Research Article

The Boundedness on Mixed Hardy Spaces

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The boundedness of operators on Hardy spaces is usually given by atomic decomposition. In this paper, we obtain the boundedness of singular integral operators in mixed Journé class on mixed Hardy spaces by a direct method.

1. Introduction

Different from the classical harmonic analysis, more general nonisotropic groups of dilations are considered on multiparameter harmonic analysis. In the past decades, many multiparameter theories of harmonic analysis have been developed, such as [1–15].

Discrete Littlewood–Paley theory is a useful tool to study multiparameter harmonic analysis, which is developed by Han, Lu, and Sawyer in [16] to study flag Hardy spaces. Recently, the authors in [17] introduced a new Hardy space as follows:

Let $\psi_0^{(1)}, \psi^{(1)} \in \mathcal{S}(\mathbb{R}^{n_1})$ with

$$\text{supp} \widehat{\psi_0^{(1)}} \subseteq \{\xi \in \mathbb{R}^{n_1} : |\xi| \leq 2\}; \quad \widehat{\psi_0^{(1)}}(\xi) = 1, \text{ if } |\xi| \leq 1, \quad (1)$$

$$\text{supp} \widehat{\psi^{(1)}} \subseteq \left\{ \xi \in \mathbb{R}^{n_1} : \frac{1}{2} \leq |\xi| \leq 2 \right\}, \quad (2)$$

$$\left| \widehat{\psi_0^{(1)}}(\xi) \right|^2 + \sum_{j=1}^{\infty} \left| \widehat{\psi^{(1)}}(2^{-j}\xi) \right|^2 = 1, \text{ for all } \xi \in \mathbb{R}^{n_1}. \quad (3)$$

Let $\psi^{(2)} \in \mathcal{S}(\mathbb{R}^{n_2})$ with

$$\text{supp} \widehat{\psi^{(2)}} \subseteq \left\{ \xi \in \mathbb{R}^{n_2} : \frac{1}{2} \leq |\xi| \leq 2 \right\}, \quad (4)$$

$$\sum_{j \in \mathbb{Z}} \left| \widehat{\psi^{(2)}}(2^{-j}\xi) \right|^2 = 1, \text{ for all } \frac{\xi \in \mathbb{R}^{n_2}}{\{0\}}. \quad (5)$$

Then, for $j, k \in \mathbb{Z}, j \geq 1$, set $\psi_j^{(1)}(x) = 2^{jn_1} \psi^{(1)}(2^j x)$, $\psi_k^{(2)}(x) = 2^{kn_2} \psi^{(2)}(2^k x)$, $\psi_{j,k}(x, y) = \psi_j^{(1)}(x) \psi_k^{(2)}(y)$, and $\psi_{0,k}(x, y) = \psi_0^{(1)}(x) \psi_k^{(2)}(y)$.

Denote $\mathcal{S}_0(\mathbb{R}^{n_1+n_2}) = \{f \in \mathcal{S}(\mathbb{R}^{n_1+n_2}) : \int_{\mathbb{R}^{n_2}} f(x) x^\alpha dx = 0, \forall |\alpha| \geq 0\}$, $\mathcal{S}_N(\mathbb{R}^n) = \{\phi \in \mathcal{S}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \phi(x) x^\alpha dx = 0, \forall |\alpha| \leq N\}$, and $\mathcal{S}_\infty(\mathbb{R}^n) = \{\phi \in \mathcal{S}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \phi(x) x^\alpha dx = 0, \forall |\alpha| > 0\}$. For $i = 1, 2$ and any $j \in \mathbb{Z}$, denote $\Pi_j^{n_i} = \{I : I \text{ are dyadic cubes in } \mathbb{R}^{n_i} \text{ with the side length } l(I) = 2^{-j}, \text{ and the left lower corners of } I \text{ are } x_I = 2^{-j} \ell, \ell \in \mathbb{Z}^{n_i}\}$, $\mathcal{D}_{n_i} = \cup_j \Pi_j^{n_i}$ and $\mathcal{D} = \mathcal{D}_{n_1} \times \mathcal{D}_{n_2}$.

Then, the following multiparameter mixed Hardy space is introduced in [17].

Definition 1. Let $0 < p < \infty$, $\psi_0^{(1)}, \psi^{(1)} \in \mathcal{S}(\mathbb{R}^{n_1})$, and $\psi^{(2)} \in \mathcal{S}(\mathbb{R}^{n_2})$ are functions satisfying conditions (1)–(5),

respectively. The multiparameter mixed Hardy space $H_{\text{mix}}^p = H_{\text{mix}}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ is defined by

$$H_{\text{mix}}^p = \left\{ f \in \mathcal{S}'_0(\mathbb{R}^{n_1+n_2}): \|f\|_{H_{\text{mix}}^p} < \infty \right\}, \quad (6)$$

where

$$\|f\|_{H_{\text{mix}}^p} = \left\| \left(\sum_{j \in \mathbb{N}, k \in \mathbb{Z}} \sum_{I \times J \in \Pi_j^{n_1} \times \Pi_k^{n_2}} |\psi_{j,k} * f(x_I, x_J)|^2 \chi_I(x) \chi_J(y) \right)^{1/2} \right\|_{L^p(\mathbb{R}^{n_1+n_2})}. \quad (7)$$

The symbol $\mathbb{N}$ means the set of natural numbers including zero. The multiparameter mixed Hardy space is well defined, that is, H_{mix}^p is independent of the choice of functions $\psi_0^{(1)}$, $\psi^{(1)}$, and $\psi^{(2)}$, which has been proved in [17]. The main purpose of this paper is to establish the boundedness of operators on H_{mix}^p .

Firstly, let us recall something about singular integrals of nonconvolution type. A locally integral function $\mathcal{K}(x, y)$ defined away from the diagonal $x = y$ in $\mathbb{R}^n \times \mathbb{R}^n$ is called a one-parameter nonconvolution Calderón–Zygmund kernel with a regularity exponent $\varepsilon > 0$ if $\mathcal{K}(x, y)$ satisfies, for some constant $C > 0$, the size condition

$$|\mathcal{K}(x, y)| \leq C \frac{1}{|x - y|^n}, \quad \text{for } x \neq y, \quad (8)$$

and the regularity conditions

$$|\mathcal{K}(x, y) - \mathcal{K}(x, y')| \leq C \frac{|y - y'|^\varepsilon}{|x - y|^{n+\varepsilon}}, \quad (9)$$

whenever $|y - y'| \leq (1/2)|x - y|$, and

$$|\mathcal{K}(x, y) - \mathcal{K}(x', y)| \leq C \frac{|x - x'|^\varepsilon}{|x - y|^{n+\varepsilon}}, \quad (10)$$

whenever $|x - x'| \leq (1/2)|x - y|$. The smallest such constant C is denoted by $\|\mathcal{K}\|_{\text{CZ}}$. We call an operator T one-parameter nonconvolution Calderón–Zygmund operator if T is a singular integral operator associated with a one-parameter nonconvolution Calderón–Zygmund kernel $\mathcal{K}(x, y)$ given by $\langle T(f), g \rangle = \int \mathcal{K}(x, y) f(y) g(x) dx dy$ for all $f, g \in \mathcal{S}(\mathbb{R}^n)$ with disjoint supports. Define $\|T\|_{\text{CZ}}$ by $\|T\|_{\text{CZ}} = \|\mathcal{K}\|_{\text{CZ}} + \|T\|_{L^2 \rightarrow L^2}$. Then, T is bounded on $L^p(\mathbb{R}^n)$ ($1 < p < +\infty$), if $\|T\|_{\text{CZ}} < +\infty$. However, by a classical result, H^p , $0 < p \leq 1$, boundedness of T implies that the operator must satisfy additional condition: $\int T(\varphi)(x) dx = 0$ for any $\varphi \in \mathcal{S}_\infty(\mathbb{R}^n)$, named $T^*(1) = 0$. On the contrary, to ensure the boundedness of T on local Hardy spaces h^p , $0 < p \leq 1$, additional condition $|\mathcal{K}(x, y)| \leq C(1)/|x - y|^{n+\delta}$, for some $\delta > 0$, should be included [18]. According to these, we extend this definition of singular integral operators in Journé class on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ suitable for our multiparameter mixed Hardy spaces, named mixed Journé class.

Definition 2. A singular integral operator T is said to be in mixed Journé class on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ with regularity exponent $\varepsilon \in (0, 1]$ and $\delta > 0$ if

$$T(f)(x_1, x_2) = \int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} \mathcal{K}(x_1, y_1, x_2, y_2) f(y_1, y_2) dy_1 dy_2, \quad (11)$$

where the kernel $\mathcal{K}$ satisfies the following conditions.

For each $x_1, y_1 \in \mathbb{R}^{n_1}$, set $\tilde{\mathcal{K}}^1(x_1, y_1)$ to be the singular integral operator acting on functions on $\mathbb{R}^{n_2}$ with the kernel

$$\tilde{\mathcal{K}}^1(x_1, y_1)(x_2, y_2) = \mathcal{K}(x_1, y_1, x_2, y_2), \quad (12)$$

and similarly,

$$\tilde{\mathcal{K}}^2(x_2, y_2)(x_1, y_1) = \mathcal{K}(x_1, y_1, x_2, y_2). \quad (13)$$

Then there exists a constant $C > 0$ such that

- (1) T is bounded on $L^2(\mathbb{R}^{n_1+n_2})$
- (2) $\|\tilde{\mathcal{K}}^1(x_1, y_1)\|_{\text{CZ}} \leq C \min\{(1/|x_1 - y_1|^{n_1}), (1/|x_1 - y_1|^{n_1+\delta})\}$, if $|x_1 - y_1| > 0$, $\|\tilde{\mathcal{K}}^1(x_1, y_1) - \tilde{\mathcal{K}}^1(x_1, y'_1)\|_{\text{CZ}} \leq C(|y_1 - y'_1|^\varepsilon/|x_1 - y_1|^{n_1+\varepsilon})$, if $|y_1 - y'_1| \leq (1/2)|x_1 - y_1|$, and $\|\tilde{\mathcal{K}}^1(x_1, y_1) - \tilde{\mathcal{K}}^1(x'_1, y_1)\|_{\text{CZ}} \leq C(|x_1 - x'_1|^\varepsilon/|x_1 - y_1|^{n_1+\varepsilon})$, if $|x_1 - x'_1| \leq (1/2)|x_1 - y_1|$
- (3) $\|\tilde{\mathcal{K}}^2(x_2, y_2)\|_{\text{CZ}} \leq C(1/|x_2 - y_2|^{n_2})$, if $|x_2 - y_2| > 0$, $\|\tilde{\mathcal{K}}^2(x_2, y_2) - \tilde{\mathcal{K}}^2(x_2, y'_2)\|_{\text{CZ}} \leq C(|y_2 - y'_2|^\varepsilon/|x_2 - y_2|^{n_2+\varepsilon})$, if $|y_2 - y'_2| \leq (1/2)|x_2 - y_2|$, and $\|\tilde{\mathcal{K}}^2(x_2, y_2) - \tilde{\mathcal{K}}^2(x'_2, y_2)\|_{\text{CZ}} \leq C(|x_2 - x'_2|^\varepsilon/|x_2 - y_2|^{n_2+\varepsilon})$, if $|x_2 - x'_2| \leq (1/2)|x_2 - y_2|$

The smallest such constant C is denoted by $\|\mathcal{K}\|_{\text{CZ}}$. The norm of the operator T is defined by $\|T\|_{\text{CZ}} = \|T\|_{L^2(\mathbb{R}^{n_1+n_2}) \rightarrow L^2(\mathbb{R}^{n_1+n_2})} + \|\mathcal{K}\|_{\text{CZ}}$.

Theorem 1. Suppose that T is a singular integral operator in mixed Journé class on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ with the regularity exponent $\varepsilon \in (0, 1]$, $\delta > 0$. Then, T is bounded on H_{mix}^p for $\max\{n_1/(n_1 + \delta), n_1/(n_1 + \varepsilon), n_2/(n_2 + \varepsilon)\} < p \leq 1$ if $T_1(1) = T_2(1) = T_1^*(1) = T_2^*(1) = 0$.

Remark 1. The meaning of $T_1(1) = 0$ is understood as follows: for any fixed $y_2 \in \mathbb{R}^{n_2}$,

$$\int \mathcal{K}(x_1, y_1, x_2, y_2) \psi_1(x_1) \psi_2(x_2) dx_1 dx_2 dy_1 = 0, \quad (14)$$

for any $\psi_i \in \mathcal{S}_M(\mathbb{R}^{n_i})$.

Similarly, $T_2(1) = 0$, where $M = M_{p, n_1, n_2}$ is a large positive integer dependent only on p, n_1, n_2 .

$T_1^*(1) = 0$ is known as for any fixed $x_2 \in \mathbb{R}^{n_2}$,

$$\int \mathcal{K}(x_1, y_1, x_2, y_2) \psi_1(y_1) \psi_2(y_2) dy_1 dy_2 dx_1 = 0, \quad (15)$$

for any $\psi_i \in \mathcal{S}_M(\mathbb{R}^{n_i}), i = 1, 2$,

similarly for $T_2^*(1) = 0$.

The organization of this paper is as follows. In section 2, we recall a discrete Calderón-type identity on $L^2(\mathbb{R}^{n_1+n_2}) \cap H_{\text{mix}}^p$ which is proved in [17]. In section 3, we prove Theorem 1.

Finally, we make some conventions. Throughout the paper, C denotes a positive constant that is independent of the main parameters involved, but whose value may vary from line to line. Constants with subscript, such as C_1 , do not change in different occurrences. We denote $f \leq Cg$ by $f \lesssim g$. If $f \lesssim g \lesssim f$, we write $f \approx g$.

2. Calderón-Type Identity

To show Theorem 1, we need a new discrete Calderón-type identity on $L^2(\mathbb{R}^{n_1+n_2}) \cap H_{\text{mix}}^p$. To do this, for any large integer $D = D_{p, n_1, n_2}$, let $\phi_0^{(1)}, \phi^{(1)} \in \mathcal{S}(\mathbb{R}^{n_1})$ with

$$\text{supp} \phi_0^{(1)} \subseteq \{x \in \mathbb{R}^{n_1} : |x| \leq 1\}; \quad \int \phi_0^{(1)} = 1, \quad (16)$$

$$\text{supp} \phi^{(1)} \subseteq \{x \in \mathbb{R}^{n_1} : |x| \leq 1\}; \quad \int \phi^{(1)}(x) x^\alpha dx = 0, \quad (17)$$

for all $|\alpha| \leq D$,

$$\left| \widehat{\phi_0^{(1)}}(\xi) \right|^2 + \sum_{j=1}^{\infty} \left| \widehat{\phi^{(1)}}(2^{-j}\xi) \right|^2 = 1, \quad \text{for all } \xi \in \mathbb{R}^{n_1}. \quad (18)$$

Let $\phi^{(2)} \in \mathcal{S}(\mathbb{R}^{n_2})$ with

$$\text{supp} \phi^{(2)} \subseteq \{x \in \mathbb{R}^{n_2} : |x| \leq 1\}; \quad \int \phi^{(2)}(x) x^\alpha dx = 0, \quad (19)$$

for all $|\alpha| \leq D$,

$$\sum_{j \in \mathbb{Z}} \left| \widehat{\phi^{(2)}}(2^{-j}\xi) \right|^2 = 1, \quad \text{for all } \frac{\xi \in \mathbb{R}^{n_2}}{\{0\}}. \quad (20)$$

Set $\phi_{0,k} = \phi_0^{(1)} * \phi_k^{(2)}$ and $\phi_{j,k} = \phi_j^{(1)} * \phi_k^{(2)}$ if $j > 1$. Then, we have the following discrete Calderón-type identity on $L^2(\mathbb{R}^{n_1+n_2}) \cap H_{\text{mix}}^p$ [17].

Theorem 2. For $0 < p \leq 1$, let $\phi_0^{(1)}, \phi^{(1)}$ and $\phi^{(2)}$ satisfy conditions (16)–(20), respectively. Then, for any $f \in L^2(\mathbb{R}^{n_1+n_2}) \cap H_{\text{mix}}^p$, there exists a positive integer N such that

$$f(x) = \sum_{j \in \mathbb{N}, k \in \mathbb{Z}} \sum_{I \times J \in \Pi_{j+N}^{n_1} \times \Pi_{k+N}^{n_2}} |I||J| (\phi_{j,k} * h)(x_I, x_J) \times \phi_{j,k}(x_1 - x_I, x_2 - x_J), \quad (21)$$

where the series converges in $L^2(\mathbb{R}^{n_1+n_2})$ and H_{mix}^p , and $h \in L^2(\mathbb{R}^{n_1+n_2}) \cap H_{\text{mix}}^p$ satisfying

$$\begin{aligned} \|h\|_{L^2(\mathbb{R}^{n_1+n_2})} &\approx f_{L^2(\mathbb{R}^{n_1+n_2})}, \\ \|h\|_{H_{\text{mix}}^p} &\approx f_{H_{\text{mix}}^p}. \end{aligned} \quad (22)$$

Corollary 1 (see [17]). Let $0 < p \leq 1$. Suppose that $\phi_{j,k}$ satisfies the same conditions as in Theorem 2. Then, for $f \in L^2(\mathbb{R}^{n_1+n_2}) \cap H_{\text{mix}}^p$,

$$\|f\|_{H_{\text{mix}}^p} \approx \left\| \left(\sum_{j \in \mathbb{N}, k \in \mathbb{Z}} \sum_{I \times J \in \Pi_{j+N}^{n_1} \times \Pi_{k+N}^{n_2}} \left| (\phi_{j,k} * f)(x_I, x_J) \right|^2 \chi_I \chi_J \right)^{1/2} \right\|_{L^p(\mathbb{R}^{n_1+n_2})}, \quad (23)$$

where N is the same as in Theorem 2.

3. Boundedness

Now, we are ready to prove Theorem 1. For $f \in L^2(\mathbb{R}^{n_1+n_2}) \cap H_{\text{mix}}^p$, by Theorem 2, one has

$$\begin{aligned} f(x, y) &= \sum_{j \in \mathbb{N}, k \in \mathbb{Z}} \sum_{I \times J \in \Pi_{j+N}^{n_1} \times \Pi_{k+N}^{n_2}} |I||J| (\psi_{j,k} * h)(x_I, x_J) \\ &\quad \times \psi_{j,k}(x - x_I, y - x_J), \end{aligned} \quad (24)$$

in $L^2(\mathbb{R}^{n_1+n_2})$, where $\psi_{j,k}$ satisfy the same conditions as in Theorem 2. By the L^2 boundedness of T ,

$$\begin{aligned} T(f)(x, y) &= \sum_{j \in \mathbb{N}, k \in \mathbb{Z}} \sum_{I \times J \in \Pi_{j+N}^{n_1} \times \Pi_{k+N}^{n_2}} |I||J| (\psi_{j,k} * h)(x_I, x_J) \\ &\quad \times T(\psi_{j,k}(\cdot - x_I, \cdot - x_J))(x, y). \end{aligned} \quad (25)$$

Then, Corollary 1 gives that

$$\begin{aligned}
\|T(f)\|_{H_{\min}^p} &\leq C \left\| \left(\sum_{j' \in \mathbb{N}, k' \in \mathbb{Z}} \sum_{l' \times j' \in \Pi_{j'+N}^{n_1} \times \Pi_{k'+N}^{n_2}} |\phi_{j',k'} * (Tf)(x_{l'}, x_{j'})|^2 \chi_{l'}(x) \chi_{j'}(y) \right)^{1/2} \right\|_{L^p(\mathbb{R}^{n_1+n_2})} \\
&= C \left\| \left(\sum_{j' \in \mathbb{N}, k' \in \mathbb{Z}} \sum_{l' \times j' \in \Pi_{j'+N}^{n_1} \times \Pi_{k'+N}^{n_2}} \left| \sum_{j \in \mathbb{N}, k \in \mathbb{Z}} \sum_{l \times j \in \Pi_{j+N}^{n_1} \times \Pi_{k+N}^{n_2}} |I||J|(\psi_{j,k} * h)(x_l, x_j) \times \phi_{j',k'} * T(\psi_{j,k}(\cdot - x_l, \cdot - x_j))(x_{l'}, x_{j'})) \right|^2 \right. \right. \\
&\quad \left. \cdot \chi_{l'}(x) \chi_{j'}(y) \right)^{1/2} \right\|_{L^p(\mathbb{R}^{n_1+n_2})}, \tag{26}
\end{aligned}$$

where $\phi_{j',k'}$ and $\psi_{j,k}$ satisfy the conditions in Theorem 2. Formally, one has

$$\begin{aligned}
&\phi_{j',k'} * T(\psi_{j,k}(\cdot - u_1, \cdot - u_2))(x_1, x_2) \\
&= \int \phi_{j'}^{(1)}(x_1 - y_1) \phi_{k'}^{(2)}(x_2 - y_2) \mathcal{K}(y_1, z_1, y_2, z_2) \psi_j^{(1)} \\
&\quad \cdot (z_1 - u_1) \psi_k^{(2)}(z_2 - u_2) dy_1 dy_2 dz_1 dz_2. \tag{27}
\end{aligned}$$

To deal with the case that $|y_1 - z_1|$ or $|y_2 - z_2|$ is small, we use the argument in [19]. For $0 < \eta < 1$, let $C_0^\eta(\mathbb{R}^n)$ denote the space of continuous functions f with compact supports such that

$$\|f\|_{\text{Lip}_\eta(\mathbb{R}^n)} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\eta} \tag{28}$$

is finite. Note that if $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with $\text{supp } \varphi \subseteq Q$ for some cube Q in $\mathbb{R}^n$, then $\varphi \in \text{Lip}_\eta(\mathbb{R}^n)$ for any $0 < \eta < 1$, and

$$\|\varphi\|_{\text{Lip}_\eta(\mathbb{R}^n)} < +\infty, \tag{29}$$

and for any $j \in \mathbb{Z}$,

$$\|\varphi_j\|_{\text{Lip}_\eta(\mathbb{R}^n)} \leq C 2^{jn+\eta n}. \tag{30}$$

A linear one-parameter operator $\mathcal{T}$ from $C_0^\eta(\mathbb{R}^n)$ to $(C_0^\eta(\mathbb{R}^n))'$, $0 < \eta < 1$, is said to satisfy the weak boundedness property if

$$|\langle \varphi, \mathcal{T}(\psi) \rangle| \leq C |Q|^{1+2\eta/n} \|\varphi\|_{\text{Lip}_\eta(\mathbb{R}^n)} \|\psi\|_{\text{Lip}_\eta(\mathbb{R}^n)}, \tag{31}$$

for any $\varphi, \psi \in C_0^\eta(\mathbb{R}^n)$ with supports in cube $Q \subseteq \mathbb{R}^n$. If $\varphi \in C_0^\eta(\mathbb{R}^n)$ with support in Q , using $\|\varphi\|_{L^2(\mathbb{R}^n)} \leq C |Q|^{(1/2)+(\eta/n)} \|\varphi\|_{\text{Lip}_\eta(\mathbb{R}^n)}$, one can see that $L^2(\mathbb{R}^n)$ boundedness of $\mathcal{T}$ implies that $\mathcal{T}$ satisfies the weak boundedness property. This property can be generalized to multiparameter operators.

Lemma 1. Suppose that $\varphi_1, \varphi_2 \in C_0^\eta(\mathbb{R}^{n_1})$ with supports in cube $Q_1 \subseteq \mathbb{R}^{n_1}$, $\psi_1, \psi_2 \in C_0^\eta(\mathbb{R}^{n_2})$ with supports in cube $Q_2 \subseteq \mathbb{R}^{n_2}$. If linear two-parameter operator T is $L^2(\mathbb{R}^{n_1+n_2})$

bounded, then T satisfies the following weak boundedness property:

$$\begin{aligned}
|\langle \varphi_1 \psi_1, T(\varphi_2 \psi_2) \rangle| &\leq C \|T\|_{L^2(\mathbb{R}^{n_1+n_2}) \rightarrow L^2(\mathbb{R}^{n_1+n_2})} |Q_1|^{1+2\eta/n_1} \\
&\quad \cdot |Q_2|^{1+2\eta/n_2} \\
&\quad \times \|\varphi_1\|_{\text{Lip}_\eta(\mathbb{R}^{n_1})} \|\psi_1\|_{\text{Lip}_\eta(\mathbb{R}^{n_2})} \|\varphi_2\|_{\text{Lip}_\eta(\mathbb{R}^{n_1})} \\
&\quad \cdot \|\psi_2\|_{\text{Lip}_\eta(\mathbb{R}^{n_2})}. \tag{32}
\end{aligned}$$

Next, we need some almost orthogonality estimates about $\phi_{j',k'} * T(\psi_{j,k}(\cdot - u_1, \cdot - u_2))(x_1, x_2)$. From now on, $\phi_{j',k'}$ and $\psi_{j,k}$ satisfy the conditions of Theorem 2, and T satisfies the conditions of Theorem 1. In Definition 2, without loss of generality, we may assume $\delta \geq \varepsilon$.

Lemma 2. For any $k, k' \in \mathbb{Z}$ and any $0 < \varepsilon' < \varepsilon$, there exists a positive constant C such that

$$\begin{aligned}
&|\phi_{0,k'} * T(\psi_{0,k}(\cdot - u_1, \cdot - u_2))(x_1, x_2)| \\
&\leq C 2^{-|k'-k|\varepsilon'} \frac{1}{(1+|x_1-u_1|)^{n_1+\varepsilon}} \frac{2^{-(k \wedge k')\varepsilon}}{(2^{-(k \wedge k')\varepsilon} + |x_2-u_2|)^{n_2+\varepsilon}}. \tag{33}
\end{aligned}$$

Proof. We divide the estimates into eight cases:

- Case I: $k' > k, 2^k|x_2 - u_2| \leq 10, |x_1 - u_1| \leq 8$
- Case II: $k' > k, 2^k|x_2 - u_2| > 10, |x_1 - u_1| \leq 8$
- Case III: $k' \leq k, 2^{k'}|x_2 - u_2| \leq 10, |x_1 - u_1| \leq 8$
- Case IV: $k' \leq k, 2^{k'}|x_2 - u_2| > 10, |x_1 - u_1| \leq 8$
- Case V: $k' > k, 2^k|x_2 - u_2| \leq 10, |x_1 - u_1| > 8$
- Case VI: $k' > k, 2^k|x_2 - u_2| > 10, |x_1 - u_1| > 8$
- Case VII: $k' \leq k, 2^{k'}|x_2 - u_2| \leq 10, |x_1 - u_1| > 8$
- Case VIII: $k' \leq k, 2^{k'}|x_2 - u_2| > 10, |x_1 - u_1| > 8$

Now, we deal with case I. Since $T_2(1) = 0$, we have

$$\begin{aligned}
& \phi_{0,k'} * T(\psi_{0,k}(\cdot - u_1, \cdot - u_2)(x_1, x_2)) \\
&= \int \phi_0^{(1)}(x_1 - y_1) \phi_{k'}^{(2)}(x_2 - y_2) \mathcal{K}(y_1, z_1, y_2, z_2) \psi_0^{(1)} \\
&\quad \cdot (z_1 - u_1) \psi_k^{(2)}(z_2 - u_2) dy_1 dy_2 dz_1 dz_2 \\
&= \int \phi_0^{(1)}(x_1 - y_1) \phi_{k'}^{(2)}(x_2 - y_2) \mathcal{K}(y_1, z_1, y_2, z_2) \psi_0^{(1)} \\
&\quad \cdot (z_1 - u_1) (\psi_k^{(2)}(z_2 - u_2) - \psi_k^{(2)}(x_2 - u_2)) dy_1 dy_2 dz_1 dz_2.
\end{aligned} \tag{34}$$

Let $\rho_0 \in C^\infty(\mathbb{R}^{n_2})$ be 1 on $B(0, 2)$ and 0 outside its double. Then,

$$\begin{aligned}
& \phi_{0,k'} * T(\psi_{0,k}(\cdot - u_1, \cdot - u_2)(x_1, x_2)) \\
&= \int \phi_0^{(1)}(x_1 - y_1) \phi_{k'}^{(2)}(x_2 - y_2) \mathcal{K}(y_1, z_1, y_2, z_2) \psi_0^{(1)} \\
&\quad \cdot (z_1 - u_1) \\
&\quad [\psi_k^{(2)}(z_2 - u_2) - \psi_k^{(2)}(x_2 - u_2)] \rho_0(2^{k'}(z_2 - x_2)) dy dz \\
&+ \int \phi_0^{(1)}(x_1 - y_1) \phi_{k'}^{(2)}(x_2 - y_2) \mathcal{K}(y_1, z_1, y_2, z_2) \psi_0^{(1)} \\
&\quad \cdot (z_1 - u_1) \\
&\quad [\psi_k^{(2)}(z_2 - u_2) - \psi_k^{(2)}(x_2 - u_2)] [1 - \rho_0(2^{k'}(z_2 - x_2))] dy dz \\
&= A + B.
\end{aligned} \tag{35}$$

One can check that

$$\|\phi_{k'}^{(2)}\|_{\text{Lip}_\eta(\mathbb{R}^{n_2})} \leq C 2^{k'n_2+k'\eta}, \tag{36}$$

$$\begin{aligned}
& \left\| [\psi_k^{(2)}(\cdot - u_2) - \psi_k^{(2)}(x_2 - u_2)] \rho_0(2^{k'}(\cdot - x_2)) \right\|_{\text{Lip}_\eta(\mathbb{R}^{n_2})} \\
& \leq C 2^{-(k'-k)} 2^{kn_2} 2^{k'\eta},
\end{aligned} \tag{37}$$

and $\|\phi_0^{(1)}\|_{\text{Lip}_\eta(\mathbb{R}^{n_1})} + \|\psi_0^{(1)}\|_{\text{Lip}_\eta(\mathbb{R}^{n_1})} \leq C$ [19]. Hence, by the weak boundedness of T on $\mathbb{R}^{n_1+n_2}$, we have

$$\begin{aligned}
|A| &\leq C 2^{-k'(n_2+2\eta)} \|\phi_{k'}^{(2)}\|_{\text{Lip}_\eta(\mathbb{R}^{n_2})} \\
&\quad \cdot \left\| [\psi_k^{(2)}(\cdot - u_2) - \psi_k^{(2)}(x_2 - u_2)] \rho_0(2^{k'}(\cdot - x_2)) \right\|_{\text{Lip}_\eta(\mathbb{R}^{n_2})} \\
&\leq C 2^{-(k'-k)} 2^{kn_2} \leq C 2^{-(k'-k)\varepsilon} 2^{kn_2}.
\end{aligned} \tag{38}$$

On the contrary, using $\int \phi^{(2)} = 0$, one has

$$\begin{aligned}
B &= \int \phi_0^{(1)}(x_1 - y_1) \phi_{k'}^{(2)}(x_2 - y_2) \\
&\quad \cdot [\mathcal{K}(y_1, z_1, y_2, z_2) - \mathcal{K}(y_1, z_1, x_2, z_2)] \\
&\quad \psi_0^{(1)}(z_1 - u_1) [\psi_k^{(2)}(z_2 - u_2) - \psi_k^{(2)}(x_2 - u_2)] \\
&\quad \cdot \left[1 - \rho_0(2^{k'}(z_2 - x_2)) \right] dy dz,
\end{aligned} \tag{39}$$

which implies that

$$\begin{aligned}
|B| &\leq C \iint |\phi_{k'}^{(2)}(x_2 - y_2)| \|\tilde{\mathcal{K}}^2(y_2, z_2) - \tilde{\mathcal{K}}^2(x_2, z_2)\|_{CZ} \\
&\quad \cdot |\psi_k^{(2)}(z_2 - u_2) - \psi_k^{(2)}(x_2 - u_2)| \\
&\quad \cdot \left[1 - \rho_0(2^{k'}(z_2 - x_2)) \right] dz_2 dy_2,
\end{aligned} \tag{40}$$

by the weak boundedness of operator $\tilde{\mathcal{K}}^2(y_2, z_2) - \tilde{\mathcal{K}}^2(x_2, z_2)$ on $\mathbb{R}^{n_1}$. Combing the smooth condition of $\tilde{\mathcal{K}}^2$ with

$$|\psi_k^{(2)}(z_2 - u_2) - \psi_k^{(2)}(x_2 - u_2)| \leq C \left(\frac{|z_2 - x_2|}{2^{-k} + |z_2 - x_2|} \right)^\varepsilon 2^{kn_2}, \tag{41}$$

we have

$$\begin{aligned}
|B| &\leq C \iint_{|x_2 - z_2| \geq 2^{-k'+1}} |\psi_{k'}^{(2)}(x_2 - y_2)| \\
&\quad \cdot \frac{|y_2 - x_2|^\varepsilon}{|z_2 - x_2|^{n_2+\varepsilon}} \left(\frac{|z_2 - x_2|}{2^{-k} + |z_2 - x_2|} \right)^\varepsilon 2^{kn_2} dy_2 dz_2 \\
&\leq C 2^{kn_2} \iint_{|x_2 - z_2| \geq 2^{-k'+1}} |\psi_{k'}^{(2)}(x_2 - y_2)| \\
&\quad \cdot \frac{2^{-k'\varepsilon}}{|z_2 - x_2|^{n_2}} dy_2 \frac{1}{(2^{-k} + |z_2 - x_2|)^\varepsilon} dz_2 \\
&\leq C 2^{kn_2} \int_{|x_2 - z_2| \geq 2^{-k'+1}} \frac{2^{-k'\varepsilon}}{|z_2 - x_2|^{n_2+\varepsilon}} \frac{1}{(2^{-k} + |z_2 - x_2|)^\varepsilon} dz_2 \\
&\leq C 2^{-k'\varepsilon} 2^{kn_2} \int_{|x_2 - z_2| > 2^{-k}} \frac{1}{|z_2 - x_2|^{n_2+\varepsilon}} dz_2 \\
&\quad + C 2^{-(k'-k)\varepsilon} 2^{kn_2} \int_{2^{-k'+1} \leq |x_2 - z_2| \leq 2^{-k}} \frac{1}{|z_2 - x_2|^{n_2}} dz_2 \\
&\leq C 2^{-k'\varepsilon} 2^{kn_2} 2^{k\varepsilon} + C 2^{-(k'-k)\varepsilon} 2^{kn_2} (k' - k) \leq C 2^{-(k'-k)\varepsilon'} 2^{kn_2}.
\end{aligned} \tag{42}$$

Since $|x_2 - u_2| \leq 10 \cdot 2^{-k}$ and $|x_1 - u_1| \leq 8$, one has $2^{-k} \approx 2^{-k} + |x_2 - u_2|$ and $1 + |x_1 - u_1| \approx 1$, which give that

$$\begin{aligned}
& |\phi_{0,k'} * T(\psi_{0,k}(\cdot - u_1, \cdot - u_2)(x_1, x_2))| \\
& \leq C 2^{-(k'-k)\varepsilon'} 2^{kn_2} \\
& \approx C 2^{-(k'-k)\varepsilon'} 2^{kn_2} \left(\frac{2^{-k}}{2^{-k} + |x_2 - u_2|} \right)^\varepsilon \\
& \approx C 2^{-(k'-k)\varepsilon'} \frac{1}{(1 + |x_1 - u_1|)^{n_1 + \varepsilon}} \frac{2^{-k\varepsilon}}{(2^{-k} + |x_2 - u_2|)^{n_2 + \varepsilon}}. \quad (43)
\end{aligned}$$

Now, we deal with case II. Using $\int \phi^{(2)} = 0$, one has

$$\begin{aligned}
& \phi_{0,k'} * T(\psi_{0,k}(\cdot - u_1, \cdot - u_2)(x_1, x_2)) \\
& = \int \phi_0^{(1)}(x_1 - y_1) \phi_{k'}^{(2)}(x_2 - y_2) \mathcal{K}(y_1, z_1, y_2, z_2) \psi_0^{(1)} \\
& \quad \cdot (z_1 - u_1) \psi_k^{(2)}(z_2 - u_2) dy_1 dy_2 dz_1 dz_2 \\
& = \int \phi_0^{(1)}(x_1 - y_1) \phi_{k'}^{(2)}(x_2 - y_2) \\
& \quad \cdot [\mathcal{K}(y_1, z_1, y_2, z_2) - \mathcal{K}(y_1, z_1, x_2, z_2)] \\
& \quad \times \psi_0^{(1)}(z_1 - u_1) \psi_k^{(2)}(z_2 - u_2) dy_1 dy_2 dz_1 dz_2. \quad (44)
\end{aligned}$$

By the weak boundedness of $\tilde{\mathcal{K}}^2(y_2, z_2) - \tilde{\mathcal{K}}^2(x_2, z_2)$, it follows that

$$\begin{aligned}
& |\phi_{0,k'} * T(\psi_{0,k}(\cdot - u_1, \cdot - u_2)(x_1, x_2))| \\
& \leq C \int |\phi_{k'}^{(2)}(x_2 - y_2)| \|\tilde{\mathcal{K}}^2(y_2, z_2) - \tilde{\mathcal{K}}^2(x_2, z_2)\|_{CZ} \\
& \quad \cdot |\psi_k^{(2)}(z_2 - u_2)| dz_2 dy_2. \quad (45)
\end{aligned}$$

In the above integral, one can check that $|y_2 - z_2| \approx |x_2 - u_2|$ and $|x_2 - u_2| \approx 2^{-k} + |x_2 - u_2|$, which together with the smooth condition of $\tilde{\mathcal{K}}^2$ yield that

$$\begin{aligned}
& |\phi_{0,k'} * T(\psi_{0,k}(\cdot - u_1, \cdot - u_2)(x_1, x_2))| \\
& \leq C \int |\phi_{k'}^{(2)}(x_2 - y_2)| \frac{|y_2 - x_2|^\varepsilon}{|y_2 - z_2|^{n_2 + \varepsilon}} |\psi_k^{(2)}(z_2 - u_2)| dz_2 dy_2 \\
& \leq C \frac{2^{-k'\varepsilon}}{(2^{-k} + |x_2 - u_2|)^{n_2 + \varepsilon}} \int |\phi_{k'}^{(2)}(x_2 - y_2)| \\
& \quad \cdot |\psi_k^{(2)}(z_2 - u_2)| dz_2 dy_2 \\
& \leq C 2^{-(k'-k)\varepsilon} \frac{1}{(1 + |x_1 - u_1|)^{n_1 + \varepsilon}} \frac{2^{-k\varepsilon}}{(2^{-k} + |x_2 - u_2|)^{n_2 + \varepsilon}}. \quad (46)
\end{aligned}$$

By symmetry, with a similar proof, one can discuss case III and case IV since $T_2^*(1) = 0$. Now, we turn to case V. Firstly, proceeding as case I, one can divide $\phi_{0,k'} * T(\psi_{0,k}(\cdot - u_1, \cdot - u_2)(x_1, x_2))$ into A and B , that is,

$$\phi_{0,k'} * T(\psi_{0,k}(\cdot - u_1, \cdot - u_2)(x_1, x_2)) = A + B. \quad (47)$$

Then, by the weak boundedness of operator $\tilde{\mathcal{K}}^1(y_1, z_1)$, one has

$$\begin{aligned}
|A| & = \left| \int \phi_0^{(1)}(x_1 - y_1) \phi_{k'}^{(2)}(x_2 - y_2) \mathcal{K}(y_1, z_1, y_2, z_2) \psi_0^{(1)}(z_1 - u_1) \right. \\
& \quad \cdot [\psi_k^{(2)}(z_2 - u_2) - \psi_k^{(2)}(x_2 - u_2)] \rho_0(2^{k'}(z_2 - x_2)) dy dz \Big| \\
& \leq C \int |\phi_0^{(1)}(x_1 - y_1)| \|\tilde{\mathcal{K}}^1(y_1, z_1)\|_{CZ} |\psi_0^{(1)}(z_1 - u_1)| dy_1 dz_1 \\
& \quad \times 2^{-k'(n_2 + 2\eta)} \|\phi_{k'}^{(2)}\|_{\text{Lip}_\eta} \left\| [\psi_k^{(2)}(\cdot - u_2) - \psi_k^{(2)}(x_2 - u_2)] \rho_0(2^{k'}(\cdot - x_2)) \right\|_{\text{Lip}_\eta} \\
& \leq C 2^{-(k'-k)\varepsilon} 2^{kn_2} \int |\phi_0^{(1)}(x_1 - y_1)| \|\tilde{\mathcal{K}}^1(y_1, z_1)\|_{CZ} |\psi_0^{(1)}(z_1 - u_1)| dy_1 dz_1. \quad (48)
\end{aligned}$$

Using the support condition of $\phi_0^{(1)}$ and $\psi_0^{(1)}$ and the size condition of $\tilde{\mathcal{K}}^1(y_1, z_1)$ when $|y_1 - z_1| > 1$, we can obtain that

$$|A| \leq C 2^{-(k'-k)\varepsilon} 2^{kn_2} \int \left| \phi_0^{(1)}(x_1 - y_1) \right| \frac{1}{|y_1 - z_1|^{n_1+\varepsilon}}$$

since $2^{-k} \approx 2^{-k} + |x_2 - u_2|$ in this case.

With similar estimates in (42), using $\int \phi^{(2)} = 0$, one has

$$\begin{aligned} & \cdot \left| \psi_0^{(1)}(z_1 - u_1) \right| dy_1 dz_1 \\ & \leq C \frac{1}{|x_1 - u_1|^{n_1+\varepsilon}} 2^{-(k'-k)\varepsilon} 2^{kn_2} \\ & \approx C 2^{-(k'-k)\varepsilon} \frac{1}{(1 + |x_1 - u_1|)^{n_1+\varepsilon}} \frac{2^{-k\varepsilon}}{(2^{-k} + |x_2 - u_2|)^{n_2+\varepsilon}}, \end{aligned} \quad (49)$$

$$\begin{aligned} |B| &= \left| \int \phi_0^{(1)}(x_1 - y_1) \phi_{k'}^{(2)}(x_2 - y_2) [\mathcal{K}(y_1, z_1, y_2, z_2) - \mathcal{K}(y_1, z_1, x_2, z_2)] \right. \\ & \quad \times \psi_0^{(1)}(z_1 - u_1) [\psi_k^{(2)}(z_2 - u_2) - \psi_k^{(2)}(x_2 - u_2)] \left. \left[1 - \rho_0(2^{k'}(z_2 - x_2)) \right] dy dz \right| \\ &\leq C \int \left| \phi_0^{(1)}(x_1 - y_1) \right| \left| \phi_{k'}^{(2)}(x_2 - y_2) \right| \|\tilde{\mathcal{K}}^{-1}\|(y_1, z_1) \|_{CZ} \frac{|y_2 - x_2|^\varepsilon}{|z_2 - x_2|^{n_2+\varepsilon}} \\ & \quad \times \left| \psi_0^{(1)}(z_1 - u_1) \right| \left| \psi_k^{(2)}(z_2 - u_2) - \psi_k^{(2)}(x_2 - u_2) \right| \left| 1 - \rho_0(2^{k'}(z_2 - x_2)) \right| dy dz \\ &\leq C \int \left| \phi_0^{(1)}(x_1 - y_1) \right| \frac{1}{|y_1 - z_1|^{n_1+\varepsilon}} \left| \psi_0^{(1)}(z_1 - u_1) \right| dy_1 dz_1 \\ & \quad \times \int \left| \phi_{k'}^{(2)}(x_2 - y_2) \right| \frac{|y_2 - x_2|^\varepsilon}{|z_2 - x_2|^{n_2+\varepsilon}} \left| \psi_k^{(2)}(z_2 - u_2) - \psi_k^{(2)}(x_2 - u_2) \right| \\ & \quad \cdot \left| 1 - \rho_0(2^{k'}(z_2 - x_2)) \right| dy_2 dz_2 \\ &\leq C \frac{1}{(1 + |x_1 - u_1|)^{n_1+\varepsilon}} 2^{-(k'-k)\varepsilon'} 2^{kn_2} \\ &\approx C 2^{-(k'-k)\varepsilon'} \frac{1}{(1 + |x_1 - u_1|)^{n_1+\varepsilon}} \frac{2^{-k\varepsilon}}{(2^{-k} + |x_2 - u_2|)^{n_2+\varepsilon}}. \end{aligned} \quad (50)$$

In case VI, similar as estimates (46), using $\int \phi^{(2)} = 0$, one has

$$\begin{aligned}
& \left| \phi_{0,k'} * T(\psi_{0,k}(\cdot - u_1, \cdot - u_2))(x_1, x_2) \right| \\
&= \left| \int \phi_0^{(1)}(x_1 - y_1) \phi_{k'}^{(2)}(x_2 - y_2) [\mathcal{K}(y_1, z_1, y_2, z_2) - \mathcal{K}(y_1, z_1, x_2, z_2)] \times \psi_0^{(1)}(z_1 - u_1) \psi_k^{(2)}(z_2 - u_2) dy_1 dy_2 dz_1 dz_2 \right| \\
&\leq C \int \left| \phi_0^{(1)}(x_1 - y_1) \right| \left| \phi_{k'}^{(2)}(x_2 - y_2) \right| \left\| \tilde{\mathcal{K}}^1(y_1, z_1) \right\|_{CZ} \frac{|y_2 - x_2|^\varepsilon}{|y_2 - z_2|^{n_2 + \varepsilon}} \\
&\quad \times \left| \psi_0^{(1)}(z_1 - u_1) \right| \left| \psi_k^{(2)}(z_2 - u_2) \right| dy dz \\
&\leq C \int \left| \phi_0^{(1)}(x_1 - y_1) \right| \frac{1}{|y_1 - z_1|^{n_1 + \varepsilon}} \left| \psi_0^{(1)}(z_1 - u_1) \right| dy_1 dz_1 \\
&\quad \times \int \left| \phi_{k'}^{(2)}(x_2 - y_2) \right| \frac{|y_2 - x_2|^\varepsilon}{|y_2 - z_2|^{n_2 + \varepsilon}} \left| \psi_k^{(2)}(z_2 - u_2) \right| dz_2 dy_2 \\
&\leq C 2^{-(k'-k)\varepsilon} \frac{1}{(1 + |x_1 - u_1|)^{n_1 + \varepsilon}} \frac{2^{-k\varepsilon}}{(2^{-k} + |x_2 - u_2|)^{n_2 + \varepsilon}},
\end{aligned} \tag{51}$$

since $|x_1 - u_1| > 8$ which implies that $|z_1 - y_1| > 1$.

By symmetry, we complete the proof. $\square$

Lemma 3. Suppose that $j, j', k, k' \in \mathbb{Z}$, $j' \geq j \geq 0$, $j' > 0$, and $k' \geq k$, then for any $0 < \varepsilon' < \varepsilon$, there exists a constant $C > 0$ such that

$$\begin{aligned}
& \left| \phi_{j',k'} * T(\psi_{j,k}(\cdot - u_1, \cdot - u_2))(x_1, x_2) \right| \\
&\leq C 2^{-(j'-j)\varepsilon'} 2^{-(k'-k)\varepsilon'} \frac{2^{-j\varepsilon}}{(2^{-j} + |x_1 - u_1|)^{n_1 + \varepsilon}} \frac{2^{-k\varepsilon}}{(2^{-k} + |x_2 - u_2|)^{n_2 + \varepsilon}}.
\end{aligned} \tag{52}$$

Proof. We divide the estimates into four cases: $2^j|x_1 - u_1| \leq 10$, $2^k|x_2 - u_2| \leq 10$; $2^j|x_1 - u_1| \leq 10, 2^k|x_2 - u_2| > 10$; $2^j|x_1 - u_1| > 10, 2^k|x_2 - u_2| \leq 10$; $2^j|x_1 - u_1| > 10, 2^k|x_2 - u_2| > 10$.

We now discuss the first case: $2^j|x_1 - u_1| \leq 10$, $2^k|x_2 - u_2| \leq 10$. Since $T_1(1) = T_2(1) = 0$,

$$\begin{aligned}
& \phi_{j',k'} * T(\psi_{j,k}(\cdot - u_1, \cdot - u_2))(x_1, x_2) \\
&= \int \phi_{j'}^1(x_1 - y_1) \phi_{k'}^2(x_2 - y_2) \mathcal{K}(y_1, z_1, y_2, z_2) \psi_j^1(z_1 - u_1) \psi_k^2(z_2 - u_2) dy_1 dy_2 dz_1 dz_2 \\
&= \int \phi_{j'}^1(x_1 - y_1) \phi_{k'}^2(x_2 - y_2) \mathcal{K}(y_1, z_1, y_2, z_2) \\
&\quad \times [\psi_j^1(z_1 - u_1) - \psi_j^1(x_1 - u_1)] [\psi_k^2(z_2 - u_2) - \psi_k^2(x_2 - u_2)] dy dz.
\end{aligned} \tag{53}$$

Let $\tau_0, \rho_0 \in C^\infty(\mathbb{R}^{n_2})$ be 1 on $B(0, 2)$ and 0 outside its double. Set $\rho_1 = 1 - \rho_0$, $\tau_1 = 1 - \tau_0$. Then,

$$\begin{aligned}
& T(\psi_{j,k}(\cdot - u_1, \cdot - u_2)(x_1, x_2)) \\
&= \int \phi_{j'}^1(x_1 - y_1) \phi_{k'}^2(x_2 - y_2) \mathcal{K}(y_1, z_1, y_2, z_2) [\psi_j^1(z_1 - u_1) - \psi_j^1(x_1 - u_1)] \\
&\quad \times [\psi_k^2(z_2 - u_2) - \psi_k^2(x_2 - u_2)] \tau_0(2^{j'}(z_1 - x_1)) \rho_0(2^{k'}(z_2 - x_2)) dy dz \\
&\quad + \int \phi_{j'}^1(x_1 - y_1) \phi_{k'}^2(x_2 - y_2) \mathcal{K}(y_1, z_1, y_2, z_2) [\psi_j^1(z_1 - u_1) - \psi_j^1(x_1 - u_1)] \\
&\quad \times [\psi_k^2(z_2 - u_2) - \psi_k^2(x_2 - u_2)] \tau_1(2^{j'}(z_1 - x_1)) \rho_0(2^{k'}(z_2 - x_2)) dy dz \\
&\quad + \int \phi_{j'}^1(x_1 - y_1) \phi_{k'}^2(x_2 - y_2) \mathcal{K}(y_1, z_1, y_2, z_2) [\psi_j^1(z_1 - u_1) - \psi_j^1(x_1 - u_1)] \\
&\quad \times [\psi_k^2(z_2 - u_2) - \psi_k^2(x_2 - u_2)] \tau_0(2^{j'}(z_1 - x_1)) \rho_1(2^{k'}(z_2 - x_2)) dy dz \\
&\quad + \int \phi_{j'}^1(x_1 - y_1) \phi_{k'}^2(x_2 - y_2) \mathcal{K}(y_1, z_1, y_2, z_2) [\psi_j^1(z_1 - u_1) - \psi_j^1(x_1 - u_1)] \\
&\quad \times [\psi_k^2(z_2 - u_2) - \psi_k^2(x_2 - u_2)] \tau_1(2^{j'}(z_1 - x_1)) \rho_1(2^{k'}(z_2 - x_2)) dy dz \\
&= A + B + D + E.
\end{aligned} \tag{54}$$

By (36) and (37), we have

$$\begin{aligned}
|A| &\leq C 2^{-k'(n_2+2\eta)} 2^{-j'(n_1+2\eta)} \|\phi_{k'}^{(2)}\|_{\text{Lip}_\eta} \left\| [\psi_k^{(2)}(\cdot - u_2) - \psi_k^{(2)}(x_2 - u_2)] \rho_0(2^{k'}(\cdot - x_2)) \right\|_{\text{Lip}_\eta} \\
&\quad \times \|\phi_{j'}^{(1)}\|_{\text{Lip}_\eta} \left\| [\psi_j^{(1)}(\cdot - u_1) - \psi_j^{(1)}(x_1 - u_1)] \tau_0(2^{j'}(\cdot - x_1)) \right\|_{\text{Lip}_\eta} \\
&\leq C 2^{-(k'-k)} 2^{kn_2} 2^{-(j'-j)} 2^{jn_1} \\
&\leq C 2^{-(j'-j)\varepsilon} 2^{jn_1} 2^{-(k'-k)\varepsilon} 2^{kn_2} \\
&\approx C 2^{-(j'-j)\varepsilon} 2^{-(k'-k)\varepsilon} \frac{2^{-j\varepsilon}}{(2^{-j} + |x_1 - u_1|)^{n_1+\varepsilon}} \frac{2^{-k\varepsilon}}{(2^{-k} + |x_2 - u_2|)^{n_2+\varepsilon}}.
\end{aligned} \tag{55}$$

Using $\int \phi^1 = 0$, one has

$$\begin{aligned}
B &= \int \phi_{j'}^1(x_1 - y_1) \phi_{k'}^2(x_2 - y_2) [\mathcal{K}(y_1, z_1, y_2, z_2) - \mathcal{K}(x_1, z_1, y_2, z_2)] \\
&\quad \times [\psi_j^1(z_1 - u_1) - \psi_j^1(x_1 - u_1)] \times [\psi_k^2(z_2 - u_2) - \psi_k^2(x_2 - u_2)] \\
&\quad \times \tau_1(2^{j'}(z_1 - x_1)) \rho_0(2^{k'}(z_2 - x_2)) dy dz.
\end{aligned} \tag{56}$$

Hence, by (42)

$$\begin{aligned}
|B| &\leq C 2^{-k'} (n_2 + 2\eta) \left\| \phi_{k'}^{(2)} \right\|_{\text{Lip}_\eta} \left\| \left[\psi_k^{(2)}(\cdot - u_2) - \psi_k^{(2)}(x_2 - u_2) \right] \rho_0 \left(2^{k'}(\cdot - x_2) \right) \right\|_{\text{Lip}_\eta} \\
&\quad \times \int \left| \phi_{j'}^1(x_1 - y_1) \right| \left\| \tilde{\mathcal{K}}^1(y_1, z_1) - \tilde{\mathcal{K}}^1(x_1, z_1) \right\|_{CZ} \left| \psi_j^1(z_1 - u_1) - \psi_j^1(x_1 - u_1) \right| \\
&\quad \times \left| \tau_1 \left(2^{j'}(z_1 - x_1) \right) \right| dy_1 dz_1 \\
&\leq C 2^{-(k'-k)} 2^{kn_2} \int \left| \phi_{j'}^1(x_1 - y_1) \right| \frac{|y_1 - x_1|^\varepsilon}{|z_1 - x_1|^{n_1 + \varepsilon}} \left| \psi_j^1(z_1 - u_1) - \psi_j^1(x_1 - u_1) \right| \\
&\quad \times \left| \tau_1 \left(2^{j'}(z_1 - x_1) \right) \right| dy_1 dz_1 \\
&\leq C 2^{-(k'-k)} 2^{kn_2} 2^{-(j'-j)\varepsilon'} 2^{jn_1},
\end{aligned} \tag{57}$$

which implies that

$$\begin{aligned}
|B| &\leq C 2^{-(j'-j)\varepsilon'} 2^{-(k'-k)\varepsilon'} \frac{2^{-j\varepsilon}}{(2^{-j} + |x_1 - u_1|)^{n_1 + \varepsilon}} \\
&\quad \cdot \frac{2^{-k\varepsilon}}{(2^{-k} + |x_2 - u_2|)^{n_2 + \varepsilon}}.
\end{aligned} \tag{58}$$

Similarly, one has

$$\begin{aligned}
|D| &\leq C 2^{-(j'-j)\varepsilon'} 2^{-(k'-k)\varepsilon'} \frac{2^{-j\varepsilon}}{(2^{-j} + |x_1 - u_1|)^{n_1 + \varepsilon}} \\
&\quad \cdot \frac{2^{-k\varepsilon}}{(2^{-k} + |x_2 - u_2|)^{n_2 + \varepsilon}}.
\end{aligned} \tag{59}$$

For E , since $\int \phi^1 = \int \phi^2 = 0$,

$$\begin{aligned}
E &= \int \phi_{j'}^1(x_1 - y_1) \phi_{k'}^2(x_2 - y_2) \left[\mathcal{K}(y_1, z_1, y_2, z_2) - \mathcal{K}(x_1, z_1, y_2, z_2) \right. \\
&\quad \left. - (\mathcal{K}(y_1, z_1, x_2, z_2) - \mathcal{K}(x_1, z_1, x_2, z_2)) \right] \left[\psi_j^1(z_1 - u_1) - \psi_j^1(x_1 - u_1) \right] \\
&\quad \times \left[\psi_k^2(z_2 - u_2) - \psi_k^2(x_2 - u_2) \right] \tau_1 \left(2^{j'}(z_1 - x_1) \right) \rho_1 \left(2^{k'}(z_2 - x_2) \right) dy dz.
\end{aligned} \tag{60}$$

So,

$$\begin{aligned}
|E| &\leq C \int \left| \phi_{j'}^1(x_1 - y_1) \right| \left\| \mathcal{K}^1(y_1, z_1) - \mathcal{K}^1(x_1, z_1) \right\|_{CZ} \frac{|y_2 - x_2|^\varepsilon}{|z_2 - x_2|^{n_2 + \varepsilon}} \\
&\quad \cdot \left| \left[\psi_j^1(z_1 - u_1) - \psi_j^1(x_1 - u_1) \right] \left[\psi_k^2(z_2 - u_2) - \psi_k^2(x_2 - u_2) \right] \right| \\
&\quad \times \left| \tau_1 \left(2^{j'}(z_1 - x_1) \right) \rho_1 \left(2^{k'}(z_2 - x_2) \right) \right| dy dz \\
&\leq C \int \left| \phi_{j'}^1(x_1 - y_1) \right| \frac{|y_1 - x_1|^\varepsilon}{|z_1 - x_1|^{n_1 + \varepsilon}} \frac{|y_2 - x_2|^\varepsilon}{|z_2 - x_2|^{n_2 + \varepsilon}} \left| \psi_j^1(z_1 - u_1) - \psi_j^1(x_1 - u_1) \right| \\
&\quad \cdot \left| \left[\psi_k^2(z_2 - u_2) - \psi_k^2(x_2 - u_2) \right] \times \tau_1 \left(2^{j'}(z_1 - x_1) \right) \rho_1 \left(2^{k'}(z_2 - x_2) \right) \right| dy dz \\
&\leq C 2^{-(j'-j)\varepsilon'} 2^{-(k'-k)\varepsilon'} \frac{2^{-j\varepsilon}}{(2^{-j} + |x_1 - u_1|)^{n_1 + \varepsilon}} \frac{2^{-k\varepsilon}}{(2^{-k} + |x_2 - u_2|)^{n_2 + \varepsilon}}.
\end{aligned} \tag{61}$$

The remaining cases are similar but easier. Thus, we complete the proof.

By symmetry, from Lemma 2 and Lemma 3, one can obtain the following result. $\square$

Corollary 2. For any $j, j' \in \mathbb{N}, k, k' \in \mathbb{Z}$, then for any $0 < \varepsilon' < \varepsilon$, there exists a constant $C > 0$ such that

$$\begin{aligned} & \left| \phi_{j',k'} * T(\psi_{j,k}(\cdot - u_1, \cdot - u_2))(x_1, x_2) \right| \\ & \leq C 2^{-|j'-j|\varepsilon'} 2^{-|k'-k|\varepsilon'} \frac{2^{-(j \wedge j')\varepsilon}}{(2^{-(j \wedge j')} + |x_1 - u_1|)^{n_1+\varepsilon}} \frac{2^{-(k \wedge k')\varepsilon}}{(2^{-(k \wedge k')} + |x_2 - u_2|)^{n_2+\varepsilon}}. \end{aligned} \quad (62)$$

Proof of Theorem 1. Substituting the inequality in Corollary 2 to (26) and by a standard estimate, we have

$$\begin{aligned} & \left| \sum_{j' \in \mathbb{N}, k' \in \mathbb{Z}} \sum_{I' \times J' \in \Pi_{j'+N}^{n_1} \times \Pi_{k'+N}^{n_2}} \left| \phi_{j',k'} * (Tf)(x_{I'}, x_{J'}) \right|^2 \chi_{I'}(x) \chi_{J'}(y) \right| \\ & \leq \sum_{j' \in \mathbb{N}, k' \in \mathbb{Z}} \sum_{I' \times J' \in \Pi_{j'+N}^{n_1} \times \Pi_{k'+N}^{n_2}} \left| \sum_{j \in \mathbb{N}, k \in \mathbb{Z}} \sum_{I \times J \in \Pi_{j+N}^{n_1} \times \Pi_{k+N}^{n_2}} |I||J| (\psi_{j,k} * h)(x_I, x_J) \right. \\ & \quad \times \left. \phi_{j',k'} * T(\psi_{j,k}(\cdot - x_I, \cdot - x_J))(x_{I'}, x_{J'}) \right|^2 \chi_{I'}(x) \chi_{J'}(y) \\ & \leq C \sum_{j', k' \in \mathbb{Z}, j' \geq 0} \left[\sum_{j, k \in \mathbb{Z}, j \geq 0} 2^{-|j'-j|\varepsilon'} 2^{-|k'-k|\varepsilon'} C_1 \left\{ \mathcal{M}_s \left(\sum_{R \in \Pi_{j+N}^{n_1} \times \Pi_{k+N}^{n_2}} |\psi_{j,k} * h(x_I, x_J)|^2 \chi_I \chi_J \right)^{r/2} (u, v) \right\}^{1/r} \right]^2, \end{aligned} \quad (63)$$

where $\max\{n_1/(n_1 + \varepsilon), n_2/(n_2 + \varepsilon)\} < r \leq 1$ and $C_1 = C 2^{(1-1/r)[n_1(j' \wedge j - j) + n_2(k' \wedge k - k)]}$. It is easy to have

$$2^{-|j'-j|\varepsilon'} 2^{-|k'-k|\varepsilon'} C_1 \leq C 2^{-|j'-j|[\varepsilon' - ((1/r)-1)n_1]} 2^{-|k'-k|[\varepsilon' - ((1/r)-1)n_2]}, \quad (64)$$

since $|j' \wedge j - j| \leq |j' - j|$, $|k' \wedge k - k| \leq |k' - k|$. One can choose $\varepsilon' < \varepsilon$ close to ε sufficiently such that $\varepsilon' - ((1/r) - 1)n_1, \varepsilon' - ((1/r) - 1)n_2 > 0$. Hence, there exists some $\varepsilon'' > 0$ such that

$$2^{-|j'-j|\varepsilon'} 2^{-|k'-k|\varepsilon'} C_1 \leq C 2^{-|j'-j|\varepsilon''} 2^{-|k'-k|\varepsilon''}. \quad (65)$$

Hence, we obtain

$$\|T(f)\|_{H_{\text{mix}}^p} \leq C \|h\|_{H_{\text{mix}}^p} \approx f_{H_{\text{mix}}^p}. \quad (66)$$

Since $L^2(\mathbb{R}^{n_1+n_2}) \cap H_{\text{mix}}^p$ is dense in H_{mix}^p , we complete the Proof of Theorem 1. $\square$

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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References

- [1] L. Carleson, "A counterexample for measures bounded on H^p for the bidisc," Mittag-Leffler Report, No. 7, 1974.
- [2] S.-Y. A. Chang and R. Fefferman, "A continuous version of duality of H^1 with BMO on the bidisc," *The Annals of Mathematics*, vol. 112, no. 1, pp. 179–201, 1980.
- [3] S.-Y. A. Chang and R. Fefferman, "The Calderon-Zygmund decomposition on product domains," *American Journal of Mathematics*, vol. 104, no. 3, pp. 455–468, 1982.
- [4] S.-Y. A. Chang and R. Fefferman, "Some recent developments in Fourier analysis and H^p -theory on product domains," *Bulletin of the American Mathematical Society*, vol. 12, no. 1, pp. 1–44, 1985.
- [5] R. Fefferman, "Calderon-Zygmund theory for product domains: H^p spaces," *Proceedings of the National Academy of Sciences*, vol. 83, no. 4, pp. 840–843, 1986.

- [6] R. Fefferman, "Harmonic analysis on product spaces," *The Annals of Mathematics*, vol. 126, no. 1, pp. 109–130, 1987.
- [7] R. Fefferman and E. M. Stein, "Singular integrals on product spaces," *Advances in Mathematics*, vol. 45, no. 2, pp. 117–143, 1982.
- [8] S. H. Ferguson and M. T. Lacey, "A characterization of product BMO by commutators," *Acta Mathematica*, vol. 189, no. 2, pp. 143–160, 2002.
- [9] R. F. Gundy and E. M. Stein, "Hp theory for the poly-disc," *Proceedings of the National Academy of Sciences*, vol. 76, no. 3, pp. 1026–1029, 1979.
- [10] Y. Han, C. Lin, G. Lu, Z. Ruan, and E. Sawyer, "Hardy spaces associated with different homogeneities and boundedness of composition operators," *Revista Matemática Iberoamericana*, vol. 29, no. 4, pp. 1127–1157, 2013.
- [11] Y. Han and D. Yang, "Hp boundedness of Calderón-Zygmund operators on product spaces," *Mathematische Zeitschrift*, vol. 249, no. 4, pp. 869–881, 2005.
- [12] J.-L. Journé, "Calderón-zygmund operators on product spaces," *Revista Matemática Iberoamericana*, vol. 1, pp. 55–91, 1985.
- [13] J.-L. Journé, "Two problems of Calderón-Zygmund theory on product-spaces," *Annales de l'institut Fourier*, vol. 38, no. 1, pp. 111–132, 1988.
- [14] C. Muscalu, J. Pipher, T. Tao, and C. Thiele, "Multi-parameter paraproducts," *Revista Matemática Iberoamericana*, vol. 22, pp. 963–976, 2006.
- [15] J. Pipher, "Journé's covering lemma and its extension to higher dimensions," *Duke Mathematical Journal*, vol. 53, pp. 683–690, 1986.
- [16] Y. Han, G. Lu, and E. Sawyer, "Flag hardy spaces and Marcinkiewicz multipliers on the Heisenberg group," *Analysis & PDE*, vol. 7, no. 7, pp. 1465–1534, 2014.
- [17] W. Ding and Y. Zhu, "Mixed hardy spaces and their applications," *Acta Mathematica Scientia*.
- [18] D. Goldberg, "A local version of real hardy spaces," *Duke Mathematical Journal*, vol. 46, no. 1, pp. 27–42, 1979.
- [19] Y. Han and E. Sawyer, "Para-accretive functions, the weak boundedness property and the T_b Theorem," *Revista Matemática Iberoamericana*, vol. 6, pp. 17–41, 1990.