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Research Article

Rapid Convergence of Approximate Solutions for Fractional Differential Equations

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In this paper, we develop a generalized quasilinearization technique for a class of Caputo's fractional differential equations when the forcing function is the sum of hyperconvex and hyperconcave functions of order m ($m \geq 0$), and we obtain the convergence of the sequences of approximate solutions by establishing the convergence of order k ($k \geq 2$).

1. Introduction

Fractional differential equations have received attention from some researchers because they have extensive application in mechanics, biochemistry, electrical engineering, medicine, and many other fields (see [1–6]). For more information about the basic theory of fractional differential equations, we can refer to the monographs [7–9] and references cited therein. It is well known [10] that the monotone iterative technique offers an approach for obtaining approximate solutions to a wide variety of nonlinear differential equations. Recently, there are some results on the monotone sequences of approximate solutions converging uniformly to a solution of fractional differential equations by employing monotone iterative technique and generalized monotone iterative method coupled with the method of upper and lower solutions, which can be found in [11–21].

In view of applications, it is very significant to study the rate of convergence of solutions. The quasilinearization method [22] is one of the effective methods to obtain a sequence of approximate solutions with quadratic convergence, and it is extremely useful in scientific computations due to its accelerated rate of convergence as in [23, 24]. A few results of quadratic convergence for fractional differential equations were also obtained by applying quasilinearization, such as the initial value problem of Caputo's fractional differential equations [13, 25, 26], fractional differential equations via initial time different lower and upper solutions [27], and the system of fractional differential equations [28]. However, to the best of our knowledge, there are few results of rapid convergence of fractional differential equations. Recently, Wang and others obtained the results on rapid convergence of solutions for various differential equations [29–33]. Inspired and motivated by [34, 35], in the present paper, we will discuss the rapid convergence of approximate solutions of fractional differential equations when the forcing function is the sum of hyperconvex and hyperconcave functions with coupled lower and upper solutions, and construct sequences of approximate solutions that converge rapidly to the extremal solutions of (1) by using an improved quasilinearization method (rate of convergence $k \geq 2$).

2. Preliminaries

Consider the initial value problem of Caputo's fractional differential equations (IVP):

$$ {}^{c}D^{q} x = f(t, x) + g(t, x), \quad x(t_0) = x_0, \tag{1}$$

where $f, g : J \times R \longrightarrow R$ are continuous functions, $J = [t_0, T]$ and $0 < q < 1$.

A function $x(t)$ is called a solution of IVP (1) if it satisfies (1).

Firstly, we give the following definitions and lemmas.

Definition 1. The locally Hölder continuous functions α_0, $\beta_0 : J \longrightarrow R$ are coupled lower and upper solutions of type I of IVP (1) if the following inequalities hold:

$$\begin{cases} {}^cD^q\alpha_0 \le f(t,\alpha_0) + g(t,\beta_0), & \alpha_0(t_0) \le x_0, \\ {}^cD^q\beta_0 \ge f(t,\beta_0) + g(t,\alpha_0), & \beta_0(t_0) \ge x_0. \end{cases} \quad (2)$$

Definition 2. The locally Hölder continuous functions α_0, $\beta_0 : J \longrightarrow R$ are coupled lower and upper solutions of type II of IVP (1) if the following inequalities hold:

$$\begin{cases} {}^cD^q\alpha_0 \le f(t,\beta_0) + g(t,\alpha_0), & \alpha_0(t_0) \le x_0, \\ {}^cD^q\beta_0 \ge f(t,\alpha_0) + g(t,\beta_0), & \beta_0(t_0) \ge x_0. \end{cases} \quad (3)$$

Let $f^{(k)}(t,x) = \partial^k f(t,x)/\partial x^k$ denote the kth partial of f with respect to $x, k \in \mathbf{N}$. $||x|| = \sup_{t\in J}|x(t)|$.

Definition 3. A function f is called m-hyperconvex, $m \in \mathbf{N}$, if $f^{(m+1)}(t,x) \ge 0$; f is called m-hyperconcave if the inequality is reversed.

The linear IVP of Caputo's fractional differential equation is given by

$$ {}^cD^q x = \lambda x + \theta(t), \quad x(t_0) = x_0, \quad (4)$$

where $\theta : J \longrightarrow R$ is a locally Hölder continuous function, $t \in J, \lambda \in R$. The unique solution of (4) can be expressed in the following form [7]:

$$x(t) = x_0 E_q(\lambda(t-t_0)^q) + \int_{t_0}^{t} \left[(t-s)^{q-1} E_{q,q}(\lambda(t-s)^q)\theta(s)\right] ds, \quad (5)$$

where

$$E_q(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(qk+1)}, \qquad E_{q,q}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(qk+q)}, \quad (6)$$

are Mittag-Leffler's functions, and Γ denotes the *Gamma* function.

Remark 4. We note that (5) remains valid if x and $\theta(t)$ are functions mapping from $J \longrightarrow R^n$, $x_0 \in R^n$, and λ is an $n \times n$ matrix.

We need the following lemmas to prove our main results, which proofs can be found in literature [26].

Lemma 5. *Assume that $x : J \longrightarrow R$ is a locally Hölder continuous function such that for $t_1 \in (t_0, T]$, $x(t_1) = 0$ and $x(t) < 0$ for $t \in [t_0, t_1)$. Then ${}^cD^q x(t_1) \ge 0$.*

Lemma 6. *Assume that $\{x_n(t)\}$ is a family of continuous functions on J. For each $n > 0$ and the Caputo's fractional differential equations, we have the following:*

$$ {}^cD^q x_n(t) = f(t, x_n(t)), \quad x_n(t_0) = x_0, \quad (7)$$

the functions $f(t, x_n(t))$ satisfy $|f(t, x_n(t))| \le M$ for $t \in J$. Then, the family $\{x_n(t)\}$ is equicontinuous on J.

In our further investigations, we need the following comparison results.

Lemma 7. *Assume that one of the following conditions holds:*

$H_1 \alpha_0, \beta_0$ are coupled lower and upper solutions of type I of (1) and

$$\begin{aligned} f(t,x_1) - f(t,x_2) &\le L(x_1 - x_2), \\ g(t,x_1) - g(t,x_2) &\ge -L(x_1 - x_2), \end{aligned} \quad (8)$$

where $x_1 \ge x_2$, $L \ge 0$ is a constant

$H_2 \alpha_0, \beta_0$ are coupled lower and upper solutions of type II of (1) and

$$\begin{aligned} f(t,x_1) - f(t,x_2) &\ge -L(x_1 - x_2), \\ g(t,x_1) - g(t,x_2) &\le L(x_1 - x_2), \end{aligned} \quad (9)$$

where $x_1 \ge x_2$, $L \ge 0$ is a constant

Then, $\alpha_0(t_0) \le \beta_0(t_0)$ implies $\alpha_0(t) \le \beta_0(t)$ on J.

Proof. Firstly, we prove that the conclusion is valid when H_1 holds. To do this, let $\tilde{\beta}_0(t) = \beta_0(t) + \varepsilon E_q(3L(t-t_0)^q)$ and $\tilde{\alpha}_0(t) = \alpha_0(t) - \varepsilon E_q(3L(t-t_0)^q)$ for any small $\varepsilon > 0$ so that $\tilde{\beta}_0(t) > \beta_0(t)$, $\tilde{\alpha}_0(t) < \alpha_0(t)$, and $\tilde{\beta}_0(t_0) > \beta_0(t_0) \ge \alpha_0(t_0) > \tilde{\alpha}_0(t_0)$. Then, in view of H_1, we have

$$\begin{aligned} {}^cD^q\tilde{\alpha}_0 &\le f(t,\alpha_0) + g(t,\beta_0) - 3L\varepsilon E_q(3L(t-t_0)q) \\ &< f(t,\tilde{\alpha}_0) + g\left(t,\tilde{\beta}_0\right). \end{aligned} \quad (10)$$

Similarly, we have ${}^cD^q\tilde{\beta}_0 > f(t,\tilde{\beta}_0) + g(t,\tilde{\alpha}_0)$.

We next show that $\tilde{\alpha}_0(t) < \tilde{\beta}_0(t)$ on J, which proves the conclusion as $\varepsilon \longrightarrow 0$. Letting $Q(t) = \tilde{\alpha}_0(t) - \tilde{\beta}_0(t)$, suppose that $Q(t) < 0$ is not true on J. Then, there exists a $t_1 \in (t_0, T]$ such that $Q(t_1) = 0$ and $Q(t) < 0$ for $t \in [t_0, t_1)$. In view of Lemma 5, ${}^cD^q Q(t_1) \ge 0$, that is, ${}^cD^q\tilde{\alpha}_0(t_1) \ge {}^cD^q\tilde{\beta}_0(t_1)$. Thus, we arrive at the following contradiction:

$$\begin{aligned} f(t_1,\tilde{\alpha}_0(t_1)) + g\left(t_1,\tilde{\beta}_0(t_1)\right) &> {}^cD^q\tilde{\alpha}_0(t_1) \ge {}^cD^q\tilde{\beta}_0(t_1) \\ &> f\left(t_1,\tilde{\beta}_0(t_1)\right) + g(t_1,\tilde{\alpha}_0(t_1)). \end{aligned} \quad (11)$$

Similar to the proof process above, we can obtain the result of Lemma 7 when H_2 holds.

Lemma 8. *Assume that* $f, g \in C[\Omega, R]$*, where* $\Omega = \{(t, x)\colon \alpha_0(t) \le x \le \beta_0(t),\ t \in J\}$*, and one of the following conditions holds:*

H_3 α_0, β_0 are coupled lower and upper solutions of type I of (1) such that $\alpha_0(t) \le \beta_0(t)$ on J, and $g(t, x)$ is monotone nonincreasing in x for $t \in J$

H_4 α_0, β_0 are coupled lower and upper solutions of type II of (1) such that $\alpha_0(t) \le \beta_0(t)$ on J, and $f(t, x)$ is monotone nonincreasing in x for $t \in J$

Then, there exists a solution $x(t)$ of (1) satisfying $\alpha_0(t) \le x(t) \le \beta_0(t)$ on J.

Proof. Suppose that H_3 holds. Consider the mapping $P : J \times R \longrightarrow R$ defined by

$$P(t, x) = \max \{\alpha_0(t), \min [x(t), \beta_0(t)]\}. \tag{12}$$

Then

$${}^cD^q x = f(t, P(t, x)) + g(t, P(t, x)), \tag{13}$$

and it has a solution $x(t)$ on J with $x(t_0) = x_0$.

Firstly, we prove that $\alpha_0(t) \le x(t)$ on J. Letting $\bar{\alpha}_0(t) = \alpha_0(t) - \varepsilon E_q(L(t - t_0)^q)$, for any small $\varepsilon > 0$ such that $\bar{\alpha}_0(t) < \alpha_0(t)$ on J. We can prove that $\bar{\alpha}_0(t) < x(t)$ on J, which shows that $\alpha_0(t) \le x(t)$ as $\varepsilon \longrightarrow 0$. Setting $m(t) = \bar{\alpha}_0(t) - x(t)$, suppose $m(t) < 0$ is not true on J, then there exists a $t_2 \in (t_0, T]$ such that $m(t_2) = 0$ and $m(t) < 0$ for $t \in [t_0, t_2)$. From Lemma 5, it follows that ${}^cD^q m(t_2) \ge 0$, that is, ${}^cD^q \bar{\alpha}_0(t_2) \ge {}^cD^q x(t_2)$. Therefore,

$${}^cD^q x(t_2) \le {}^cD^q \bar{\alpha}_0(t_2) < {}^cD^q \alpha_0(t_2) \le f(t_2, \alpha_0(t_2)) + g(t_2, \beta_0(t_2)). \tag{14}$$

On the other hand, we have

$$\begin{aligned} {}^cD^q x(t_2) &= f(t_2, P(t_2, x)) + g(t_2, P(t_2, x)) \\ &\ge f(t_2, \alpha_0(t_2)) + g(t_2, \beta_0(t_2)), \end{aligned} \tag{15}$$

and it contradicts with (14). This contradiction proves the claim.

Similarly, letting $\bar{\beta}_0(t) = \beta_0(t) + \varepsilon E_q(L(t - t_0)^q)$, we can find that $x(t) \le \beta_0(t)$.

It is easy to construct the proofs of the results relative to H_4. We omit the details.

To obtain the results of this paper, we need to consider two-dimensional Caputo's fractional differential systems:

$${}^cD^q X = H(t, X), \quad X(t_0) = X_0, \tag{16}$$

where $H \in C[J \times R^2, R^2]$ and $X : J \longrightarrow R^2$ is a locally Hölder continuous function.

Lemma 9. *Assume that $v, w : J \longrightarrow R^2$ are locally Hölder continuous functions satisfying the following:*

$$\begin{aligned} &{}^cD^q v \le H(t, v), \\ &v(t_0) \le X_0, \\ &{}^cD^q w \ge H(t, w), \\ &w(t_0) \ge X_0, \end{aligned} \tag{17}$$

and whenever $X \ge Y$,

$$H_i(t, x_1, x_2) - H_i(t, y_1, y_2) \le L[(x_1 - y_1) + (x_2 - y_2)], \tag{18}$$

where $L \ge 0$ is a constant, $i = 1, 2$. Then $v(t_0) \le w(t_0)$ implies $v(t) \le w(t)$ on J.

Proof. Let $\bar{w}(t) = w(t) + \varepsilon E_q(2L(t - t_0)^q)$ and $\bar{v}(t) = v(t) - \varepsilon E_q(2L(t - t_0)^q)$ for any small $\varepsilon > 0$ and $\varepsilon = (\varepsilon_1, \varepsilon_2)$ so that $\bar{w}(t) > w(t)$, $\bar{v}(t) < v(t)$, and $\bar{w}(t_0) > w(t_0) \ge v(t_0) > \bar{v}(t_0)$. Consequently, we obtain, for each i

$$\begin{aligned} {}^cD^q \bar{v}_i &= {}^cD^q v_i - 2L\varepsilon_i E_q(2L(t - t_0)^q) \\ &\le H_i(t, \bar{v}_1, \bar{v}_2) - L\varepsilon_i E_q(3L(t - t_0)^q) \\ &< H_i(t, \bar{v}_1, \bar{v}_2). \end{aligned} \tag{19}$$

Similarly, we have

$${}^cD^q \bar{w}_i = {}^cD^q w_i + 2L\varepsilon_i E_q(2L(t - t_0)^q) > H_i(t, \bar{w}_1, \bar{w}_2). \tag{20}$$

We next prove that $\bar{v}(t) < \bar{w}(t)$ on J, which shows the required conclusion as $\varepsilon \longrightarrow 0$. Suppose that $\bar{v}(t) < \bar{w}(t)$ is not true on J, then there exists an index j, and a $t_3 \in (t_0, T]$ such that $\bar{v}_j(t_3) = \bar{w}_j(t_3)$ and $\bar{v}_j(t) < \bar{w}_j(t)$ for $t \in [t_0, t_3)$. Set $m(t) = \bar{v}_j(t) - \bar{w}_j(t)$, it then follows from Lemma 5 that ${}^cD^q m(t_3) \ge 0$, that is ${}^cD^q \bar{v}_j(t_3) \ge {}^cD^q \bar{w}_j(t_3)$. Furthermore,

$$H_j(t_3, \bar{v}_1(t_3), \bar{v}_2(t_3)) > {}^cD^q \bar{v}_j(t_3) > H_j(t_3, \bar{w}_1(t_3), \bar{w}_2(t_3)), \tag{21}$$

which leads to a contradiction. This completes the proof.

3. Main Results

In this section, we consider that $f(t, x)$ and $g(t, x)$ are hyperconvex and hyperconcave in x of order $m - 1$, respectively. We first give some inequalities depending on whether m is even or odd [34]:

(i) $m = 2k$

$$f(t,\eta) \geq \sum_{i=0}^{2k-1} \frac{f^{(i)}(t,\xi)(\eta-\xi)^i}{i!}, \tag{22}$$

$$f(t,\eta) \leq \sum_{i=0}^{2k-2} \frac{f^{(i)}(t,\xi)(\eta-\xi)^i}{i!} + \frac{f^{(2k-1)}(t,\eta)(\eta-\xi)^{2k-1}}{(2k-1)!}, \tag{23}$$

$$g(t,\eta) \leq \sum_{i=0}^{2k-1} \frac{g^{(i)}(t,\xi)(\eta-\xi)^i}{i!}, \tag{24}$$

$$g(t,\eta) \geq \sum_{i=0}^{2k-2} \frac{g^{(i)}(t,\xi)(\eta-\xi)^i}{i!} + \frac{g^{(2k-1)}(t,\eta)(\eta-\xi)^{2k-1}}{(2k-1)!}. \tag{25}$$

(ii) $m = 2k+1$

$$f(t,\eta) \leq \sum_{i=0}^{2k-1} \frac{f^{(i)}(t,\xi)(\eta-\xi)^i}{i!} + \frac{f^{(2k)}(t,\eta)(\eta-\xi)^{2k}}{(2k)!}, \quad \eta \geq \xi, \tag{26}$$

$$f(t,\eta) \geq \sum_{i=0}^{2k-1} \frac{f^{(i)}(t,\xi)(\eta-\xi)^i}{i!} + \frac{f^{(2k)}(t,\eta)(\eta-\xi)^{2k}}{(2k)!}, \quad \eta \leq \xi, \tag{27}$$

$$g(t,\eta) \geq \sum_{i=0}^{2k-1} \frac{g^{(i)}(t,\xi)(\eta-\xi)^i}{i!} + \frac{g^{(2k)}(t,\eta)(\eta-\xi)^{2k}}{(2k)!}, \quad \eta \geq \xi, \tag{28}$$

$$g(t,\eta) \leq \sum_{i=0}^{2k-1} \frac{g^{(i)}(t,\xi)(\eta-\xi)^i}{i!} + \frac{g^{(2k)}(t,\eta)(\eta-\xi)^{2k}}{(2k)!}, \quad \eta \leq \xi, \tag{29}$$

Based on the above inequalities, we have the following result which is relative to the coupled lower and upper solutions of type I in Definition 1 when m is even.

Theorem 10. *Consider the following assumptions:*

A_1 α_0, β_0 are coupled lower and upper solutions of type I of (1) with $\alpha_0 \leq \beta_0$ on J

A_2 $f, g \in C^{2k}[\Omega, R]$ such that $f(t,x)$ and $g(t,x)$ are the hyperconvex and hyperconcave in x of the order $2k-1$, respectively

A_3 $g_x(t,x)$ satisfies

$$g_x(t,x) \leq \min\left[g^{(2k)}(t,x)\right] \frac{(\beta_0-\alpha_0)^{2k-1}}{(2k-2)!} \leq 0 \text{ on } \Omega \tag{30}$$

Then, there exist monotone sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ converging uniformly to the solution $x(t)$ of (1) on J and the convergence is of the order $2k$.

Proof. It follows from assumption A_2 that the inequalities (22), (23), (24), and (25) hold. Consider the following fractional differential equations:

$$\begin{aligned} {}^cD^q v(t) &= F(t,\alpha,\beta;v,w) \\ &\triangleq \sum_{i=0}^{2k-1} \frac{f^{(i)}(t,\alpha)(v-\alpha)^i}{i!} \\ &\quad + \sum_{i=0}^{2k-2} \frac{g^{(i)}(t,\beta)(w-\beta)^i}{i!} \\ &\quad + \frac{g^{(2k-1)}(t,\alpha)(w-\beta)^{2k-1}}{(2k-1)!}, \quad v(t_0)=x_0, \end{aligned} \tag{31}$$

$$\begin{aligned} {}^cD^q w(t) &= G(t,\alpha,\beta;w,v) \\ &\triangleq \sum_{i=0}^{2k-2} \frac{f^{(i)}(t,\beta)(w-\beta)^i}{i!} \\ &\quad + \sum_{i=0}^{2k-1} \frac{g^{(i)}(t,\alpha)(v-\alpha)^i}{i!} \\ &\quad + \frac{f^{(2k-1)}(t,\alpha)(w-\beta)^{2k-1}}{(2k-1)!}, \quad w(t_0)=x_0. \end{aligned} \tag{32}$$

Firstly, applying (31) and (32) and taking $\alpha = \alpha_0$, $\beta = \beta_0$, we obtain

$${}^cD^q v(t) = F(t,\alpha_0,\beta_0;v,w), \quad v(t_0)=x_0, \tag{33}$$

$${}^cD^q w(t) = G(t,\alpha_0,\beta_0;w,v), \quad w(t_0)=x_0. \tag{34}$$

Condition A_1 and inequalities (22), (23), (24), and (25) imply

$$\begin{aligned} {}^cD^q\alpha_0 &\leq f(t,\alpha_0)+g(t,\beta_0) = F(t,\alpha_0,\beta_0;\alpha_0,\beta_0), \quad \alpha_0(t_0)\leq x_0, \\ {}^cD^q\beta_0 &\geq \sum_{i=0}^{2k-1} \frac{f^{(i)}(t,\alpha_0)(\beta_0-\alpha_0)^i}{i!} \\ &\quad + \sum_{i=0}^{2k-2} \frac{g^{(i)}(t,\beta_0)(\alpha_0-\beta_0)^i}{i!} \\ &\quad + \frac{g^{(2k-1)}(t,\alpha_0)(\alpha_0-\beta_0)^{2k-1}}{(2k-1)!} \\ &= F(t,\alpha_0,\beta_0;\beta_0,\alpha_0), \quad \beta_0(t_0)\geq x_0, \\ {}^cD^q\alpha_0 &\leq \sum_{i=0}^{2k-2} \frac{f^{(i)}(t,\beta_0)(\alpha_0-\beta_0)^i}{i!} \\ &\quad + \frac{f^{(2k-1)}(t,\alpha_0)(\alpha_0-\beta_0)^{2k-1}}{(2k-1)!} \\ &\quad + \sum_{i=0}^{2k-1} \frac{g^{(i)}(t,\alpha_0)(\alpha_0-\beta_0)^i}{i!} \\ &= G(t,\alpha_0,\beta_0;\alpha_0,\beta_0), \quad \alpha_0(t_0)\leq x_0, \\ {}^cD^q\beta_0 &\geq f(t,\beta_0)+g(t,\alpha_0) = G(t,\alpha_0,\beta_0;\beta_0,\alpha_0), \quad \beta_0(t_0)\geq x_0. \end{aligned} \tag{35}$$

Employing assumption A_3 and the Taylor series expansion with the Lagrange remainder, we get

$$F_w(t,\alpha_0,\beta_0;v,w)=g_w(t,w)-\frac{g^{(2k)}(t,\xi_2)(w-\beta_0)^{2k-2}(\xi_1-\alpha_0)}{(2k-2)!}\le 0,$$
$$G_v(t,\alpha_0,\beta_0;w,v)=g_v(t,v)-\frac{g^{(2k)}(t,\eta_1)(v-\alpha_0)^{2k-1}}{(2k-1)!}\le 0, \tag{36}$$

where $\alpha_0\le\xi_2\le\xi_1\le\beta_0$ and $\alpha_0\le\eta_1\le\beta_0$. Therefore, $F(t,\alpha_0,\beta_0;v,w)$ and $G(t,\alpha_0,\beta_0;w,v)$ are nonincreasing in w and v, respectively. By Lemma 8, there exist solutions $\alpha_1(t)$ and $\beta_1(t)$ of (33), (34) on J such that $\alpha_0(t)\le\alpha_1(t)\le\beta_0(t)$ and $\alpha_0(t)\le\beta_1(t)\le\beta_0(t)$. Furthermore, in view of the inequalities (22), (23), (24), and (25) and condition A_2, we have

$$\begin{aligned}{}^cD^q\alpha_1&=F(t,\alpha_0,\beta_0;\alpha_1,\beta_1)\le f(t,\alpha_1)+g(t,\beta_1),\\ &\qquad\alpha_1(t_0)=x_0,\\ {}^cD^q\beta_1&=G(t,\alpha_0,\beta_0;\beta_1,\alpha_1)\ge f(t,\beta_1)+g(t,\alpha_1),\\ &\qquad\beta_1(t_0)=x_0,\end{aligned} \tag{37}$$

and in view of H_1, we have $\alpha_1(t)\le\beta_1(t)$ for $t\in J$. Hence, $\alpha_0(t)\le\alpha_1(t)\le\beta_1(t)\le\beta_0(t)$.

By induction, for all n, we can obtain that

$$\alpha_0(t)\le\alpha_1(t)\le\cdots\le\alpha_n(t)\le\beta_n(t)\le\cdots\le\beta_1(t)\le\beta_0(t), \tag{38}$$

where α_n and β_n are solutions of

$$\begin{aligned}{}^cD^q v(t)&=F(t,\alpha_{n-1},\beta_{n-1};v,w),\quad v(t_0)=x_0,\\ {}^cD^q w(t)&=G(t,\alpha_{n-1},\beta_{n-1};w,v),\quad w(t_0)=x_0,\end{aligned} \tag{39}$$

and

$$\begin{aligned}{}^cD^q\alpha_n&=F(t,\alpha_{n-1},\beta_{n-1};\alpha_n,\beta_n)\le f(t,\alpha_n)+g(t,\beta_n),\\ &\qquad\alpha_n(t_0)=x_0,\end{aligned} \tag{40}$$

$$\begin{aligned}{}^cD^q\beta_n&=G(t,\alpha_{n-1},\beta_{n-1};\beta_n,\alpha_n)\ge f(t,\beta_n)+g(t,\alpha_n),\\ &\qquad\beta_n(t_0)=x_0.\end{aligned} \tag{41}$$

According to (40) and (41), α_n and β_n are coupled lower and upper solutions of type I of (1). We have $g_x\le 0$ on Ω from assumption A_3. It then follows from Lemma 8 that $x(t)$ is a solution of (1) on J satisfying $\alpha_n(t)\le x(t)\le\beta_n(t)$. Hence, we have

$$\alpha_0(t)\le\alpha_1(t)\le\cdots\le\alpha_n(t)\le x\le\beta_n(t)\le\cdots\le\beta_1(t)\le\beta_0(t). \tag{42}$$

By (42), the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are uniformly bounded on J. From Lemma 6, the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are equicontinuous on J. Consequently, by employing the Ascoli-Arzela Theorem, sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are uniformly convergent on J.

Finally, we prove that the convergence of $\{\alpha_n\}$ and $\{\beta_n\}$ is of the order $2k$.

Set $W_n(t)=x(t)-\alpha_n(t)\ge 0$ and $V_n(t)=\beta_n(t)-x(t)\ge 0$ for $t\in J$ with $W_n(t_0)=V_n(t_0)=0$. Using the known conditions and the mean value theorem, we obtain

$$\begin{aligned}{}^cD^qW_{n+1}(t)&=f(t,x)+g(t,x)\\ &\quad-\Bigg[\sum_{i=0}^{2k-1}\frac{f^{(i)}(t,\alpha_n)(\alpha_{n+1}-\alpha_n)^i}{i!}\\ &\quad+\sum_{i=0}^{2k-2}\frac{g^{(i)}(t,\beta_n)(\beta_{n+1}-\beta_n)^i}{i!}\\ &\quad+\frac{g^{(2k-1)}(t,\alpha_n)(\beta_{n+1}-\beta_n)^{2k-1}}{(2k-1)!}\Bigg]\\ &=f(t,x)+g(t,x)\\ &\quad-\Bigg[f(t,\alpha_{n+1})-\frac{f^{(2k)}(t,\delta_1)(\alpha_{n+1}-\alpha_n)^{2k}}{(2k)!}\\ &\quad+g(t,\beta_{n+1})-\frac{g^{(2k-1)}(t,\delta_2)(\beta_{n+1}-\beta_n)^{2k-1}}{(2k-1)!}\\ &\quad+\frac{g^{(2k-1)}(t,\alpha_n)(\beta_{n+1}-\beta_n)^{2k-1}}{(2k-1)!}\Bigg]\\ &=f_x(t,\delta_3)(x-\alpha_{n+1})-g_x(t,\delta_4)(\beta_{n+1}-x)\\ &\quad+\frac{f^{(2k)}(t,\delta_1)(\alpha_{n+1}-\alpha_n)^{2k}}{(2k)!}\\ &\quad-\frac{g^{(2k)}(t,\delta_5)(\beta_n-\beta_{n+1})^{2k-1}(\delta_2-\alpha_n)}{(2k-1)!}\\ &\le f_x(t,\delta_3)W_{n+1}(t)-g_x(t,\delta_4)V_{n+1}(t)\\ &\quad+\frac{f^{(2k)}(t,\delta_1)(x-\alpha_n)^{2k}}{(2k)!}\\ &\quad-\frac{g^{(2k)}(t,\delta_5)(\beta_n-x)^{2k-1}[(\beta_n-x)+(x-\alpha_n)]}{(2k-1)!}\\ &=f_x(t,\delta_3)W_{n+1}-g_x(t,\delta_4)V_{n+1}\\ &\quad+\frac{f^{(2k)}(t,\delta_1)W_n^{2k}}{(2k)!}\\ &\quad-\frac{g^{(2k)}(t,\delta_5)V_n^{2k-1}(V_n+W_n)}{(2k-1)!}\\ &\le k_1W_{n+1}+k_2V_{n+1}+k_3W_n^{2k}+k_4V_n^{2k-1}(V_n+W_n),\end{aligned} \tag{43}$$

where $\alpha_n\le\delta_1\le\alpha_{n+1}\le\delta_3\le x\le\delta_4\le\beta_{n+1}\le\delta_2\le\beta_n$, $\alpha_n\le\delta_5\le\delta_2$, $0\le f_x(t,x)\le k_1$, and $0\le -g_x(t,x)\le k_2$ and it follows from assumption A_2 that $f^{(2k)}$ and $g^{(2k)}$ are bounded on Ω, that is, $0\le f^{(2k)}(t,x)/(2k)!\le k_3$, $0\le -g^{(2k)}(t,x)/(2k-1)!\le k_4$.

Similarly, we have

$$\begin{aligned}
{}^cD^qV_{n+1}(t) = &\sum_{i=0}^{2k-2}\frac{f^{(i)}(t,\beta_n)(\beta_{n+1}-\beta_n)^i}{i!} \\
&+\frac{f^{(2k-1)}(t,\alpha_n)(\beta_{n+1}-\beta_n)^{2k-1}}{(2k-1)!} \\
&+\sum_{i=0}^{2k-1}\frac{g^{(i)}(t,\alpha_n)(\alpha_{n+1}-\alpha_n)^i}{i!} \\
&-f(t,x)-g(t,x) \\
=& f(t,\beta_{n+1})-\frac{f^{(2k-1)}(t,\sigma_1)(\beta_{n+1}-\beta_n)^{2k-1}}{(2k-1)!} \\
&+\frac{f^{(2k-1)}(t,\alpha_n)(\beta_{n+1}-\beta_n)^{2k-1}}{(2k-1)!} \\
&+g(t,\alpha_{n+1})-\frac{g^{(2k)}(t,\sigma_2)(\alpha_{n+1}-\alpha_n)^{2k}}{(2k)!} \\
&-f(t,x)-g(t,x)=f_x(t,\sigma_3)(\beta_{n+1}-x) \\
&-g_x(t,\sigma_4)(x-\alpha_{n+1}) \\
&+\frac{f^{(2k)}(t,\sigma_5)(\beta_n-\beta_{n+1})^{2k-1}(\sigma_1-\alpha_n)}{(2k-1)!} \\
&-\frac{g^{(2k)}(t,\sigma_2)(\alpha_{n+1}-\alpha_n)^{2k}}{(2k)!} \\
\le& f_x(t,\sigma_3)V_{n+1}-g_x(t,\sigma_4)W_{n+1} \\
&-\frac{g^{(2k)}(t,\sigma_2)(x-\alpha_n)^{2k}}{(2k)!} \\
&+\frac{f^{(2k)}(t,\sigma_5)(\beta_n-x)^{2k-1}[(\beta_n-x)+(x-\alpha_n)]}{(2k-1)!} \\
=& f_x(t,\sigma_3)V_{n+1}-g_x(t,\sigma_4)W_{n+1} \\
&+\frac{f^{(2k)}(t,\sigma_5)V_n^{2k-1}(V_n+W_n)}{(2k-1)!} \\
&-\frac{g^{(2k)}(t,\sigma_2)W_n^{2k}}{(2k)!} \\
\le& k_1V_{n+1}+k_2W_{n+1}+k_5V_n^{2k-1}(V_n+W_n) \\
&+k_6W_n^{2k},
\end{aligned} \tag{44}$$

where

$$\begin{aligned}
&\alpha_n\le\sigma_2\le\alpha_{n+1}\le\sigma_4\le x\le\sigma_3\le\beta_{n+1}\le\sigma_1\le\beta_n, \\
&\alpha_n\le\sigma_5\le\sigma_1, \\
&0\le\frac{f^{(2k)}(t,x)}{(2k-1)}\le k_5, \\
&0\le-\frac{g^{(2k)}(t,x)}{(2k)!}\le k_6.
\end{aligned} \tag{45}$$

For the following inequalities

$$\begin{aligned}
{}^cD^qW_{n+1}(t)\le{}& k_1W_{n+1}+k_2V_{n+1} \\
&+\left\|k_3W_n^{2k}+k_4V_n^{2k-1}(V_n+W_n)\right\|, \quad W_{n+1}(t_0)=0, \\
{}^cD^qV_{n+1}(t)\le{}& k_1V_{n+1}+k_2W_{n+1} \\
&+\left\|k_5V_n^{2k-1}(V_n+W_n)+k_6W_n^{2k}\right\|, \quad V_{n+1}(t_0)=0,
\end{aligned} \tag{46}$$

we can get

$${}^cD^q\Phi\le\lambda\Phi+\Psi, \quad \Phi(t_0)=\Psi_0, \tag{47}$$

where

$$\begin{aligned}
\Phi&=\begin{pmatrix}W_{n+1}\\ V_{n+1}\end{pmatrix}, \\
\lambda&=\begin{pmatrix}k_1 & k_2\\ k_2 & k_1\end{pmatrix}, \\
\Psi&=\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}, \\
\Psi_0&=\begin{pmatrix}0\\ 0\end{pmatrix},
\end{aligned} \tag{48}$$

in which

$$\begin{aligned}
\Psi_1&=\left\|k_3W_n^{2k}+k_4V_n^{2k-1}(V_n+W_n)\right\|, \\
\Psi_2&=\left\|k_5V_n^{2k-1}(V_n+W_n)+k_6W_n^{2k}\right\|.
\end{aligned} \tag{49}$$

Formulas (4) and (5) and Lemma 9 imply

$$\Phi(t)\le\int_{t_0}^t(t-s)^{q-1}E_{q,q}(\lambda(t-s)^q)\,ds\,\Psi\le M\Psi, \tag{50}$$

where

$$\begin{aligned}
M&=\begin{pmatrix}\frac{(T-t_0)^q}{q}E_{q,q}(k_1(T-t_0)^q) & \frac{(T-t_0)^q}{q}E_{q,q}(k_2(T-t_0)^q)\\ \frac{(T-t_0)^q}{q}E_{q,q}(k_2(T-t_0)^q) & \frac{(T-t_0)^q}{q}E_{q,q}(k_1(T-t_0)^q)\end{pmatrix} \\
&=\begin{pmatrix}M_1 & M_2\\ M_2 & M_1\end{pmatrix}.
\end{aligned} \tag{51}$$

Hence, we have

$$\begin{aligned} W_{n+1}(t) &\leq M_1 \left\| k_3 W_n^{2k} + k_4 V_n^{2k-1}(V_n + W_n) \right\| \\ &\quad + M_2 \left\| k_5 V_n^{2k-1}(V_n + W_n) + k_6 W_n^{2k} \right\| \\ &\leq M_1 \left(k_3 \| W_n \|^{2k} + k_4 \| V_n \|^{2k-1} (\| V_n \| + \| W_n \|) \right) \\ &\quad + M_2 \left(k_5 \| V_n \|^{2k-1} (\| V_n \| + \| W_n \|) + k_6 \| W_n \|^{2k} \right) \\ &= c_1 \| W_n \|^{2k} + c_2 \| V_n \|^{2k-1} (\| V_n \| + \| W_n \|), \end{aligned} \tag{52}$$

where $c_1 = M_1 k_3 + M_2 k_6$, $c_2 = M_1 k_4 + M_2 k_5$. Similarly, we get

$$V_{n+1}(t) \leq c_3 \| W_n \|^{2k} + c_4 \| V_n \|^{2k-1} (\| V_n \| + \| W_n \|). \tag{53}$$

Employing the Binomial Theorem

$$\begin{aligned} (a+b)^{2k} &= C_{2k}^0 a^{2k} + C_{2k}^1 a^{2k-1} b + \cdots + C_{2k}^r a^{2k-r} b^r + \cdots + C_{2k}^{2k} b^{2k} \\ &\geq a^{2k} + b^{2k} + a^{2k-1} b, \end{aligned} \tag{54}$$

and the inequalities (52) and (53), we obtain

$$\begin{aligned} W_{n+1} + V_{n+1}(t) &\leq (c_1 + c_3) \| W_n \|^{2k} \\ &\quad + (c_2 + c_4) \| V_n \|^{2k-1} (\| V_n \| + \| W_n \|) \\ &= c_5 \| W_n \|^{2k} + c_6 \| V_n \|^{2k} + c_6 \| V_n \|^{2k-1} \| W_n \| \\ &\leq c_7 (\| W_n \| + \| V_n \|)^{2k}, \end{aligned} \tag{55}$$

that is,

$$\| x(t) - \alpha_{n+1}(t) \| + \| \beta_{n+1}(t) - x(t) \| \leq c_7 [\| x(t) - \alpha_n(t) \| + \| \beta_{n+1}(t) - x(t) \|]^{2k}, \tag{56}$$

where $c_1 + c_3 = c_5$, $c_2 + c_4 = c_6$, $c_7 = \max \{c_5, c_6\}$.

The following theorem is relative to the coupled lower and upper solutions of type II in Definition 1 when m is odd.

Theorem 11. *Consider the following assumptions:*

D_1 α_0, β_0 are coupled lower and upper solutions of type II of (1) with $\alpha_0 \leq \beta_0$ on J.

D_2 $f, g \in C^{2k+1}[\Omega, R]$ such that $f(t,x)$ and $g(t,x)$ are hyperconvex and hyperconcave in x of order 2k, respectively.

D_3 $f_x(t,x)$ satisfies

$$f_x(t,x) \leq -\max \left[f^{(2k+1)}(t,x) \right] \frac{(\beta_0 - \alpha_0)^{2k}}{(2k-1)!} \leq 0 \text{ on } \Omega. \tag{57}$$

Then, there exist monotone sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ converging uniformly to the solution $x(t)$ of (1) on J and the convergence is of the order 2k.

Proof. It follows from assumption D_2 that the inequalities (26), (27), (28), and (29) hold. In order to construct the sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$, consider the following fractional differential equations:

$$\begin{aligned} {}^c D^q \alpha_n(t) &= F(t, \alpha_{n-1}, \beta_{n-1}; \beta_n, \alpha_n) \\ &\triangleq \sum_{i=0}^{2k-1} \frac{f^{(i)}(t, \beta_{n-1})(\beta_n - \beta_{n-1})^i}{i!} \\ &\quad + \frac{f^{(2k)}(t, \alpha_{n-1})(\beta_n - \beta_{n-1})^{2k}}{(2k)!} \\ &\quad + \sum_{i=0}^{2k-1} \frac{g^{(i)}(t, \alpha_{n-1})(\alpha_n - \alpha_{n-1})^i}{i!} \\ &\quad + \frac{g^{(2k)}(t, \beta_{n-1})(\alpha_n - \alpha_{n-1})^{2k}}{(2k)!}, \quad \alpha_n(t_0) = x_0, \\ {}^c D^q \beta_n(t) &= G(t, \alpha_{n-1}, \beta_{n-1}; \alpha_n, \beta_n) \\ &\triangleq \sum_{i=0}^{2k-1} \frac{f^{(i)}(t, \alpha_{n-1})(\alpha_n - \alpha_{n-1})^i}{i!} \\ &\quad + \frac{f^{(2k)}(t, \beta_{n-1})(\alpha_n - \alpha_{n-1})^{2k}}{(2k)!} \\ &\quad + \sum_{i=0}^{2k-1} \frac{g^{(i)}(t, \beta_{n-1})(\beta_n - \beta_{n-1})^i}{i!} \\ &\quad + \frac{g^{(2k)}(t, \alpha_{n-1})(\beta_n - \beta_{n-1})^{2k}}{(2k)!}, \quad \beta_n(t_0) = x_0. \end{aligned} \tag{58}$$

Similar to the proof of Theorem 10, the conclusion can be obtained. We omit the details.

Data Availability

Data sharing not applicable to this article as no data sets were generated or analysed during the current study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors completed the paper together. All authors read and approved the final manuscript.

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