

Research Article

Pointwise Estimates of Solutions for the Viscous Cahn-Hilliard Equation with Inertial Term

Nianying Li, Li Yin , and Honglian You

College of Science, Binzhou University, Binzhou City, Shandong Province 256603, China

Correspondence should be addressed to Li Yin; yinli_79@163.com

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In this paper, we study the pointwise estimates of solutions to the viscous Cahn-Hilliard equation with the inertial term in multidimensions. We use Green's function method. Our approach is based on a detailed analysis on the Green's function of the linear system. And we get the solution's L^p convergence rate.

1. Introduction

In this paper, we study the pointwise estimates of the solution $\rho(x, t)$ to the Cauchy problem:

$$\begin{cases} \eta\rho_{tt} + \rho_t + \Delta^2\rho - k\Delta\rho_t - \Delta f(\rho) = 0, & (x, t) \in \mathbb{R}^n \times (0, \infty), \\ \rho(x, t)|_{t=0} = \rho_0(x), \rho_t(x, t)|_{t=0} = \rho_1(x), \end{cases} \quad (1)$$

where $n \geq 4$. $f(\rho)$ is the intrinsic chemical potential which is smooth in the small neighborhood of the origin, and $f(\rho) = O(\rho^{1+\theta})$ when $|\rho| \leq 1$ and θ is a positive integer. When $\eta = 0$, Eq. (1) is the well-known Cahn-Hilliard equation. When $\eta \neq 0$, $\eta\rho_{tt}$ is the inertial term. When $k \neq 0$, $-k\Delta\rho_t$ is the viscous term. Without loss of generality, we let $\eta = 1$ and $k = 1$.

The classical Cahn-Hilliard equation was proposed in the sixties by Cahn and Hilliard which describes the phase separation in materials science, and it has been widely studied. The reader may see references ([1–6]) and the related references therein. The Cahn-Hilliard equations with inertial term model nonequilibrium decompositions caused by deep supercooling in certain glasses. As we know, the well-known Cahn-Hilliard equation is a parabolic equation, but the Cahn-Hilliard equation with the inertial term is a hyper-

bolic equation with relaxation which brings many mathematical difficulties to study. For which, without smallness assumption on initial data, [7] got the global existence of the classical solution. [8] obtained the global existence and the optimal decay rate of the classical solution by the Fourier splitting method. Wang and Wu [9] obtained the global existence and optimal decay rate of the classical solution by long wave-short wave method. Li and Mi [10] got the pointwise estimates and the L^p ($1 \leq p \leq \infty$) convergence rate of the solution by Green's function method. Some other works on the Cahn-Hilliard equation with the inertial term can be seen in [11–13].

For viscous Cahn-Hilliard equation, [14] discussed the large time behavior of solutions when the dimension $n \leq 5$. For the viscous Cahn-Hilliard equation with the inertial term, it describes the early stages of spinodal decomposition in certain glasses (see [15–16]). And for which, [17] established the existence of families of exponential attractors and inertial manifolds; [18] studied the long time dynamic of the system in three-dimensional. In this paper, we are interested in the viscous Cahn-Hilliard equation with the inertial term. Under the smallness assumption on initial data, based on the detailed analysis of the Green's function, we get the pointwise estimates of solutions. From the representation of the symbol value to the Green's function for the linear problem of Eq. (1), we also find that the decay rate mainly depends on the lower-frequency part, i.e., the long wave part.

It is shown that the solution's decay rate is the same as [10]. Our study bases on Section 4 in [9].

To the best of our knowledge, this is the first time to obtain the pointwise estimates of the solution to Eq. (1).

Throughout this paper, C denotes the generic positive constants. $W^{m,p} = W^{m,p}(\mathbb{R}^n)$ ($m \in \mathbb{Z}_+, p \in [1, \infty]$) denote the usual Lebesgue space with norms $\|\cdot\|_{L^p}$ and the usual Sobolev space with its norm

$$\|f\|_{W^{m,p}} = \sum_{|\alpha|=0}^m \|\partial_x^\alpha f\|_{L^p}. \quad (2)$$

In particular, we use $W^{m,2} = H^m$.

The main result can be stated as following Theorem 1:

Theorem 1. *If $\|\rho_0\|_{H^{s+1} \cap W^{1,1}} + \|\rho_1\|_{H^s \cap L^1} \leq \varepsilon$, $s \geq \max\{[n/2] + 5, 2n\}$, and for any multi-index β , $|\beta| < s - (n/2)$, there exists a constant $d > (n/2)$, such that*

$$\left| D_x^\beta \rho_0(x) \right| + \left| D_x^{\beta+2} \rho_0(x) \right| + \left| D_x^\beta \rho_1(x) \right| \leq CE(1 + |x|^4)^{-d}, \quad (3)$$

then for $|\beta| < n$, the solution to Eq. (1) has the following estimates:

$$\left| D_x^\beta \rho(x, t) \right| \leq C(1+t)^{-(n+|\beta|/4)} B_d(|x|^2, t), \quad (4)$$

where ε and E is sufficiently small positive constants, $[m] = \max\{a \mid a \in \mathbb{Z}, a \leq m\}$, $B_d(|x|^2, t) = (1 + (|x|^2/1 + t))^{-d}$.

Corollary 2. *Under the assumptions of Theorem 1, for $p \geq 1$, $|\beta| < n$, we have that*

$$\left\| D_x^\beta \rho(x, t) \right\|_{L^p} \leq C(1+t)^{-(n/4)(1-(1/p)) - |\beta|/4}. \quad (5)$$

Remark 3. We get the same decay rate of the solution as [10].

Remark 4. Our study bases on [9, 10], where the spacial dimension $n \geq 4$. Then in this paper, we have the same assumptions for the spacial dimension.

2. The Green Function

We first study the Green's function to Eq. (1) which satisfies

$$\begin{cases} G_{tt} + G_t + \Delta^2 G - \Delta G_t = 0, & (x, t) \in \mathbb{R}^n \times (0, \infty), \\ G(x, t)|_{t=0} = 0, G_t(x, t)|_{t=0} = \delta(x). \end{cases} \quad (6)$$

We apply the Fourier transform $\hat{f}(\xi, t) = \int_{\mathbb{R}^n} f(x, t) e^{-ix \cdot \xi} dx$ and the inverse Fourier transform $(\mathcal{F}^{-1} \hat{f})(x, t) = (2\pi)^{-2n} \int_{\mathbb{R}^n} \hat{f}(\xi, t) e^{ix \cdot \xi} d\xi$.

By applying the Fourier transform with respect to the variable x , we get

$$\begin{cases} \left[\partial_t^2 + \partial_t (1 + |\xi|^2) + |\xi|^4 \right] \widehat{G}(\xi, t) = 0, \\ \widehat{G}(\xi, t)|_{t=0} = 0, \widehat{G}_t(\xi, t)|_{t=0} = 1, \end{cases} \quad (7)$$

the symbol of which is

$$\nu^2 + \nu(1 + |\xi|^2) + |\xi|^4 = 0. \quad (8)$$

Here, ν and ξ correspond to ∂_t and $D_{x_j} = (1/i)\partial_{x_j}$, $j = 1, 2, \dots, n$. By a direct calculation, we have

$$\nu = \nu_{\pm}(\xi) = \frac{-(1 + |\xi|^2) \pm \sqrt{(1 + |\xi|^2)^2 - 4|\xi|^4}}{2}, \quad (9)$$

$$\widehat{G}(\xi, t) = \left((1 + |\xi|^2)^2 - 4|\xi|^4 \right)^{-1/2} \left(e^{\nu_+(\xi)t} - e^{\nu_-(\xi)t} \right). \quad (10)$$

By Duhamel's principle, we get the solution of the non-linear problem (1)

$$\begin{aligned} \rho(x, t) &= G(t) * (\rho_0 + \rho_1 - \Delta \rho_0) + \partial_t G * \rho_0 \\ &\quad + \int_0^t G(t-\tau) * \Delta f(\rho)(\tau) d\tau. \end{aligned} \quad (11)$$

Now we decompose $\widehat{G}(\xi, t) = \widehat{G}^+(\xi, t) + \widehat{G}^-(\xi, t)$, where

$$\widehat{G}^{\pm}(\xi, t) = \pm \nu_0(\xi) e^{\nu_{\pm}(\xi)t}, \nu_0(\xi) = \left((1 + |\xi|^2)^2 - 4|\xi|^4 \right)^{-1/2}. \quad (12)$$

Let

$$\Gamma_1(\xi) = \begin{cases} 1, & |\xi| \leq s, \\ 0, & |\xi| > 2s \end{cases}, \Gamma_3(\xi) = \begin{cases} 1, & |\xi| > R+1, \\ 0, & |\xi| \leq R \end{cases} \quad (13)$$

and $\Gamma_2(\xi) = 1 - \Gamma_1(\xi) - \Gamma_3(\xi)$ be smooth cut-off functions, where $s, R > 0$, $2s < R$.

Set

$$\widehat{G}_i^{\pm}(\xi, t) = \Gamma_i(\xi) \widehat{G}^{\pm}(\xi, t), i = 1, 2, 3. \quad (14)$$

Now we estimate the Green's function $G(x, t)$.

2.1. Lower-Frequency Part. First, we give the following Lemma.

Lemma 5. *If $\widehat{y}(\xi, t)$ has compact support in the variable ξ , N is a positive integer, and there is a constant $u > 0$, such that*

$$\left| D_{\xi}^{\beta}(\xi^{\alpha} \widehat{y}(\xi, t)) \right| \leq C \left[|\xi|^{(|\alpha|-|\beta|)_+} + |\xi|^{|\alpha|} t^{|\beta|/4} \right] \left(1 + t|\xi|^4 \right)^k e^{-u|\xi|^4 t}, \quad (15)$$

for any multi-indexes α, β with $|\beta| \leq 4N$, then

$$|D_x^{\alpha} y(x, t)| \leq Ct^{-n+|\alpha|/4} B_N(|x|^2, t), \quad (16)$$

where k is any fixed positive number, $(s)_+ = \max\{0, s\}$.

The proof of Lemma (9) can be seen in [10].

For $|\xi|$ is sufficiently small, from (9) and the Taylor expansion, we have

$$\begin{aligned} \nu_+(\xi) &= -|\xi|^4 + O(|\xi|^6), \nu_-(\xi) = -1 - |\xi|^2 + |\xi|^4 \\ &\quad + O(|\xi|^6), \nu_0(\xi) = 1 + O(|\xi|^2), \\ e^{\nu_+(\xi)t} &= e^{-|\xi|^4 t} \left(1 + O(|\xi|^8 t) \right), e^{\nu_-(\xi)t} \\ &= e^{-t} e^{-|\xi|^2 t} e^{|\xi|^4 t} \left(1 + O(|\xi|^6 t) \right). \end{aligned} \quad (17)$$

Then

$$\begin{aligned} \widehat{G}^+(\xi, t) &= \nu_0(\xi) e^{\nu_+(\xi)t} = \left(1 + O(|\xi|^2) \right) \\ &\quad \cdot \left(1 + O(|\xi|^6 t) \right) e^{-|\xi|^4 t} = \left[1 + O(|\xi|^2) \right. \\ &\quad \left. + O(|\xi|^6 t) \right] e^{-|\xi|^4 t}. \end{aligned} \quad (18)$$

Since $G^{\pm}(\xi, t)$ are smooth functions to variable ξ near $|\xi| = 0$, we obtain that when $|\beta| \leq 4N$,

$$\begin{aligned} \left| D_{\xi}^{\beta}(\xi^{\alpha} \widehat{G}_1^+(\xi, t)) \right| &= \left| \sum_{\mu_1 + \mu_2 = \beta} \frac{\beta!}{\mu_1! \mu_2!} \left(D_{\xi}^{\mu_1} \chi_1(\xi) \right) \right. \\ &\quad \cdot \left(D_{\xi}^{\mu_2}(\xi^{\alpha} \widehat{G}^+(\xi, t)) \right) \left| \leq C \left[|\xi|^{(|\alpha|-|\mu_1|)} \right. \right. \\ &\quad \left. \left. + |\xi|^{|\alpha|} t^{|\mu_2|/4} \right] \left(1 + t|\xi|^4 \right)^{(3/4)|\mu_2|} e^{-|\xi|^4 t} \right. \\ &\leq C \left[|\xi|^{(|\alpha|-|\beta|)_+} + |\xi|^{|\alpha|} t^{|\beta|/4} \right] \\ &\quad \cdot \left(1 + t|\xi|^4 \right)^{(3/4)|\beta|} e^{-|\xi|^4 t}. \end{aligned} \quad (19)$$

By Lemma (9), we get

$$|D_x^{\alpha} G_1^+(x, t)| \leq Ct^{-(n+|\alpha|/4)} B_N(|x|^2, t). \quad (20)$$

For $\widehat{G}^-(\xi, t)$, we have

$$\begin{aligned} \widehat{G}^-(\xi, t) &= -\nu_0(\xi) e^{\lambda_-(\xi)t} = -\left(1 + O(|\xi|^2) \right) \\ &\quad \cdot \left(1 + O(|\xi|^6 t) \right) e^{-t} e^{-|\xi|^2 t} e^{|\xi|^4 t} \\ &= -\left[1 + O(|\xi|^2) + O(|\xi|^6 t) \right] e^{-t} e^{-|\xi|^2 t} e^{|\xi|^4 t}. \end{aligned} \quad (21)$$

Then, we have

$$\begin{aligned} \left| D_{\xi}^{\beta}(\xi^{\alpha} \widehat{G}_1^-(\xi, t)) \right| &= \left| \sum_{\mu_1 + \mu_2 + \mu_3 = \beta} \frac{\beta!}{\mu_1! \mu_2! \mu_3!} \left(D_{\xi}^{\mu_1} \chi_1(\xi) \right) \right. \\ &\quad \cdot \left(D_{\xi}^{\mu_2}(\xi^{\alpha}) \right) \left(D_{\xi}^{\mu_3}(\widehat{G}^-(\xi, t)) \right) \left| \right. \\ &\leq C \left[|\xi|^{(|\alpha|-|\beta|)_+} + |\xi|^{|\alpha|} t^{|\beta|/4} \right] \\ &\quad \cdot \left(1 + t|\xi|^2 \right)^{(3/4)|\beta|+1} e^{-(1/2-|\xi|^2+|\xi|^4)t} \\ &\leq C \left[|\xi|^{(|\alpha|-|\beta|)_+} + |\xi|^{|\alpha|} t^{|\beta|/4} \right] \\ &\quad \cdot \left(1 + t|\xi|^4 \right)^{(3/4)|\beta|} e^{-(1/2-|\xi|^4)t} e^{-(r/2)t}, \end{aligned} \quad (22)$$

where $r > 0$.

By Lemma (9), we get

$$\begin{aligned} |D_x^{\alpha} G_1^-(x, t)| &\leq Ce^{-(r/2)t} t^{-n+|\alpha|/2} B_N(|x|^2, t) \\ &\leq Ct^{-n+|\alpha|/4} B_N(|x|^2, t). \end{aligned} \quad (23)$$

From (20)–(23), we have the following proposition:

Proposition 6. *For sufficiently small s , we have*

$$|D_x^{\alpha} G_1(x, t)| \leq Ct^{-n+|\alpha|/4} B_N(|x|^2, t). \quad (24)$$

2.2. Middle-Frequency Part. We can get the following proposition.

Proposition 7. *For fixed s and R , there exist positive numbers m and C such that*

$$|D_x^{\alpha} G_2(x, t)| \leq Ce^{-mt} B_N(|x|^2, t). \quad (25)$$

The proof of Proposition 7 is similar to that of Proposition 12 in [10], so we omit it.

2.3. *Higher-Frequency Part.* For $|\xi|$ is large enough, we have

$$\begin{aligned} \sqrt{(1+|\xi|^2)^2-4|\xi|^4} &= |\xi|^2 \sqrt{2|\xi|^{-2}+|\xi|^{-4}-3} \\ &= \sqrt{3}i \left(|\xi|^2 - \frac{1}{3}|\xi|^{-2} + O(|\xi|^{-4}) \right), \\ v_0(\xi) &= \frac{1}{\sqrt{(1+|\xi|^2)^2-4|\xi|^4}} \\ &= |\xi|^{-2} \frac{1}{\sqrt{2|\xi|^{-2}+|\xi|^{-4}-3}} = \frac{\sqrt{3}}{3} i \left(|\xi|^{-2} + O(|\xi|^{-4}) \right). \end{aligned} \quad (26)$$

Then, we have

$$\begin{aligned} \widehat{G}_3^+(\xi, t) &= v_0(\xi) e^{v_+(\xi)t} = \frac{1}{\sqrt{3}} i \left(|\xi|^{-2} + O(|\xi|^{-4}) \right) \\ &\quad \cdot e^{[-(1/2)-(1/2)(1-\sqrt{3}i)|\xi|^2 - (\sqrt{3}i/3)|\xi|^{-2} + O(|\xi|^{-4})]t} \\ &= e^{[-(1/2)-(1/2)(1-\sqrt{3}i)|\xi|^2]t} \left[\sum_{j=0}^k B_j^+(t) |\xi|^{-2j} \right. \\ &\quad \left. + B_{k+1}^+(t) O(|\xi|^{-2(k+1)}) \right] \\ \widehat{G}_3^-(\xi, t) &= -v_0(\xi) e^{v_-(\xi)t} = -\frac{1}{\sqrt{3}} i \left(|\xi|^{-2} + O(|\xi|^{-4}) \right) \\ &\quad \cdot e^{[-(1/2)-(1/2)(1+\sqrt{3}i)|\xi|^2 + (\sqrt{3}i/3)|\xi|^{-2} + O(|\xi|^{-4})]t} \\ &= -e^{[-(1/2)-(1/2)(1+\sqrt{3}i)|\xi|^2]t} \left[\sum_{j=0}^k B_j^-(t) |\xi|^{-2j} \right. \\ &\quad \left. + B_{k+1}^-(t) O(|\xi|^{-2(k+1)}) \right], \end{aligned} \quad (27)$$

where $B_j^\pm(t)$ are polynomials in t with degree no more than j .

Let

$$\begin{aligned} \widehat{R}^+(\xi, t) &= \Gamma_3(\xi) e^{[-(1/2)-(1/2)(1-\sqrt{3}i)|\xi|^2]t} \sum_{j=0}^{|\alpha|+(n+1/2)} B_j^+(t) |\xi|^{-2j}, \\ \widehat{R}^-(\xi, t) &= -\Gamma_3(\xi) e^{[-(1/2)-(1/2)(1+\sqrt{3}i)|\xi|^2]t} \sum_{j=0}^{|\alpha|+(n+1/2)} B_j^-(t) |\xi|^{-2j}, \\ \widehat{R}(\xi, t) &= \widehat{R}^+(\xi, t) + \widehat{R}^-(\xi, t). \end{aligned} \quad (28)$$

Because

$$\left| x^{2\gamma} D_x^\alpha (G_3^\pm - R^\pm)(x, t) \right| \leq \int_{\mathbb{R}^n} \left| D_\xi^{2\gamma} \xi^\alpha \left(\widehat{G}_3^\pm - \widehat{R}^\pm \right) \right| d\xi \leq C e^{-bt}, \quad (29)$$

taking $|\gamma| = 0$ or $|\gamma| = 2N$, we get the following proposition:

Proposition 8. For R being sufficiently large, we have

$$\left| D_x^\alpha (G_3^\pm - R^\pm)(x, t) \right| \leq C e^{-mt} B_N(|x|^2, t), \quad (30)$$

where $m > 0$.

Combining Proposition 6–8, we obtain the following estimate of the Green's function:

Proposition 9. For any multi-index α , we have

$$\left| D_x^\alpha (G - R)(x, t) \right| \leq C(1+t)^{-n+|\alpha|/4} B_N(|x|^2, t). \quad (31)$$

3. The Proof of Theorem 1

In this section, we shall give the pointwise estimates of the solution to the problem (1). From (3), we have

$$\rho(x, t) = \Phi_1 - \Phi_2 + \Phi_3. \quad (32)$$

where

$$\begin{aligned} \Phi_1 &= G(t) * (\rho_0 + \rho_1), \\ \Phi_2 &= G(t) * \Delta \rho_0, \\ \Phi_3 &= \partial_t G * \rho_0 + \int_0^t G(t-\tau) * \Delta f(\rho)(\tau) d\tau. \end{aligned} \quad (33)$$

For Φ_1 and Φ_3 , we have the following proposition:

Proposition 10.

$$\begin{aligned} \left| D_x^\alpha \Phi_1 \right| &\leq C E(1+t)^{-n+|\alpha|/4} B_d(|x|^2, t), \\ \left| D_x^\alpha \Phi_3 \right| &\leq C \Theta(t)^{\theta+1} (1+t)^{-n+|\alpha|/4} B_d(|x|^2, t), \end{aligned} \quad (34)$$

where $|\alpha| < n$ and $n \geq 4$, $\psi(x, t) = (1+t)^{n+|\alpha|/4} (B_d(|x|^2, t))^{-1}$,

$$\Theta(t) = \sup_{(x,\tau) \in \mathbb{R}^n \times [0,t], |\alpha| < n} \left| D_x^\alpha \rho(x, \tau) \right| \psi(x, \tau). \quad (35)$$

The proof of the above proposition is similar to proposition 4.1-4.2 in [10], so we omit it.

Next, we give a Lemma which is important to estimate Φ_2 and has been proved in [10].

Lemma 11. If $a, b > (n/2)$, $c = \min(a, b)$, we have

$$\int_{\mathbb{R}^n} \left(1 + \frac{|x-y|^4}{1+t} \right)^{-a} (1+|y|^4)^{-b} dy \leq C \left(1 + \frac{|x|^4}{1+t} \right)^{-c}. \quad (36)$$

We write

$$\Phi_2 = G * \Delta \rho_0 = (G - R) * \Delta \rho_0 + R * \Delta \rho_0 =: S_1 + S_2. \quad (37)$$

Making use of (31), Lemma 11 and (3), we have

$$\begin{aligned} |D_x^\alpha S_1| &\leq C(1+t)^{-n+|\alpha|/4} B_N(|x|^2, t) * \Delta \rho_0(x) \\ &\leq CE(1+t)^{-n+|\alpha|/4} \int_{\mathbb{R}^n} B_N(|x-y|^2, t) (1+|y|^4)^{-d} dy \\ &\leq CE(1+t)^{-n+|\alpha|/4} B_d(|x|^2, t), \end{aligned} \tag{38}$$

where $d > (n/2)$.

From the definition of $R(x, t)$, we have

$$|x^\gamma D_x^\alpha R(x, t)| \leq C \int_{\mathbb{R}^n} \left| D_\xi^\gamma \xi^\alpha \widehat{R}(\xi, t) \right| d\xi \leq Ce^{-mt}, \tag{39}$$

taking $|\gamma| = 0$ or $|\gamma| = 4N$, we obtain

$$|D_x^\alpha R(x, t)| \leq Ce^{-mt} B_N(|x|^2, t), \tag{40}$$

then, we get

$$\begin{aligned} |D_x^\alpha R(x, t) * \rho_{i-1}(x)| &\leq Ce^{-mt} B_N(|x|^2, t) * (1+|x|^4)^{-d} \\ &\leq Ce^{-mt} B_r(|x|^2, t), i = 1, 2. \end{aligned} \tag{41}$$

Thus, we obtain

$$|D_x^\alpha S_2| \leq Ce^{-mt} B_d(|x|^2, t). \tag{42}$$

Together with (38) and (42), we obtain the following result:

Proposition 12. *If $|\alpha| < n$, then*

$$|D_x^\alpha \Phi_2| \leq CE(1+t)^{-n+|\alpha|/4} B_d(|x|^2, t). \tag{43}$$

Combining Proposition 10–12, we have the following result:

Proposition 13.

$$|D_x^\alpha \rho(x, t)| \leq C \left(E + \Theta(t)^{\theta+1} \right) (1+t)^{-n+|\alpha|/4} B_d(|x|^2, t), \tag{44}$$

where $|\alpha| < n$.

By the smallness of E and the continuity of $\Theta(t)$, we have

$$|D_x^\alpha \rho(x, t)| \leq CE(1+t)^{-n+|\alpha|/4} B_d(|x|^2, t). \tag{45}$$

Thus, we complete the proof of Theorem 1.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the manuscript and read and approved the final manuscript.

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