_{11}(t_2,s)}{1+t_2^{\alpha_1-1}}-\frac{G_{11}(t_1,s)}{1+t_1^{\alpha_1-1}}\right| f_1(s,u(s),v(s))\mathrm{d}s \\
&\quad\le \frac{1}{\Gamma(\alpha_1)}\int_0^{t_1}\left(\left|\frac{t_2^{\alpha_1-1}}{1+t_2^{\alpha_1-1}}-\frac{t_1^{\alpha_1-1}}{1+t_1^{\alpha_1-1}}\right|+\left|\frac{(t_2-s)^{\alpha_1-1}}{1+t_2^{\alpha_1-1}}-\frac{(t_1-s)^{\alpha_1-1}}{1+t_1^{\alpha_1-1}}\right|\right) f_1(s,u(s),v(s))\mathrm{d}s \\
&\qquad+\frac{1}{\Gamma(\alpha_1)}\int_{t_1}^{t_2}\left(\left|\frac{t_2^{\alpha_1-1}}{1+t_2^{\alpha_1-1}}-\frac{t_1^{\alpha_1-1}}{1+t_1^{\alpha_1-1}}\right|+\left|\frac{(t_2-s)^{\alpha_1-1}}{1+t_2^{\alpha_1-1}}\right|\right) f_1(s,u(s),v(s))\mathrm{d}s \\
&\qquad+\frac{1}{\Gamma(\alpha_1)}\int_{t_2}^{\infty}\left|\frac{t_2^{\alpha_1-1}}{1+t_2^{\alpha_1-1}}-\frac{t_1^{\alpha_1-1}}{1+t_1^{\alpha_1-1}}\right| f_1(s,u(s),v(s))\mathrm{d}s \\
&\quad\le \frac{1}{\Gamma(\alpha_1)}\int_0^{\kappa}\left(\left|\frac{t_2^{\alpha_1-1}}{1+t_2^{\alpha_1-1}}-\frac{t_1^{\alpha_1-1}}{1+t_1^{\alpha_1-1}}\right|+\left|\frac{(t_2-s)^{\alpha_1-1}}{1+t_2^{\alpha_1-1}}-\frac{(t_1-s)^{\alpha_1-1}}{1+t_1^{\alpha_1-1}}\right|\right) f_1(s,u(s),v(s))\mathrm{d}s \\
&\qquad+\frac{1}{\Gamma(\alpha_1)}\int_{\kappa}^{t_1}\left(\left|\frac{t_2^{\alpha_1-1}}{1+t_2^{\alpha_1-1}}-\frac{t_1^{\alpha_1-1}}{1+t_1^{\alpha_1-1}}\right|+\left|\frac{(t_2-s)^{\alpha_1-1}}{1+t_2^{\alpha_1-1}}-\frac{(t_1-s)^{\alpha_1-1}}{1+t_1^{\alpha_1-1}}\right|\right) f_1(s,u(s),v(s))\mathrm{d}s \\
&\qquad+\frac{1}{\Gamma(\alpha_1)}\int_{\kappa}^{\infty}\left(\left|\frac{t_2^{\alpha_1-1}}{1+t_2^{\alpha_1-1}}-\frac{t_1^{\alpha_1-1}}{1+t_1^{\alpha_1-1}}\right|+1\right) f_1(s,u(s),v(s))\mathrm{d}s \\
&\qquad+\frac{1}{\Gamma(\alpha_1)}\int_{\kappa}^{\infty}\left|\frac{t_2^{\alpha_1-1}}{1+t_2^{\alpha_1-1}}-\frac{t_1^{\alpha_1-1}}{1+t_1^{\alpha_1-1}}\right| f_1(s,u(s),v(s))\mathrm{d}s \\
&\quad\le \frac{\max_{s\in[0,\kappa],\,(u,v)\in\Omega}\left|f_1(s,u(s),v(s))\right|}{\Gamma(\alpha_1)}\int_0^{\kappa}\left(\left|\frac{t_2^{\alpha_1-1}}{1+t_2^{\alpha_1-1}}-\frac{t_1^{\alpha_1-1}}{1+t_1^{\alpha_1-1}}\right|+\left|\frac{(t_2-s)^{\alpha_1-1}}{1+t_2^{\alpha_1-1}}-\frac{(t_1-s)^{\alpha_1-1}}{1+t_1^{\alpha_1-1}}\right|\right)\mathrm{d}s \\
&\qquad+\frac{1}{\Gamma(\alpha_1)}\int_{\kappa}^{\infty}\left(\left|\frac{t_2^{\alpha_1-1}}{1+t_2^{\alpha_1-1}}-\frac{t_1^{\alpha_1-1}}{1+t_1^{\alpha_1-1}}\right|+\left|\frac{(t_2-s)^{\alpha_1-1}}{1+t_2^{\alpha_1-1}}-\frac{(t_1-s)^{\alpha_1-1}}{1+t_1^{\alpha_1-1}}\right|\right) f_1(s,u(s),v(s))\mathrm{d}s \\
&\qquad+\frac{1}{\Gamma(\alpha_1)}\int_{\kappa}^{\infty}\left(\left|\frac{t_2^{\alpha_1-1}}{1+t_2^{\alpha_1-1}}-\frac{t_1^{\alpha_1-1}}{1+t_1^{\alpha_1-1}}\right|+1\right) f_1(s,u(s),v(s))\mathrm{d}s \\
&\qquad+\frac{1}{\Gamma(\alpha_1)}\int_{\kappa}^{\infty}\left|\frac{t_2^{\alpha_1-1}}{1+t_2^{\alpha_1-1}}-\frac{t_1^{\alpha_1-1}}{1+t_1^{\alpha_1-1}}\right| f_1(s,u(s),v(s))\mathrm{d}s \\
&\quad\le \frac{\max_{s\in[0,\kappa],\,(u,v)\in\Omega}\left|f_1(s,u(s),v(s))\right|}{\Gamma(\alpha_1)}\int_0^{\kappa}\left(\left|\frac{t_2^{\alpha_1-1}}{1+t_2^{\alpha_1-1}}-\frac{t_1^{\alpha_1-1}}{1+t_1^{\alpha_1-1}}\right|+\left|\frac{(t_2-s)^{\alpha_1-1}}{1+t_2^{\alpha_1-1}}-\frac{(t_1-s)^{\alpha_1-1}}{1+t_1^{\alpha_1-1}}\right|\right)\mathrm{d}s \\
&\qquad+\frac{9}{\Gamma(\alpha_1)}\int_{\kappa}^{\infty} f_1(s,u(s),v(s))\mathrm{d}s \\
&\quad\le \frac{\max_{s\in[0,\kappa],\,(u,v)\in\Omega}\left|f_1(s,u(s),v(s))\right|}{\Gamma(\alpha_1)}2\kappa\varepsilon+\frac{9}{\Gamma(\alpha_1)}\varepsilon.
\end{aligned}
\tag{48}
$$

By (47) and (48), we have

$$\left|\frac{T_1(u,v)(t_2)}{1+t_2^{\alpha_1-1}}-\frac{T_1(u,v)(t_1)}{1+t_1^{\alpha_1-1}}\right| \le \frac{\max_{s\in[0,\kappa],(u,v)\in\Omega}\left|f_1(s,u(s),v(s))\right|}{\Gamma(\alpha_1)}2\kappa\varepsilon+\frac{9}{\Gamma(\alpha_1)}\varepsilon + \varepsilon\int_0^{\infty}\frac{\mu_1\int_0^{\infty}a_1(t)G_{11}(t,s)\mathrm{d}A_1(t)}{\Gamma(\alpha_1)-\mu_1\int_0^{\infty}a_1(t)t^{\alpha_1-1}\mathrm{d}A_1(t)}f_1(s,u(s),v(s))\mathrm{d}s. \tag{49}$$

So, $T_1(u,v)$ is equicontinuous on $+\infty$, proof similar to (49), and we know $T_2(u,v)$ is equicontinuous on $+\infty$; thus, $T(u,v)$ is equicontinuous on $+\infty$. It follows from Lemma 4 that $T:X\longrightarrow X$ is relatively compact. Therefore, $T:X\longrightarrow X$ is completely continuous. The proof is completed. □

Theorem 2. *Assume that (**H**$_1$) holds; then, system (1) has at least one positive solution if* $\varpi(g_1^*+g_2^*+h_1^*+h_2^*)<1$.

Proof. Let

$$r\ge\frac{\varpi(p_1^*+p_2^*)}{1-\varpi(g_1^*+g_2^*+h_1^*+h_2^*)}, \quad K=\{(u,v)\in X,\|(u,v)\|\le r\}. \tag{50}$$

Now, we illustrate $T(K)\subset K$ for any $(u,v)\in K$ and $t\in[0,+\infty)$; by Lemma 3 and Remark 1, we have

$$\begin{aligned}\left|\frac{T_1(u,v)(t)}{1+t^{\alpha_1-1}}\right| &= \left|\int_0^{\infty}\frac{G_1(t,s)}{1+t^{\alpha_1-1}}f_1(s,u(s),v(s))\mathrm{d}s\right| \\ &\le \varpi\int_0^{\infty}\left|f_1(s,u(s),v(s))\right|\mathrm{d}s \\ &\le \varpi\left(p_1^*+(g_1^*+h_1^*)\|(u,v)\|\right) \\ &\le \varpi\left(p_1^*+(g_1^*+h_1^*)r\right)\le r.\end{aligned} \tag{51}$$

Similarly, we have

$$\left|\frac{T_2(u,v)(t)}{1+t^{\alpha_2-1}}\right|\le\varpi\left(p_2^*+(g_2^*+h_2^*)r\right)\le r. \tag{52}$$

By (51) and (52),

$$\|T(u,v)\|=\|T_1(u,v)\|+\|T_2(u,v)\|\le\varpi\left[\left(p_1^*+(g_1^*+h_1^*)r\right)+\left(p_2^*+(g_2^*+h_2^*)r\right)\right]\le r. \tag{53}$$

Therefore, $T(K)\subset K$. By Theorem 1, we know that $T: K\longrightarrow K$ is completely continuous. So, by the Schauder fixed-point theorem, system (1) has at least one positive solution. The proof is completed. □

Theorem 3. *Assume that (**H**$_2$) holds; then, system (1) has a unique positive solution if* $\varpi(g_1^*+g_2^*+h_1^*+h_2^*)<1$.

Proof. From (**H**$_2$), we know

$$\left|f_i(t,u,v)\right|\le g_i(t)|u|+h_i(t)|v|+\left|f_i(t,0,0)\right|. \tag{54}$$

So, for any $(u,v)\in X$, we have

$$\begin{aligned}&\int_0^{\infty}\left|f_i(t,u(t),v(t))\right|\mathrm{d}t \\ &\le\int_0^{\infty}\left(\left(g_i(t)|u(t)|+h_i(t)|v(t)|+\left|f_i(t,0,0)\right|\right)\right)\mathrm{d}t \\ &=\int_0^{\infty}\left(\left(1+t^{\alpha_1-1}\right)g_i(t)\frac{|u(t)|}{1+t^{\alpha_1-1}}+\left(1+t^{\alpha_2-1}\right)h_i(t)\frac{|v(t)|}{1+t^{\alpha_2-1}}+\left|f_i(t,0,0)\right|\right)\mathrm{d}t \\ &\le\int_0^{\infty}\left(\left(1+t^{\alpha_1-1}\right)g_i(t)\|u(t)\|+\left(1+t^{\alpha_2-1}\right)h_i(t)\|v(t)\|+\left|f_i(t,0,0)\right|\right)\mathrm{d}t \\ &=\int_0^{\infty}\left(1+t^{\alpha_1-1}\right)g_i(t)\mathrm{d}t\|u(t)\|+\int_0^{\infty}\left(1+t^{\alpha_2-1}\right)h_i(t)\mathrm{d}t\|v(t)\|+\int_0^{\infty}\left|f_i(t,0,0)\right|\mathrm{d}t \\ &=g_i^*\|u\|+h_i^*\|v\|+\int_0^{\infty}\left|f_i(t,0,0)\right|\mathrm{d}t\le\left(g_i^*+h_i^*\right)\|(u,v)\|+\int_0^{\infty}\left|f_i(t,0,0)\right|\mathrm{d}t.\end{aligned} \tag{55}$$

For any $(u_1, v_1), (u_2, v_2) \in X$ and $t \in [0, +\infty)$, by Lemma 3, we have

$$\left|\frac{T_1(u_1,v_1)(t) - T_1(u_2,v_2)(t)}{1+t^{\alpha_1-1}}\right|$$
$$= \left|\int_0^\infty \frac{G_1(t,s)}{1+t^{\alpha_1-1}} \left|f_1(s,u_1(s),v_1(s)) - f_1(s,u_2(s),v_2(s))\right| ds\right|$$
$$\le \varpi \int_0^\infty \left(g_1(s)\left|(u_1(s)-u_2(s))\right| + h_1(s)\left|(v_1(s)-v_2(s))\right|\right) ds$$
$$\le \varpi \int_0^\infty \left(\left(1+s^{\alpha_1-1}\right) g_1(s) \frac{\left|(u_1(s)-u_2(s))\right|}{1+s^{\alpha_1-1}} + \left(1+s^{\alpha_1-1}\right) h_1 \left(s\right) \frac{\left|(v_1(s)-v_2(s))\right|}{1+s^{\alpha_1-1}}\right) ds$$
$$\le \varpi\left(g_1^* \|u_1-u_2\| + h_1^* \|v_1-v_2\|\right). \tag{56}$$

By the similar proof, we have

$$\left|\frac{T_2(u_1,v_1)(t) - T_2(u_2,v_2)(t)}{1+t^{\alpha_2-1}}\right| \le \varpi\left(g_2^* \|u_1-u_2\| + h_2^* \|v_1-v_2\|\right). \tag{57}$$

Now, inequalities (56) and (57) can show that

$$\|T(u,v)\| \le \varpi\left(g_1^* + g_2^* + h_1^* + h_2^*\right)\left(\|u_1-u_2\| + \|v_1-v_2\|\right). \tag{58}$$

Thus, by the Banach contraction mapping theorem that T has a unique fixed point in X, system (1) has a unique positive solution. The proof is completed. □

4. An Example

Consider the following fractional differential system:

$$\begin{cases} D_{0+}^{(5/2)} u(t) + f_1(t,u(t),v(t)) = 0, \\ D_{0+}^{(7/2)} v(t) + f_2(t,u(t),v(t)) = 0, \quad 0 < t < +\infty, \\ u(0) = u'(0) = 0, \\ D_{0+}^{(3/2)} u(+\infty) = \dfrac{1}{5}\displaystyle\int_0^\infty e^{-t} u(t) dt, \\ v(0) = v'(0) = v''(0) = 0, \\ D_{0+}^{(5/2)} v(+\infty) = \dfrac{1}{4}\displaystyle\int_0^\infty e^{-t} v(t) dt, \end{cases} \tag{59}$$

where $\alpha_1 = (5/2)$, $\alpha_2 = (7/2)$, $\mu_1 = (1/5)$, $\mu_2 = (1/4)$, $A_1(t) = A_2(t) = t$, $a_1(t) = e^{-t} = a_2(t) = e^{-t}$. Then, we have

$$\begin{gathered} \int_0^\infty a_1(t) dA_1(t) = \int_0^\infty a_2(t) dA_2(t) = \int_0^\infty e^{-t} dt = 1 < +\infty, \\ \int_0^\infty a_1(t) t^{\alpha_1-1} dA_1(t) = \int_0^\infty e^{-t} t^{(3/2)} dt = 2.3562 < +\infty, \\ \int_0^\infty a_2(t) t^{\alpha_2-1} dA_2(t) = \int_0^\infty e^{-t} t^{(5/2)} dt = 3.3233 < +\infty, \\ \varpi = 2.0929. \end{gathered} \tag{60}$$

Take

$$\begin{aligned} f_1(t,u(t),v(t)) &= \frac{1}{1+e^t} + \frac{u(t)|\cos t|}{\left(1+t^{(3/2)}\right)e^{7t}} + \frac{v(t)|\sin t|}{7\left(1+t^{(5/2)}\right)e^{3t}}, \\ f_2(t,u(t),v(t)) &= \frac{1}{2(1+e^t)} + \frac{u(t)}{\left(1+t^{(3/2)}\right)e^{6t}} + \frac{v(t)}{3\left(1+t^{(5/2)}\right)e^{4t}}. \end{aligned} \tag{61}$$

Let

$$\begin{aligned} g_1(t) &= \frac{1}{\left(1+t^{(3/2)}\right)e^{7t}}, \\ g_2(t) &= \frac{1}{\left(1+t^{(3/2)}\right)e^{6t}}, \\ h_1(t) &= \frac{1}{7\left(1+t^{(5/2)}\right)e^{3t}}, \\ h_2(t) &= \frac{1}{3\left(1+t^{(5/2)}\right)e^{4t}}. \end{aligned} \tag{62}$$

Through calculation, we get

$$\begin{aligned} g_1^* &= \int_0^\infty \left(1+t^{\alpha_1-1}\right) g_1(t) dt = \int_0^\infty e^{7t} dt = \frac{1}{7}, \\ g_2^* &= \int_0^\infty \left(1+t^{\alpha_1-1}\right) g_2(t) dt = \int_0^\infty e^{6t} dt = \frac{1}{6}, \\ h_1^* &= \int_0^\infty \left(1+t^{\alpha_2-1}\right) h_1(t) dt = \frac{1}{7}\int_0^\infty e^{3t} dt = \frac{1}{21}, \\ h_2^* &= \int_0^\infty \left(1+t^{\alpha_2-1}\right) h_2(t) dt = \frac{1}{3}\int_0^\infty e^{4t} dt = \frac{1}{12}, \\ &\varpi\left(g_1^* + g_2^* + h_1^* + h_2^*\right) = 2.0929\left(\frac{1}{7} + \frac{1}{6} + \frac{1}{21} + \frac{1}{12}\right) \\ &= 0.9219 < 1. \end{aligned} \tag{63}$$

Then, by Theorem 2, system (59) has at least one positive solution.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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