Research Article

Composition Operator, Boundedness, Compactness, Hyperbolic Bloch-Type Space β_μ^* , Hyperbolic-Type Space

Shuan Tang and Pengcheng Wu

School of Mathematics Sciences, Guizhou Normal University, Guiyang 550001, China

Correspondence should be addressed to Shuan Tang; tsaflyhigher@163.com

Received 26 May 2020; Accepted 9 July 2020; Published 1 August 2020

Academic Editor: John R. Akeroyd

Copyright © 2020 Shuan Tang and Pengcheng Wu. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In this paper, we obtain some characterizations of composition operators C_φ , which are induced by an analytic self-map φ of the unit disk Δ , from hyperbolic Bloch type space β_μ^* into hyperbolic type space $Q_{K,p,q}^*$.

1. Introduction and Results

Let $\Delta = \{z : |z| < 1\}$ be the unit disk in the complex plane $\mathbb{C}$, $\Delta^* = \mathbb{C} \setminus \bar{\Delta}$ and $H(\Delta)$ denote the space of all analytic functions on Δ . Let $B(\Delta)$ denote the subspace of $H(\Delta)$ consisting of those analytic self-maps φ for which $\varphi(\Delta) \subset \Delta$. For every $\varphi \in B(\Delta)$, the linear composition operator is defined as

$$C_\varphi = f \circ \varphi : H(\Delta) \rightarrow H(\Delta). \quad (1)$$

An important problem is to characterize the boundedness and compactness of C_φ in analytic function spaces by means of the geometric and analytic properties of φ . The reader is invited to the monographs [1, 2] for an excellent overview of this subject. Composition operators from the Bloch-type space into some other analytic function spaces have been much investigated in recent years, see [3–19] and references therein. Also, there are many works concerning composition operators from hyperbolic Bloch type spaces into some other hyperbolic classes, see [20–23]. This work is intended to characterize the boundedness and compactness of composition operators C_φ from hyperbolic Bloch type spaces β_μ^* into hyperbolic type spaces $Q_{K,p,q}^*$. We start with some notations and definitions.

A positive continuous function μ on the interval $[0, 1)$ is called normal if there exist constants $0 \leq \delta < 1$ and $0 < a < b$ such that

- (i) $\mu(r)/(1-r)^a$ is decreasing on $[\delta, 1)$ and $\lim_{r \rightarrow 1} (\mu(r)/(1-r)^a) = 0$;
- (ii) $\mu(r)/(1-r)^b$ is increasing on $[\delta, 1)$ and $\lim_{r \rightarrow 1} (\mu(r)/(1-r)^b) = \infty$.

It is clear that such a normal weight function μ is decreasing in the neighborhood of the point 1 and satisfies

$$\lim_{r \rightarrow 1} \mu(r) = 0. \quad (2)$$

It should be pointed out that the normal weight function μ has been usually used to define the mixed norm spaces [6]. Let μ be normal, the Bloch type space β_μ is defined as the set of all analytic functions f so that

$$\|f\|_{\beta_\mu} = |f(0)| + \sup_{z \in \Delta} \mu(|z|) |f'(z)| < \infty. \quad (3)$$

The little Bloch-type space $\beta_{\mu,0}$ consists of all $f \in \beta_\mu$ such that

$$\lim_{|z| \rightarrow 1} \mu(|z|) |f'(z)| = 0. \quad (4)$$

Let $\mu(|z|) = (1 - |z|^2)^\alpha$, $\alpha > 0$. Then, the Bloch-type space β_μ becomes the so-called α -Bloch space $\mathcal{B}_\alpha$, which

for $\alpha = 1$ reduces to the classical Bloch space $\mathcal{B}$. If $\mu(|z|) = (1 - |z|^2)^\alpha \log(e/1 - |z|^2)^s$, where $\alpha > 0$ and $s \leq 0$, then, we obtain the so-called logarithmic Bloch-type space $\beta_{\log^s}^\alpha$ which consists of all analytic functions $f \in H(\Delta)$ so that

$$\|f\|_{\beta_{\log^s}^\alpha} = |f(0)| + \sup_{z \in \Delta} (1 - |z|^2)^\alpha \log\left(\frac{e}{1 - |z|^2}\right)^s |f'(z)| < \infty. \quad (5)$$

The little logarithmic Bloch-type space $\beta_{\log^s,0}^\alpha$ consists of all analytic functions $f \in \beta_{\log^s}^\alpha$ and

$$\lim_{|z| \rightarrow 1} (1 - |z|^2)^\alpha \log\left(\frac{e}{1 - |z|^2}\right)^s |f'(z)| = 0. \quad (6)$$

For more results on β_μ space, see, e.g., [24–26] and related references therein.

Hyperbolic function classes are subset of $B(\Delta)$ and are defined by using the hyperbolic derivative

$$f^*(z) = \frac{|f'(z)|}{1 - |f(z)|^2}. \quad (7)$$

It is clear that the hyperbolic derivative of the composition $f \circ \varphi$ satisfies the chain rule

$$(f \circ \varphi)^*(z) = f^*(\varphi(z)) |\varphi'(z)|. \quad (8)$$

The hyperbolic Bloch-type space β_μ^* consists of all analytic functions $f \in B(\Delta)$ with

$$\|f\|_{\beta_\mu^*} = |f(0)| + \sup_{z \in \Delta} \mu(|z|) f^*(z) < \infty. \quad (9)$$

The little hyperbolic Bloch-type space $\beta_{\mu,0}^*$ is the set of all analytic functions $f \in B(\Delta)$ for which $f \in \beta_\mu^*$ and

$$\lim_{|z| \rightarrow 1} \mu(|z|) f^*(z) = 0. \quad (10)$$

Replacing the derivative f' by hyperbolic derivative f^* , we can define the corresponding hyperbolic α -Bloch space $\mathcal{B}^{\alpha,*}$ and hyperbolic logarithmic Bloch-type space $\beta_{\log^s}^{\alpha,*}$ and $\beta_{\log^s,0}^{\alpha,*}$.

For any $a \in \Delta$, the function

$$\varphi_a(z) = \frac{a - z}{1 - \bar{a}z}, \quad z \in \Delta, \quad (11)$$

is a Möbius map of the unit disk and interchanges the points a to 0. The Green function $g(z, a)$ of Δ is defined as

$$g(z, a) = \log \frac{1}{|\varphi_a(z)|}. \quad (12)$$

Let $p > 0$, $q > -2$ and $K : [0, \infty) \rightarrow [0, \infty)$ be right con-

tinuous and nondecreasing. The space $Q_{K,p,q}$ consists of all analytic functions f in Δ such that

$$\|f\|_{Q_{K,p,q}}^p = \sup_{a \in \Delta} \iint_{\Delta} |f'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) < \infty, \quad (13)$$

where dA is Lebesgue measure on Δ , normalized so that the unit disk has Lebesgue measure equals to 1. The definition was introduced in [23] and was motivated by the theory of Q_p , $F(p, q, s)$, and Q_K , see [16, 27–31] and references therein. We note that the case when $p = 2$ and $q = 0$ is the classic Q_K space, which was introduced by Wulan and Wu in [28] and has been much investigated in recent years. We also observe that the case when $K(t) = t^s$ is the space $F(p, q, s)$ introduced by Zhao in [31]. For $\alpha = (q + 2)/p$, the spaces $Q_{K,p,q}$ are subsets of space $\mathcal{B}^\alpha$. Throughout this paper, we assume that

$$\int_0^1 (1 - r^2)^q K\left(\log \frac{1}{r}\right) r dr < \infty. \quad (14)$$

Otherwise, $Q_{K,p,q}$ contains constant functions only (see [32]).

The corresponding hyperbolic space $Q_{K,p,q}^*$ is defined as the set of all analytic functions f in Δ such that

$$\|f\|_{Q_{K,p,q}^*}^p = \sup_{a \in \Delta} \iint_{\Delta} (f^*(z))^p (1 - |z|^2)^q K(g(z, a)) dA(z) < \infty, \quad (15)$$

where $K : [0, \infty) \rightarrow [0, \infty)$ is defined as before and satisfies the condition (14). Replacing the derivative f' by hyperbolic derivative f^* , we can define the corresponding hyperbolic space Q_p^* .

Yang, Xu, and Kotilainen [30] obtained some characterizations of the boundedness and compactness of the composition operators $C_\varphi : \beta_\mu \rightarrow Q_{K,p,q}$. For an $r \in (0, 1)$ and $\varphi \in B(\Delta)$, set $\Omega_r = \{z \in \Delta : |\varphi(z)| > r\}$ and $\Delta_r = \{z \in \Delta : |z| > r\}$. We use 1_E to denote the characteristic function of a set $E \subset \Delta$.

Theorem 1 (see [18]). *Assume that μ is a normal function and $\varphi \in B(\Delta)$. Let $p > 0$, $q > -2$ and K be nonnegative and nondecreasing in $[0, \infty)$. Then*

(i) $C_\varphi : \beta_\mu \rightarrow Q_{K,p,q}$ is bounded if and only if

$$\sup_{a \in \Delta} \iint_{\Delta} \frac{|\varphi'(z)|^p (1 - |z|^2)^q}{\mu^p(|\varphi(z)|)} K(g(z, a)) dA(z) < \infty. \quad (16)$$

(ii) $C_\varphi : \beta_\mu \rightarrow Q_{K,p,q}$ is compact if and only if

$$\lim_{r \rightarrow 1} \sup_{a \in \Delta} \iint_{\Delta} \frac{|\varphi'(z)|^p (1 - |z|^2)^q}{\mu^p(|\varphi(z)|)} K(g(z, a)) 1_{\Omega_r} dA(z) = 0. \quad (17)$$

In [11], the authors considered bounded composition operators from hyperbolic α -Bloch spaces $\mathcal{B}_\alpha$ into hyperbolic space Q_p^* . They proved the following result.

Theorem 2. Assume that $\varphi \in B(\Delta)$. Let $0 < \alpha \leq 1$, $0 \leq p \leq 1$. Then the following statements are equivalent:

- (i) $C_\varphi : \mathcal{B}_\alpha \longrightarrow Q_p$ is bounded;
- (ii) $C_\varphi : \mathcal{B}_\alpha^* \longrightarrow Q_p^*$ is bounded;
- (iii) $C_\varphi : \mathcal{B}_{\alpha,0}^* \longrightarrow Q_p^*$ is bounded.

In this paper, we extend Theorem 1 to the corresponding hyperbolic classes. For a real increasing function $h(r)$ defined on $[0, 1)$, its growth order is defined as

$$\rho_h := \limsup_{r \rightarrow 1} \frac{\log^+ h(r)}{\log(1/(1-r))}. \quad (18)$$

We obtain the following result.

Theorem 3. Suppose that μ is a normal function defined on $[0, 1)$ and the growth order of function $1/\mu$ is strictly less than 1. Let $\varphi \in B(\Delta)$, $p > 0$, $q > -2$, and K be nonnegative and nondecreasing in $[0, \infty)$. Then, the following statements are equivalent:

- (1) $C_\varphi : \beta_\mu \longrightarrow Q_{K,p,q}$ is bounded;
- (2) $C_\varphi : \beta_\mu^* \longrightarrow Q_{K,p,q}^*$ is bounded;
- (3) $C_\varphi : \beta_{\mu,0}^* \longrightarrow Q_{K,p,q}^*$ is bounded.

It should be pointed out that there are many normal functions which satisfy the condition of Theorem 3. For example, if $\mu(r) = (1 - r^2)^\alpha$ with $0 < \alpha < 1$, then the growth order of $1/\mu$ is α . As a corollary, we obtain the following result by taking $\mu(r) = (1 - r^2)^\alpha \log(e/(1 - r^2))^s$, where $1 > \alpha > 0$ and $s \leq 0$.

Corollary 4. Let $1 > \alpha > 0$, $s \leq 0$, and $\varphi \in B(\Delta)$. Let $p > 0$, $q > -2$, and K be nonnegative and nondecreasing in $[0, \infty)$. Then, the following statements are equivalent:

- (1) $C_\varphi : \beta_{\log^s}^\alpha \longrightarrow Q_{K,p,q}$ is bounded;
- (2) $C_\varphi : \beta_{\log^s}^{\alpha,*} \longrightarrow Q_{K,p,q}^*$ is bounded;
- (3) $C_\varphi : \beta_{\log^s,0}^{\alpha,*} \longrightarrow Q_{K,p,q}^*$ is bounded.

We now turn to the compactness of composition operator $C_\varphi : \beta_\mu^* \longrightarrow Q_{K,p,q}^*$. It is clear that β_μ^* is not a linear space since the sum of two functions in $B(\Delta)$ does not necessarily belong to $B(\Delta)$. It is also known that there is a natural metric in β_μ^* , which is defined by

$$d(f, g, \beta_\mu^*) := \|f - g\|_{\beta_\mu^*}^* + \|f - g\|_{\beta_\mu} + |f(0) - g(0)|. \quad (19)$$

The classes β_μ^* equipped with the metric $d(\cdot, \cdot, \beta_\mu^*)$ is a complete metric space, and $\beta_{\mu,0}^*$ is a closed subspace of β_μ^* (see [5, 9]). Also, there is a natural metric in $Q_{K,p,q}^*$ defined as

$$d(f, g, Q_{K,p,q}^*) := \|f - g\|_{Q_{K,p,q}^*}^* + \|f - g\|_{Q_{K,p,q}} + |f(0) - g(0)|. \quad (20)$$

A similar argument shows that the classes $Q_{K,p,q}^*$ equipped with the metric $d(\cdot, \cdot, Q_{K,p,q}^*)$ is a complete metric space (see [21, 22]).

Recall that an operator $C_\varphi : \beta_\mu^* \longrightarrow Q_{K,p,q}^*$ is compact if it maps any ball in β_μ^* onto a related relatively compact set in $Q_{K,p,q}^*$. This is equivalent to say that the operator $C_\varphi : \beta_\mu^* \longrightarrow Q_{K,p,q}^*$ is completely continuous. We obtain the following result.

Theorem 5. Suppose that μ is a normal function defined on $[0, 1)$ and the growth order of function $1/\mu$ is strictly less than 1. Let $\varphi \in B(\Delta)$, $p > 0$, $q > -2$, and K be nonnegative and nondecreasing in $[0, \infty)$. Then, the following statements are equivalent:

- (1) $C_\varphi : \beta_\mu^* \longrightarrow Q_{K,p,q}^*$ is compact

$$\lim_{r \rightarrow 1} \sup_{a \in \Delta} \iint_{\Delta} \frac{|\varphi'(z)|^p (1 - |z|^2)^q}{\mu^p(|\varphi(z)|)} K(g(z, a)) 1_{\Omega_r} dA(z) = 0. \quad (21)$$

Similarly, if we take $\mu(r) = (1 - r^2)^\alpha \log(e/(1 - r^2))^s$, where $1 > \alpha > 0$ and $s \leq 0$, then we have

Corollary 6. Let $1 > \alpha > 0$, $s \leq 0$, and $\varphi \in B(\Delta)$. Let $p > 0$, $q > -2$, and K be nonnegative and nondecreasing in $[0, \infty)$. Then, the following statements are equivalent:

- (1) $C_\varphi : \beta_{\log^s}^{\alpha,*} \longrightarrow Q_{K,p,q}^*$ is compact

$$\lim_{r \rightarrow 1} \sup_{a \in \Delta} \iint_{\Delta} \frac{|\varphi'(z)|^p (1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{p\alpha} \log(e/(1 - |\varphi(z)|^2))^s} K(g(z, a)) 1_{\Omega_r} dA(z) = 0. \quad (22)$$

The paper is organized as follows. We shall prove Theorem 3 in section 2 and prove Theorem 5 in section 3. Throughout this paper, we use the notation $a \lesssim b$ to denote that there is a constant $C > 0$ such that $a \leq Cb$, and the notation $a \approx b$ to indicate that $a \lesssim b \lesssim a$.

2. Proof of Theorem 1.1

To prove Theorem 3, we need the following Lemma [30], which gives some lower estimate of the β_μ norm of some functions in β_μ .

Lemma 7 (see [18]). *Suppose that μ is a normal function defined on $[0, 1]$. Then, there exist two functions $f_1, f_2 \in \beta_\mu$ such that for all $z \in \Delta$,*

$$|f_1'(z)| + |f_2'(z)| \geq \frac{C}{\mu(|z|)} \quad (23)$$

for some positive constant $C > 0$.

We shall prove a hyperbolic version of Lemma 7.

Lemma 8. *Suppose that μ is a normal function defined on $[0, 1]$ and the growth order of function $1/\mu$ is strictly less than 1. Then, there exist two functions $g_1, g_2 \in \beta_\mu^*$ such that for all $z \in \Delta$,*

$$g_1^*(z) + g_2^*(z) \geq \frac{C}{\mu(|z|)} \quad (24)$$

for some positive constant $C > 0$.

Proof. By Lemma 7, there are two functions $f_1, f_2 \in \beta_\mu$ so that

$$|f_1'(z)| + |f_2'(z)| \approx 1/\mu(|z|). \quad (25)$$

Consequently, we get

$$|f_i(z)| \leq \int_0^{|z|} |f_i'(w)| |dw| \leq \|f_i\|_{\beta_\mu} \int_0^1 \frac{1}{\mu(r)} dr, \quad i = 1, 2. \quad (26)$$

Let $h(r) = 1/\mu(r)$ and note that the growth order of $h(r)$ is less than 1, we deduce that for positive number $(1 + \rho_h)/2 > 0$, there exists r_0 with $1 > r_0 > 0$, such that for all $r > r_0$,

$$h(r) < \frac{1}{(1-r)^{(1+\rho_h)/2}}. \quad (27)$$

On the other hand, since $\mu(r)$ is a positive continuous function, there exists positive number $M > 0$ such that for all $0 \leq r \leq r_0$, $h(r) \leq M$. Thus, combining (26) and (27) yields that

$$|f_i(z)| \leq \|f_i\|_{\beta_\mu} \left(Mr_0 + \frac{1}{(1-\rho_h)/2} \right), \quad i = 1, 2, \quad (28)$$

hold for all $z \in \Delta$. This implies that the functions f_i are bounded.

Let $C_i = \|f_i\|_{\beta_\mu} (Mr_0 + (1/(1-\rho_h)/2))$ and $g_i = f_i/2C_i$, $i = 1, 2$. Then, $\|g_i\|_{H^\infty} \leq 1/2$ and $g_i^*(z) \approx |g_i'(z)|$, $i = 1, 2$. This gives the following

$$g_1^*(z) + g_2^*(z) \approx 1/\mu(|z|). \quad (29)$$

The proof follows.

We now start our proof of Theorem 1.1.

Proof of Theorem 1.1. Since $\beta_{\mu,0}^*$ is a closed subset of β_μ^* , it is obvious that (2) implies (3). Thus, it is enough to show that (1) $\implies$ (2) and (3) $\implies$ (1).

We first show that (1) $\implies$ (2). Assume that (1) holds, that is $C_\varphi : \beta_\mu \longrightarrow Q_{K,p,q}$ is bounded. Then, for any $f \in \beta_\mu^*$, it follows from Theorem 1 that

$$\begin{aligned} & \iint_\Delta |(f \circ \varphi)^*(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) \\ & \leq \|f\|_{\beta_\mu^*}^p \iint_\Delta \frac{|\varphi'(z)|^p (1 - |z|^2)^q}{\mu^p(|\varphi(z)|)} K(g(z, a)) dA(z) < \infty, \end{aligned} \quad (30)$$

uniformly for all $a \in \Delta$. This implies that $C_\varphi : \beta_\mu^* \longrightarrow Q_{K,p,q}^*$ is bounded, and we complete the proof of (1) $\implies$ (2).

We next show that (3) $\implies$ (1). By Lemma 8, there exist two functions $g_1, g_2 \in \beta_\mu^*$ such that for all $z \in \Delta$,

$$g_1^*(z) + g_2^*(z) \geq \frac{C}{\mu(|z|)} \quad (31)$$

for some positive constant $C > 0$. Consequently, by the assumption that $C_\varphi : \beta_{\mu,0}^* \longrightarrow Q_{K,p,q}^*$ is bounded, we have

$$\begin{aligned} & \iint_\Delta \frac{|\varphi'(z)|^p (1 - |z|^2)^q}{\mu^p(|\varphi(z)|)} K(g(z, a)) dA(z) \\ & \approx \iint_\Delta (g_1^*(\varphi(z)) + g_2^*(\varphi(z))^p |\varphi'(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) \\ & \leq 2^p \iint_\Delta (((g_1 \circ \varphi)^*(z))^p \\ & \quad + ((g_2 \circ \varphi)^*(z))^p (1 - |z|^2)^q K(g(z, a)) dA(z) \\ & \leq \|g_1\|_{\beta_\mu^*}^p + \|g_2\|_{\beta_\mu^*}^p < \infty. \end{aligned} \quad (32)$$

The proof of Theorem 1.1 is completed.

3. Proof of Theorem 1.3

In this section, we shall prove Theorem 5. First, we find some test functions in β_μ . Let μ be a normal weight function defined in $[0, 1]$, $q \in \mathbb{N}$ be large and

$\{b_i\} \subset (1/2, 1)$, consider a family analytic functions defined by

$$f_{i,m,\theta}(z) = \frac{1}{b_i} \sum_{j=1}^{\infty} \frac{(b_i e^{i\theta})^{q^j}}{\mu(1-1/q^j)} \frac{z^{q^j+q^m}}{q^j+q^m}, \quad (33)$$

where $0 \leq \theta \leq 2\pi$, $i, m \in \mathbb{N}$, and $b_i \rightarrow 1$ as $i \rightarrow \infty$.

It is proved in [21] that if we take the nature number q large enough, then, the sequence $\{f_{i,m,\theta}(z)\}_{m=1}^{\infty}$ weakly converges to zero in β_{μ} .

Lemma 9 (see [33]). *Let $f_{i,m,\theta}(z)$ be the functions defined in (33). Then, there are natural number $q \in \mathbb{N}$ such that $f_{i,m,\theta}(z) \in \beta_{\mu,0}$ and positive constant $M > 0$ such that $\|f_{i,m,\theta}\|_{\beta_{\mu}} \leq M$ for all i, m and $\theta \in [0, 2\pi]$. Moreover, set $g_{i,m,\theta}(z) = f_{i,m,\theta}(z)/M$, then*

$$\lim_{m \rightarrow \infty} \sup_{i, \theta} \|C_{\varphi} g_{i,m,\theta}\|_{Q_{K,p,q}}^p \geq \lim_{r \rightarrow 1} \sup_{a \in \Delta} \iint_{|\varphi(z)| > r} \frac{|\varphi'(z)|^p}{\mu(|\varphi(z)|)^p} dA_{q,K,a}(z), \quad (34)$$

where $dA_{q,K,a}(z) = (1 - |z|^2)^q K(g(z, a)) dA(z)$.

Now, we are in a position to prove Theorem 5.

Proof of Theorem 1.3. We first show that $(1) \implies (2)$. Let $f_{i,m,\theta}(z)$ be the analytic functions defined as in Lemma 9. Therefore, there is a positive constant $M > 0$ such that

$$|f'_{i,m,\theta}(z)| \leq \frac{M}{\mu(|z|)}, \quad z \in \Delta, \quad (35)$$

for all i, m , and $\theta \in [0, 2\pi]$. Observe that the growth order of function $1/\mu$ is less than 1. Arguing as we have done in Lemma 8, we can find a constant $C > 0$ such that $\|f_{i,m,\theta}\|_{H^{\infty}} < C$ for all i, m and $\theta \in [0, 2\pi]$. Take $g_{i,m,\theta} = f_{i,m,\theta}/2C$, where $i, m \in \mathbb{N}$, and $\theta \in [0, 2\pi]$. Then, $\|g_{i,m,\theta}\|_{H^{\infty}} \leq 1/2$ and $g_{i,m,\theta}^*(z) \approx |g_{i,m,\theta}'(z)|$, $i = 1, 2$.

On the other hand, by the definition of $g_{i,m,\theta}$, it is clear that the analytic function sequence $\{g_{i,m,\theta}\}_{m=1}^{\infty}$ converges uniformly on compact subset of Δ to 0. Consequently, by Lemma 9, the sequence $\{g_{i,m,\theta}\}_{m=1}^{\infty}$ belongs to $\beta_{\mu,0}^*$ and tends to 0 weakly as $m \rightarrow \infty$.

Since $C_{\varphi} : \beta_{\mu}^* \rightarrow Q_{K,p,q}^*$ is compact, we have

$$\lim_{m \rightarrow \infty} \sup_{i, \theta} \|C_{\varphi} g_{i,m,\theta}\|_{Q_{K,p,q}^*} = 0. \quad (36)$$

Thus, it follows from Lemma 9 that

$$\begin{aligned} & \lim_{r \rightarrow 1} \sup_{a \in \Delta} \iint_{|\varphi(z)| > r} \frac{|\varphi'(z)|^p}{\mu(|\varphi(z)|)^p} (1 - |z|^2)^q K(g(z, a)) dA(z) \\ & \leq \lim_{m \rightarrow \infty} \sup_{i, \theta} \|C_{\varphi} g_{i,m,\theta}\|_{Q_{K,p,q}}^p \approx \lim_{m \rightarrow \infty} \sup_{i, \theta} \|C_{\varphi} g_{i,m,\theta}\|_{Q_{K,p,q}^*}^p = 0. \end{aligned} \quad (37)$$

This completes the proof of $(1) \implies (2)$.

We now prove that $(2) \implies (1)$. Suppose that (2) holds. Let $B := \{f \in \beta_{\mu}^* : d(f, g, \beta_{\mu}^*) \leq \rho\}$, where $g \in \beta_{\mu}^*$ and $\rho > 0$, and $\{f_n\}_{n=1}^{\infty} \subset B$ be any sequence. By the definition of compactness of C_{φ} , it is sufficient to show that its image $\{C_{\varphi}(f_n)\}_{n=1}^{\infty}$ has a convergent subsequence in $Q_{K,p,q}^*$.

Since $\{f_n\}_{n=1}^{\infty} \subset B(\Delta)$ is a normal family, there is a subsequence $\{f_{n_j}\}_{j=1}^{\infty}$ which converges uniformly on compact subset of Δ to an analytic function f . It is clear that $\{f'_{n_j}\}_{j=1}^{\infty}$ also converges uniformly on compact subset of Δ to f' . Consequently, the sequences $\{f_{n_j} \circ \varphi\}_{j=1}^{\infty}$ and $\{f'_{n_j} \circ \varphi\}_{j=1}^{\infty}$ converge uniformly on compact subsets of Δ to $f \circ \varphi$ and $f' \circ \varphi$.

We claim that $f \in B \subset \beta_{\mu}^*$. Indeed, for any fixed R with $0 < R < 1$, we have

$$\begin{aligned} & \sup_{|z| < R} \left| \frac{f'(z)}{1 - |f(z)|^2} - \frac{g'(z)}{1 - |g(z)|^2} \right| \mu(|z|) + \sup_{|z| < R} |f'(z) - g'(z)| \mu(|z|) \\ & + |f(0) - g(0)| = \lim_{j \rightarrow \infty} \left(\sup_{|z| < R} \left| \frac{f'_{n_j}(z)}{1 - |f_{n_j}(z)|^2} - \frac{g'(z)}{1 - |g(z)|^2} \right| \mu(|z|) \right) \\ & + \lim_{j \rightarrow \infty} \left(\sup_{|z| < R} |f'_{n_j}(z) - g'(z)| \mu(|z|) + |f_{n_j}(0) - g(0)| \right) \leq \rho. \end{aligned} \quad (38)$$

Letting $R \rightarrow 1$ gives the desired result.

For any $\varepsilon > 0$, by the uniform convergence, there is $N_0 \in \mathbb{N}$ such that for all $j \geq N_0$,

$$|f_{n_j} \circ \varphi(0) - f \circ \varphi(0)| < \varepsilon. \quad (39)$$

By Theorem 1, the condition (4) implies that the composition operator $C_{\varphi} : \beta_{\mu} \rightarrow Q_{K,p,q}$ is compact. Therefore, by passing once more to a subsequence and adjusting the notations, we can assume that

$$\|f_{n_j} \circ \varphi - f \circ \varphi\|_{Q_{K,p,q}} < \varepsilon, \quad (40)$$

for all $j \geq N_1$, for some $N_1 \in \mathbb{N}$.

On the other hand, since (2) holds, we may find r with $0 < r < 1$ so that

$$\sup_{a \in \Delta} \iint_{\Delta} \frac{|\varphi'(z)|^p (1 - |z|^2)^q}{\mu^p(|\varphi(z)|)} K(g(z, a)) 1_{\Omega_r} dA(z) < \varepsilon. \quad (41)$$

Observe that $\{f_{n_j}\}_{j=1}^{\infty} \subset B$ and $f \in B$, it follows that

$$d(f_{n_j}, f, \beta_{\mu}^*) \leq d(f_{n_j}, g, \beta_{\mu}^*) + d(f, g, \beta_{\mu}^*) \leq 2\rho. \quad (42)$$

Consequently, we have

$$\begin{aligned} & \sup_{a \in \Delta} \iint_{\Delta} \left| \frac{f'_{n_j}(\varphi(z))}{1 - |f_{n_j}(\varphi(z))|^2} - \frac{f'(\varphi(z))}{1 - |f(\varphi(z))|^2} \right|^p |\varphi(z)'|^p 1_{\Omega_r} dA_{q,K,a}(z) \\ & \leq (2\rho)^p \sup_{a \in \Delta} \iint_{\Delta} \frac{|\varphi'(z)|^p (1 - |z|^2)^q}{\mu^p(|\varphi(z)|)} K(g(z, a)) 1_{\Omega_r} dA(z) \\ & < (2\rho)^p \varepsilon. \end{aligned} \quad (43)$$

Finally, by the uniform convergence on compact subset of Δ , we can find $N_2 \in \mathbb{N}$ so that for all $j \geq N_2$ and for all z with $|\varphi(z)| \leq r$,

$$\left| \frac{f'_{n_j}(\varphi(z))}{1 - |f_{n_j}(\varphi(z))|^2} - \frac{f'(\varphi(z))}{1 - |f(\varphi(z))|^2} \right| < \varepsilon. \quad (44)$$

Note again that the condition (4) implies that the composition operator $C_{\varphi} : \beta_{\mu} \rightarrow Q_{K,p,q}$ is compact. Thus, $C_{\varphi} : \beta_{\mu} \rightarrow Q_{K,p,q}$ is also bounded, and we get from Theorem 1 that

$$C := \sup_{a \in \Delta} \iint_{\Delta} \frac{|\varphi'(z)|^p (1 - |z|^2)^q}{\mu^p(|\varphi(z)|)} K(g(z, a)) dA(z) < \infty. \quad (45)$$

Therefore, we have for all $j \geq N_2$,

$$\begin{aligned} & \sup_{a \in \Delta} \iint_{\Delta} \left| \frac{f'_{n_j}(\varphi(z))}{1 - |f_{n_j}(\varphi(z))|^2} - \frac{f'(\varphi(z))}{1 - |f(\varphi(z))|^2} \right|^p |\varphi(z)'|^p (1 - 1_{\Omega_r}) dA_{q,K,a}(z) \\ & \leq \varepsilon^p \sup_{a \in \Delta} \iint_{\Delta} |\varphi(z)'|^p (1 - 1_{\Omega_r}) dA_{q,K,a}(z) \\ & \leq \varepsilon^p \sup_{a \in \Delta} \iint_{\Delta} \frac{|\varphi'(z)|^p (1 - |z|^2)^q}{\mu^p(|\varphi(z)|)} K(g(z, a)) (1 - 1_{\Omega_r}) dA(z) \\ & \leq \varepsilon^p. \end{aligned} \quad (46)$$

Combining (39), (40), (43), and (46) yields

$$\lim_{j \rightarrow \infty} d(f_{n_j}, f, Q_{K,p,q}^*) = 0. \quad (47)$$

The proof of Theorem 1.3 is completed.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The authors would like to thank the referees for their careful reading and valuable suggestions, which improve the quality of this paper. This work was supported by the National Natural Science Foundation of China (grant no. 11601100) and the joint foundation of the Guizhou Provincial Science and Technology Department (grant no. [2017]7337, [2017]5726).

References

- [1] C. Cowen and B. MacCluer, *Composition Operators on Spaces of Analytic Functions*, CRC Press, Boca Raton, 1995.
- [2] J. H. Shapiro, *Composition Operators and Classical Function Theory*, Universitext, Springer-Verlag, New York, NY, USA, 1993.
- [3] P. Galanopoulos, "On Blog to Qlogp pullbacks," *Journal of Mathematical Analysis and Applications*, vol. 337, no. 1, pp. 712–725, 2008.
- [4] Z. Hu and S. Wang, "Composition operators on Bloch-type spaces," *Proceedings of the Royal Society of Edinburgh*, vol. 135, no. 6, pp. 1229–1239, 2005.
- [5] L. Jiang and Y. He, "Composition operators from B_{α} to $F(p, q, s)$," *Acta Mathematica Scientia*, vol. 23, no. 2, pp. 252–260, 2003.
- [6] M. Kotilainen and J. Rättyä, "The essential norm of a composition operator mapping into Q_k type spaces," *Journal of Function Spaces and Applications*, vol. 6, no. 3, pp. 241–258, 2008.
- [7] M. Lindström, S. Makhmutov, and J. Taskinen, "The Essential Norm of a Bloch-to- Q_p Composition Operator," *Canadian Mathematical Bulletin*, vol. 47, no. 1, pp. 49–59, 2004.
- [8] K. M. Madigan and A. Matheson, "Compact composition operators on the Bloch space," *Transactions of the American Mathematical Society*, vol. 347, no. 7, pp. 2679–2687, 1995.
- [9] S. Makhmutov and M. Tjani, "Composition operators on some Möbius invariant Banach spaces," *Bulletin of the Australian Mathematical Society*, vol. 62, no. 1, pp. 1–19, 2000.
- [10] B. MacCluer and R. Zhao, "Essential norms of weighted composition operators between Bloch-type spaces," *The Rocky Mountain Journal of Mathematics*, vol. 33, no. 4, pp. 1437–1458, 2003.
- [11] J. Pau, "Composition operators between bloch-type spaces and Möbius invariant Q_k spaces," *Rocky Mountain Journal of Mathematics*, vol. 39, no. 6, pp. 2051–2065, 2009.

- [12] J. Rättyä, "The essential norm of a composition operator mapping into the Q_s -space," *Journal of Mathematical Analysis and Applications*, vol. 333, no. 2, pp. 787–797, 2007.
- [13] W. Smith and R. Zhao, "Composition operators mapping into the Q_p spaces," *Analysis*, vol. 17, no. 2-3, pp. 239–264, 1997.
- [14] H. Wulan, "Compactness of Composition Operators from the Bloch Space $\mathcal{B}$ to Q_K Spaces," *Acta Mathematica Sinica*, vol. 21, no. 6, pp. 1415–1424, 2005.
- [15] H. Wulan and P. Wu, "Composition operators from the Bloch space into the spaces QT ," *Journal of Mathematics and Mathematical Sciences*, vol. 2003, no. 31, pp. 1973–1979, 2003.
- [16] J. Xiao, "Holomorphic Q classes," in *Lecture Notes in Mathematics*, 1767, Springer-Verlag, Berlin, 2001.
- [17] J. Xiao, "Composition operators associated with Bloch-type spaces," *Complex Variables and Elliptic Equations*, vol. 46, no. 2, pp. 109–121, 2007.
- [18] C. Yang, W. Xu, and M. Kotilainen, "Composition operators from Bloch type spaces into QK type spaces," *Journal of Mathematical Analysis and Applications*, vol. 379, no. 1, pp. 26–34, 2011.
- [19] J. Zhou, "Composition operators from $\mathcal{B}^\alpha$ to Q_K type spaces," *Journal of Function Spaces and Applications*, vol. 6, pp. 89–105, 2008.
- [20] A. El-Sayed Ahmed and M. A. Bakhit, "Composition operators in hyperbolic general Besov-type spaces," *Cubo (Temuco)*, vol. 15, no. 3, pp. 19–30, 2013.
- [21] F. Pérez-González, J. Rättyä, and J. Taskinen, "Lipschitz continuous and compact composition operators in hyperbolic classes," *Mediterranean Journal of Mathematics*, vol. 8, no. 1, pp. 123–135, 2011.
- [22] A. Kamal, "Properties of superposition operators acting between $B_\mu *$ and $QK*$," *Journal of the Egyptian Mathematical Society*, vol. 23, no. 3, pp. 507–512, 2015.
- [23] X. Li, F. Pérez-González, and J. Rättyä, "Composition operators in hyperbolic Q -classes," *Annales Academiæ Scientiarum Fennicæ Mathematica*, vol. 31, pp. 391–404, 2006.
- [24] Z. Hu, "Extended Cesàro operators on mixed norm spaces," *Proceedings of the American Mathematical Society*, vol. 131, no. 7, pp. 2171–2179, 2003.
- [25] S. Stević, "Bloch-type functions with Hadamard gaps," *Applied Mathematics and Computation*, vol. 208, no. 2, pp. 416–422, 2009.
- [26] X. Zhang, "Composition type operator from Bergman space to μ -Bloch type space in C_n ," *Journal of Mathematical Analysis and Applications*, vol. 298, no. 2, pp. 710–721, 2004.
- [27] R. Aulaskari and P. Lappan, "Criteria for an analytic function to be Bloch and a harmonic or meromorphic function to be normal Complex analysis and its applications (Hong Kong, 1993)," *Pitman Research Notes in Mathematics Ser.*, vol. 305, pp. 136–146, 1994.
- [28] H. Wulan and P. Wu, "Characterizations of QT Spaces," *Journal of Mathematical Analysis and Applications*, vol. 254, no. 2, pp. 484–497, 2001.
- [29] J. Xiao, "Geometric Q functions," in *Frontiers in Mathematics*, Birkhäuser, Basel, 2006.
- [30] J. Xiao, "Carleson measure, atomic decomposition and free interpolation from Bloch space," *Annales Academiæ Scientiarum Fennicæ Series A. I. Mathematica*, vol. 19, pp. 35–46, 1994.
- [31] R. Zhao, "On a general family of function spaces," *Annales Academiæ Scientiarum Fennicæ Mathematica Dissertationes*, vol. 105, p. 56, 1996.
- [32] H. Wulan and J. Zhou, "QK type spaces of analytic functions," *Journal of Function Spaces and Applications*, vol. 4, no. 1, pp. 73–84, 2006.
- [33] S. Tang and C. Yang, *The essential norm of composition operators from Bloch type space into Q_K type spaces* preprint.