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Research Article

Some Properties of Prequasi Normed Generalized de La Vallée Poussin's Mean Sequence Space

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In this article, we study some topological properties of the multiplication operator on generalized de La Vallée Poussin's mean sequence space equipped with the prequasi norm and the prequasi operator ideal generated by s-numbers and this sequence space.

1. Introduction

All through the article, by $L(U, V)$, we signify the space of all bounded linear transformations between arbitrary Banach spaces U and V; if $U = V$, we mark $L(U)$. We will denote $F(V)$, $\Psi(V)$, and $L_c(V)$ for the space of all finite rank, approximable, and compact transformations on V, respectively. The set is composed of nonnegative integers ($\mathbb{N}$) and real numbers ($\mathbb{R}$). Moreover, $\mathbb{C}^{\mathbb{N}}$, ℓ_∞, C_0, $(s_r(A))$, $(d_r(A))$, and $(\alpha_r(A))$ will denote the space of complex, bounded, null, r-th s-numbers [1], r-th Kolmogorov, and approximation sequences, respectively, where $A \in L(U, V)$. The multiplication operators and operator ideals have a wide field of mathematics in functional analysis, for instance, in eigenvalue distributions theorem, fixed point theorem, geometry of Banach spaces, and spectral theory. Singh and Kumar [2] investigated the connection between the composition operators and the multiplication operators on L_q-spaces. They proved the equivalence between the composition operator $A \in L_c(L_q(U;\mathbb{C}))$ and $B_\beta \in L_c(L_q(U;\mathbb{C}))$, where B_β is the multiplication operator induced by $\beta = d\nu B^{-1}/d\nu$. The roots of the multiplication operators are contained in the spectral theory. According to Hilbert space theorem, there is a unitary equivalence between a multiplication operator and each normal operator on a separable Hilbert space. For more subtleties on multiplication operators, see [2–7]. In view of sequence spaces, Mursaleen and Noman [8] investigated some difference sequence spaces under compact matrix transformation. The multiplication operators belong to $L(\ell_\varphi)$, where φ is an Orlicz function, studied by Komal and Gupta [9]. Furthermore, Komal et al. [10] explained the multiplication operators in $L((\mathrm{ces}_q)_{\|\cdot\|})$, for $q \in (1, \infty)$ with the Luxemburg norm $\|.\|$. In the operator ideal theory, the scalar sequence space is sometimes used to define operator ideals in the class of Banach spaces or Hilbert spaces, since the ideal $L_c(U)$ is generated by C_0 and $(d_r(A))$. A Banach space U is called simple [11], if there is one and only one nontrivial closed ideal in $L(U)$. The quasi-ideals $S^{\mathrm{app}}_{\ell_q}$, where $q \in (0, \infty)$, is investigated by Pietsch [11]. Also, he proved that ℓ_2 and ℓ_1 generated the ideals of Hilbert Schmidt operators between Hilbert spaces and of nuclear operators, respectively. He also investigated the sufficient condition on ℓ_q so that $\bar{F(U, V)} = S^{\mathrm{app}}_{\ell_q}(U, V)$ and $L(\ell_q)$ is simple Banach space. Pietsch [12] studied the smallness of $S^{\mathrm{app}}_{\ell_q}$. For any infinite dimensional Banach spaces U and V, Makarov and Faried [13] gave the sufficient conditions for which $S^{\mathrm{app}}_{\ell_r}$ is strictly contained in $S^{\mathrm{app}}_{\ell_m}$, for every $m > r > 0$. Bakery [14] introduced some properties of $S^{\mathrm{app}}_{V(\gamma,p)}$, where $V(\gamma, p)$ is the generalized de La Vallée Poussin's mean sequence space. Some properties of s-type Orlicz-Lorentz sequence spaces are examined by Mohiuddine and Raj [15]. Faried and Bakery [16] introduced a generalization of the usual classes of operator ideal which is prequasi operator ideal. They explained some results of $S_{ces(q)}$ and S_{ℓ_φ}. We introduce in this paper the concept of prequasi norm on $V(\gamma, p)$, give the conditions on $V(\gamma, p)$ equipped with the prequasi norm to construct Banach space, and study the necessity and sufficient conditions on $V(\gamma, p)$ so that the multiplication operator defined on $V(\gamma, p)$ is bounded, invertible, approximable, closed range, and Fredholm operator. We investigate the sufficient conditions on the class $S_{V(\gamma,p)}$ to form small, simple, closed Banach prequasi operator ideal. Also, we examine the strict inclusion relation between $S_{V(\gamma,p)}$ and S_E^λ (the class of all bounded linear operators whose sequence of eigenvalues belongs to $V(\gamma, p)$).

2. Definitions and Preliminaries

We will denote $e_j = (0, 0, \cdots, 1, 0, 0, \cdots)$, with 1 in the j^{th} position for every $j \in \mathbb{N}$.

Lemma 1 (see [11]). *For $T \in L(V, W)$, if $T \notin \Psi(V, W)$, then there are operators $B \in L(W)$ and $G \in L(V)$ with $BTGe_i = e_i$ for each $i \in \mathbb{N}$.*

Theorem 2 (see [11]). *If W is a Banach space with* $\dim(W) = \infty$, *then*

$$F(W) \subsetneq \Psi(W) \subsetneq L_c(W) \subsetneq L(W). \quad (1)$$

Definition 3 (see [17]). An operator $D \in L(V)$ is called Fredholm if it satisfies $\dim(kerD) < \infty$, $\dim(R(D))^c < \infty$, and D has closed range, where $(R(D))^c$ denotes the complement of the range D.

Let $(\gamma_k) \in (\mathbb{R}^+)^{\mathbb{N}}$, where $\gamma_0 = 1$, $\gamma_{k+1} \geq \gamma_k$, $\gamma_{k+1} \leq \gamma_k + 1$, for all $k \in \mathbb{N}$, $\lim_{k\to\infty}\gamma_k = \infty$ and $(p_i) \in (\mathbb{R}^+)^{\mathbb{N}}$ with $p_i \geq 1$, for all $i \in \mathbb{N}$. Simsek et al. [18] defined the generalized de La Vallée Poussin's mean sequence space as follows:

$$V(\gamma, p) = \left\{x = (x_i) \in \mathbb{C}^{\mathbb{N}} : \exists \eta > 0 \text{ with } \sigma(\eta x) < \infty\right\} \quad \text{where } \sigma(x) = \sum_{i=0}^{\infty}\left(\frac{\sum_{k\in G_i}|x_k|}{\gamma_i}\right)^{p_i}, \quad (2)$$

and $G_i = [i - \gamma_i + 1, i]$, for $i \in \mathbb{N}$. The space $(V(\gamma, p), \|.\|)$, where

$$\|x\| = \inf\left\{\eta > 0 : \sigma\left(\frac{x}{\eta}\right) \leq 1\right\}, \quad (3)$$

is a Banach space. When $(p_n) \in \ell_\infty$, it is clear that

$$V(\gamma, p) = \left\{(x_i) \in \mathbb{C}^{\mathbb{N}} : \sum_{i=0}^{\infty}\left(\frac{\sum_{k\in G_i}|x_k|}{\gamma_i}\right)^{p_i} < \infty\right\}. \quad (4)$$

For more details on $V(\gamma, p)$, see [19, 20].

Remark 4.

(1) If $\gamma_i = i + 1$, for every $i \in \mathbb{N}$, then $V(\gamma, p) = ces((p_n))$ examined by Sanhan and Suantai [21]

(2) If $\gamma_i = i + 1$ and $p_i = p$, for all $i \in \mathbb{N}$, then $V(\gamma, p) = ces_p$. Some authors [22–24] investigated various sorts of Cesáro summable sequence spaces

Definition 5 (see [25]). A class of linear sequence spaces X is called a special space of sequences (sss) if

(1) $e_j \in X$ for each $j \in \mathbb{N}$

(2) for $u = (u_j) \in \mathbb{C}^{\mathbb{N}}$, $v = (v_j) \in X$ and $|u_j| \leq |v_j|$ for all $j \in \mathbb{N}$, then $u \in X$

(3) for $(u_j)_{j=0}^{\infty} \in X$, then $(u_{[j/2]})_{j=0}^{\infty} \in X$, wherever $[j/2]$ means the integral part of $j/2$

Theorem 6 (see [14]). *$V(\gamma, p)$ is a (sss), if*

(a1) (p_k) is increasing and $(p_k) \in \ell_\infty$ with $p_0 > 1$

(a2) $(\gamma_k) \in (\mathbb{R}^+)^{\mathbb{N}}$, where $\gamma_0 = 1$, $\gamma_{k+1} \geq \gamma_k$, $\gamma_{k+1} \leq \gamma_k + 1$, for all $k \in \mathbb{N}$, $\lim_{k\to\infty}\gamma_k = \infty$ and $(\gamma_k^{-1}) \in \ell_{(p_k)}$ are satisfied

Definition 7 (see [25]). A subclass of (sss) is called a premodular (sss) if there is a function $\sigma : X \longrightarrow [0, \infty)$ verifying the conditions

(i) $\sigma(v) \geq 0$ for all $v \in X$ and $\sigma(v) = 0 \Leftrightarrow v = \theta$, here θ is the zero element of X

(ii) there is $L \geq 1$ with $\sigma(\eta v) \leq L|\eta|\sigma(v)$ for each $v \in X$, and $\eta \in \mathbb{C}$

(iii) there is $K \geq 1$, $\sigma(u + v) \leq K(\sigma(u) + \sigma(v))$ for all u, $v \in X$

(iv) for $|u_i| \leq |v_i|$ for all $i \in \mathbb{N}$, then $\sigma((u_i)) \leq \sigma((v_i))$

(v) for some $K_0 \geq 1$, $\sigma((u_i)) \leq \sigma((u_{[i/2]})) \leq K_0\sigma((u_i))$

(vi) for all $u = (u_j)_{j=o}^{\infty} \in X$ and $\varepsilon > 0$, there is $s \in \mathbb{N}$ with $\sigma((u_j)_{j=s}^{\infty}) < \varepsilon$

(vii) there is $\xi > 0$ such that $\sigma(\eta, 0, 0, 0, \cdots) \geq \xi|\eta|\sigma(1, 0, 0, 0, \cdots)$ for each $\eta \in \mathbb{C}$

Theorem 8 (see [14]). *If conditions (a1) and (a2) are verified, then $(V(\gamma, p))_\sigma$ is a premodular (sss), with $\sigma(x) = \sum_{j=0}^{\infty}(\sum_{k\in G_j}|x_k|/\gamma_j)^{p_j}$ for each $x \in V(\gamma, p)$. The usual classes of operator ideal generalized to the prequasi operator ideal.*

Definition 9 (see [25]). A function $f : \Phi \longrightarrow [0, \infty)$ is called a prequasi norm on the ideal Φ if it satisfies the following:

(1) If $A \in \Phi(U,V)$, then $f(A) \geq 0$ and $f(A)=0$ if and only if $A=0$

(2) There is $M \geq 1$ with $f(\eta A) \leq M|\eta| f(A)$, for each $A \in \Phi(U,V)$ and $\eta \in \mathbb{C}$

(3) There is $K \geq 1$ such that $f(A_1+A_2) \leq K[f(A_1)+f(A_2)]$, for every $A_1, A_2 \in \Phi(U,V)$

(4) There is $C \geq 1$, if $A \in L(U_0, U)$, $P \in \Phi(U,V)$ and $R \in L(V, V_0)$ then $f(RPA) \leq C\|R\| f(P) \|A\|$, where U_0 and V_0 are normed spaces

Notations 10 (see [16]).

$$\begin{aligned} S_E &:= \{S_E(U,V)\,;\, U \text{and } V \text{ are Banach Spaces}\}, \\ &\quad \text{with } S_E(U,V) := \{A \in L(U,V) \colon (s_i(A))_{i=0}^{\infty} \in E\}. \\ S_E^{\text{app}} &:= \{S_E^{\text{app}}(U,V)\,;\, U \text{ and } V \text{ are Banach Spaces}\}, \\ &\quad \text{with } S_E^{\text{app}}(U,V) := \{A \in L(U,V) \colon (\alpha_i(A))_{i=0}^{\infty} \in E\}. \\ S_E^{Kol} &:= \left\{S_E^{Kol}(U,V)\,;\, U \text{ and } V \text{ are Banach Spaces}\right\}, \\ &\quad \text{with } S_E^{Kol}(U,V) := \{T \in L(U,V) \colon (d_i(A))_{i=0}^{\infty} \in E\}. \end{aligned} \tag{5}$$

Theorem 11 (see [16]). *Let E_ϱ be a premodular (sss), then the function $g(T) = \varrho(s_i(T))_{i=0}^{\infty}$ be a prequasi norm on S_{E_ϱ}.*

During this paper, $(p_i) \in (\mathbb{R}^+)^{\mathbb{N}}$ with $(p_i) \in \ell_\infty$ and the inequality [26] $|a_j + b_j|^{p_j} \leq H(|a_j|^{p_j} + |b_j|^{p_j})$, where $H = 2^{h-1}$, $h = \sup_j p_j$, and $p_j \geq 1$ for all $j \in \mathbb{N}$, will be used.

3. Main Results

We introduce a generalization of the usual norm, the concept of prequasi norm on $V(\gamma,p)$. We investigate the sufficient conditions on $V(\gamma,p)$ equipped with a prequasi norm to form Banach space.

Definition 12. Assume X is (sss). If there is a function $\sigma : X \longrightarrow [0,\infty)$ satisfying the conditions

(i) $\sigma(\nu) \geq 0$ for all $\nu \in X$ and $\sigma(\nu) = 0 \Leftrightarrow \nu = \theta$

(ii) there is $L \geq 1$ with $\sigma(\eta\nu) \leq L|\eta|\sigma(\nu)$ for every $\nu \in X$, and $\eta \in \mathbb{C}$

(iii) there is $K \geq 1$, we have $\sigma(u+\nu) \leq K(\sigma(u) + \sigma(\nu))$ for each $u, \nu \in X$

The space X_σ is called prequasi normed (sss). If X is complete with σ, hence X_σ is called a prequasi Banach (sss). We express the following two theorems without verification, since they are clear.

Theorem 13. *X is a prequasi norm (sss), if it is quasi norm (sss).*

Theorem 14. *Every premodular (sss) is prequasi normed (sss).*

Theorem 15. *If conditions (a1) and (a2) are verified, then $(V(\gamma,p))_\varrho$ is a prequasi Banach (sss), with $\varrho(x) = \sum_{i=0}^{\infty} (\sum_{j \in G_i} |x_j|/\gamma_i)^{p_i}$ for all $x \in V(\gamma,p)$.*

Proof. Let the conditions be satisfied, then from Theorem 8, the space $(V(\gamma,p))_\varrho$ is premodular (sss). From Theorem 14, we have $(V(\gamma,p))_\varrho$ which is a prequasi normed (sss). To show that $(V(\gamma,p))_\varrho$ is a prequasi Banach (sss), assume $x^n = (x_k^n)_{k=0}^{\infty}$ be a Cauchy sequence in $(V(\gamma,p))_\varrho$. Therefore, for all $\varepsilon \in (0,1)$, there is $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$, one has

$$\varrho(x^n - x^m) = \sum_{i=0}^{\infty} \left(\frac{\sum_{j \in G_i} \left| x_j^n - x_j^m \right|}{\gamma_i} \right)^{p_i} < \varepsilon^h. \tag{6}$$

Hence for $n, m \geq n_0$ and $j \in \mathbb{N}$, we get

$$\left| x_j^n - x_j^m \right| < \varepsilon. \tag{7}$$

So (x_j^m) is a Cauchy sequence in $\mathbb{C}$ for fixed $j \in \mathbb{N}$; this gives $\lim_{m\to\infty} x_j^m = x_j^0$ for fixed $j \in \mathbb{N}$. Hence, $\varrho(x^n - x^0) < \varepsilon^h$, for all $n \geq n_0$. Finally, to prove that $x^0 \in V(\gamma,p)$, we have

$$\varrho(x^0) = \varrho(x^0 - x^n + x^n) \leq H(\varrho(x^n - x^0) + \varrho(x^n)) < \infty, \tag{8}$$

so $x^0 \in V(\gamma,p)$. This completes the proof.

Corollary 16. *Let $1 < p < \infty$, then $(ces_p)_\varrho$ be a prequasi Banach (sss), with $\varrho(x) = \sum_{i=0}^{\infty} (\sum_{j=0}^{i} |x_j|/i+1)^p$ for all $x \in ces_p$.*

4. Multiplication Operator on Prequasi Normed (sss)

We define here a multiplication operator on $V(\gamma,p)$ with a prequasi norm and investigate the sufficient and necessary conditions on the multiplication operator to become bounded, closed range operator, approximable, invertible, and Fredholm.

Definition 17. If $\eta \in \mathbb{C}^{\mathbb{N}}$, $\eta \in \ell_\infty$ and X_ϱ is a prequasi normed (sss). An operator T_η is called multiplication operator if $T_\eta : X \longrightarrow X$, where $T_\eta y = \eta y = (\eta_k y_k)_{k=0}^{\infty}$, for all $y \in X$. When $T_\eta \in L(X)$, T_η is called the multiplication operator induced by η.

Theorem 18. *Let $\alpha \in \mathbb{C}^{\mathbb{N}}$, the conditions (a1) and (a2) be satisfied, hence $\alpha \in \ell_\infty$ if and only if $T_\alpha \in L((V(\gamma,p))_\varrho)$, with $\varrho(x) = \sum_{i=0}^{\infty} (\sum_{j \in G_i} |x_j|/\gamma_i)^{p_i}$ for every $x \in (V(\gamma,p))_\varrho$.*

Proof. Let the conditions be satisfied and $\alpha \in \ell_\infty$. Therefore, there is $D > 0$ with $|\alpha_k| \leq D$, for each $k \in \mathbb{N}$. For $x \in$

$(V(\gamma,p))_{\varrho}$, we have

$$\begin{aligned}\varrho(T_{\alpha}x)&=\varrho(\alpha x)=\varrho\left((\alpha_k x_k)_{k=0}^{\infty}\right)=\varrho\left((|\alpha_k||x_k|)_{k=0}^{\infty}\right)\\&=\sum_{i=0}^{\infty}\left(\frac{\sum_{j\in G_i}|\alpha_j||x_j|}{\gamma_i}\right)^{p_i}\le\sum_{i=0}^{\infty}\left(\frac{\sum_{j\in G_i}C|x_j|}{\gamma_i}\right)^{p_i}\\&\le L\varrho(x),\end{aligned}\tag{9}$$

where $L=\sup_i D^{p_i}$; this implies that $T_{\alpha}\in L((V(\gamma,p))_{\varrho})$. Conversely, let $T_{\alpha}\in L((V(\gamma,p))_{\varrho})$ to show that $\alpha\in\ell_{\infty}$. For, if $\alpha\notin\ell_{\infty}$, hence for all $n\in\mathbb{N}$, there are $i_n\in\mathbb{N}$ with $\alpha_{i_n}>n$. We have

$$\begin{aligned}\varrho\left(T_{\alpha}e_{i_n}\right)&=\varrho\left(\alpha e_{i_n}\right)=\varrho\left(\left(\alpha_j\left(e_{i_n}\right)_j\right)_{j=0}^{\infty}\right)\\&=\sum_{i=0}^{\infty}\left(\frac{\sum_{j\in G_i}|\alpha_j|\left(e_{i_n}\right)_j}{\gamma_i}\right)^{p_i}=\sum_{i=0}^{\infty}\left(\frac{\left|\alpha_{i_n}\right|}{\gamma_i}\right)^{p_i}\\&>\sum_{i=0}^{\infty}\left(\frac{n}{\gamma_i}\right)^{p_i}\ge\inf_i n^{p_i}\varrho\left(e_{i_n}\right)=n^{p_0}\varrho\left(e_{i_n}\right).\end{aligned}\tag{10}$$

This shows that $T_{\alpha}\notin L((V(\gamma,p))_{\varrho})$. So, $\alpha\in\ell_{\infty}$.

Theorem 19. *If $\alpha\in\mathbb{C}^{\mathbb{N}}$ and $(V(\gamma,p))_{\varrho}$ is a prequasi normed (sss), with $\varrho(x)=\sum_{i=0}^{\infty}(\sum_{j\in G_i}|x_j|/\gamma_i)^{p_i}$ for each $x\in(V(\gamma,p))_{\varrho}$. Hence, $|\alpha_j|=1$ for each $j\in\mathbb{N}$ if and only if T_{α} is an isometry.*

Proof. Suppose $|\alpha_j|=1$, for every $j\in\mathbb{N}$. Therefore,

$$\begin{aligned}\varrho(T_{\alpha}x)&=\varrho(\alpha x)=\varrho\left((\alpha_j x_j)_{j=0}^{\infty}\right)=\sum_{i=0}^{\infty}\left(\frac{\sum_{j\in G_i}|\alpha_j||x_j|}{\gamma_i}\right)^{p_i}\\&=\sum_{i=0}^{\infty}\left(\frac{\sum_{j\in G_i}|x_j|}{\gamma_i}\right)^{p_i}=\varrho(x),\end{aligned}\tag{11}$$

for all $x\in(V(\gamma,p))_{\varrho}$. Hence, T_{α} is an isometry. Conversely, let $|\alpha_n|<1$ for some $n=n_0$. Then,

$$\begin{aligned}\varrho\left(T_{\alpha}e_{n_0}\right)&=\varrho\left(\alpha e_{n_0}\right)=\varrho\left(\left(\alpha_j\left(e_{n_0}\right)_j\right)_{j=0}^{\infty}\right)\\&=\sum_{i=0}^{\infty}\left(\frac{\sum_{j\in G_i}\left|\alpha_j\right|\left|\left(e_{n_0}\right)_j\right|}{\gamma_i}\right)^{p_i}\\&=\sum_{i=0}^{\infty}\left(\frac{\alpha_{n_0}}{\gamma_i}\right)^{p_i}<\sum_{i=0}^{\infty}\left(\frac{1}{\gamma_i}\right)^{p_i}<\varrho\left(e_{n_0}\right).\end{aligned}\tag{12}$$

Also, if $|\alpha_{n_0}|>1$, then we get $\varrho(T_{\alpha}e_{n_0})>\varrho(e_{n_0})$. Therefore, we have a contradiction in the two cases. So, $|\alpha_j|=1$, for each $j\in\mathbb{N}$.

Theorem 20. *If $\alpha\in\mathbb{C}^{\mathbb{N}}$, the conditions (a1) and (a2) are satisfied and $T_{\alpha}\in L((V(\gamma,p))_{\varrho})$, where $\rho(x)=\sum_{r=0}^{\infty}(\sum_{j\in G_r}|x_j|/\gamma_r)^{p_r}$ for all $x\in(V(\gamma,p))_{\varrho}$. Then, $T_{\alpha}\in\Psi((V(\gamma,p))_{\varrho})$ if and only if $(\alpha_n)_{n=0}^{\infty}\in C_0$.*

Proof. Let T_{α} be an approximable operator. Hence, $T_{\alpha}\in L_c((V(\gamma,p))_{\varrho})$. To prove that $(\alpha_r)_{r=0}^{\infty}\in C_0$, assume $(\alpha_r)_{r=0}^{\infty}\notin C_0$; hence, there is $\delta>0$ with $B_{\delta}=\{r\in\mathbb{N}:|\alpha_r|\ge\delta\}$ which is an infinite set. Suppose $d_1,\cdots d_2,\cdots,d_n,\cdots$ be in B_{δ}. Then, $\{e_{d_n}:d_n\in B_{\delta}\}$ is an infinite bounded set in $(V(\gamma,p))_{\varrho}$. We have

$$\begin{aligned}&\varrho\left(T_{\alpha}e_{d_n}-T_{\alpha}e_{d_m}\right)\\&\quad=\varrho\left(\alpha e_{d_n}-\alpha e_{d_m}\right)\\&\quad=\varrho\left(\left(\alpha_k\left(\left(e_{d_n}\right)_k-\left(e_{p_m}\right)_k\right)\right)_{k=0}^{\infty}\right)\\&\quad=\sum_{i=0}^{\infty}\left(\frac{\sum_{j\in G_i}\left|\alpha_j\left(\left(e_{d_n}\right)_j-\left(e_{p_m}\right)_j\right)\right|}{\gamma_i}\right)^{p_i}\\&\quad\ge\sum_{i=0}^{\infty}\left(\frac{\sum_{j\in G_i}\left|\delta\left(\left(e_{d_n}\right)_j-\left(e_{p_m}\right)_j\right)\right|}{\gamma_i}\right)^{p_i}\\&\quad\ge\left(\inf_n\delta^{p_n}\right)\sum_{i=0}^{\infty}\left(\frac{\sum_{j\in G_i}\left|\left(\left(e_{d_n}\right)_j-\left(e_{p_m}\right)_j\right)\right|}{\gamma_i}\right)^{p_i}\\&\quad=\left(\inf_n\delta^{p_n}\right)\rho\left(e_{d_n}-e_{d_m}\right),\end{aligned}\tag{13}$$

for all $d_n,d_m\in B_{\delta}$. This proves $\{e_{d_n}:d_n\in B_{\delta}\}\in\ell_{\infty}$ which cannot have a convergent subsequence under T_{α}. Therefore, this gives $T_{\alpha}\notin L_c((V(\gamma,p))_{\varrho})$, so $T_{\alpha}\notin\Psi((V(\gamma,p))_{\varrho})$; this implies a contradiction. Therefore, $\lim_{n\to\infty}\alpha_n=0$. Conversely, suppose $\lim_{n\to\infty}\alpha_n=0$. Hence, for all $\delta>0$, the set $B_{\delta}=\{n\in\mathbb{N}:|\alpha_n|\ge\delta\}$ is finite. We have

$$\left((V(\gamma,p))_{\varrho}\right)_{B_{\delta}}=\left\{x=(x_n)\in(V(\gamma,p))_{\varrho}:n\in B_{\delta}\right\},\tag{14}$$

which is a finite dimensional space for all $\delta>0$. Hence, $T_{\alpha}|((V(\gamma,p))_{\varrho})_{B_{\delta}}$ is a finite rank operator. Define $\alpha_i\in\mathbb{C}^{\mathbb{N}}$ as follows

$$(\alpha_i)_j=\begin{cases}\alpha_j, & j\in B_{1/i},\\0, & \text{otherwise}.\end{cases}\tag{15}$$

It is clear that since T_{α_i} is a finite rank operator, then the space $((V(\gamma, p))_\varrho)_{B_{1/i}}$ is finite dimensional for each $i \in \mathbb{N}$. Therefore, by using conditions (a1) and (a2), we have

$$\begin{aligned}
&\varrho\left(\left(T_\alpha - T_{\alpha_n}\right)x\right) \\
&= \varrho\left(\left(\left(\alpha_j - (\alpha_n)_j\right)x_j\right)_{j=0}^{\infty}\right) \\
&= \sum_{i=0, i\in B_{1/n}}^{\infty}\left(\frac{\sum_{j\in G_i}\left|\left(\alpha_j - (\alpha_n)_j\right)x_j\right|}{\gamma_i}\right)^{p_i} \\
&\quad + \sum_{i=0, i\notin B_{1/n}}^{\infty}\left(\frac{\sum_{j\in G_i}\left|\left(\alpha_j - (\alpha_n)_j\right)x_j\right|}{\gamma_i}\right)^{p_i} \\
&= \sum_{i=0, i\in B_{1/n}}^{\infty}\left(\frac{\sum_{j\in G_i, j\notin B_{1/n}}\left|\alpha_j x_j\right|}{\gamma_i}\right)^{p_i} \\
&\quad + \sum_{i=0, i\notin B_{1/n}}^{\infty}\left(\frac{\sum_{j\in G_i, j\notin B_{1/n}}\left|\alpha_j x_j\right|}{\gamma_i}\right)^{p_i} \\
&< \frac{1}{n}\sum_{i=0}^{\infty}\left(\frac{\sum_{j\in G_i}\left|x_j\right|}{\gamma_i}\right)^{p_i} = \frac{1}{n}\varrho(x).
\end{aligned} \tag{16}$$

This proves that $\|T_\alpha - T_{\alpha_n}\| \le 1/n$. So, T_α is a limit of finite rank operators. This finishes the proof.

Theorem 21. *If $\alpha \in \mathbb{C}^{\mathbb{N}}$, the conditions (a1) and (a2) are satisfied and $T_\alpha \in L((V(\gamma, p))_\varrho)$, where $\rho(x) = \sum_{i=0}^{\infty}(\sum_{j\in G_i}|x_j|/\gamma_i)^{p_i}$ for all $x \in (V(\gamma, p))_\varrho$. Then, $T_\alpha \in L_c((V(\gamma, p))_\varrho)$ if and only if $(\alpha_n)_{n=0}^{\infty} \in C_0$.*

Proof. It is clear so omitted.

Corollary 22. *If conditions (a1) and (a2) are satisfied, we have*

$$L_c\left((V(\gamma, p))_\varrho\right) \subsetneqq L\left((V(\gamma, p))_\varrho\right), \tag{17}$$

with $\varrho(x) = \sum_{i=0}^{\infty}(\sum_{j\in G_i}|x_j|/\gamma_i)^{p_i}$ for each $x \in V(\gamma, p)$.

Proof. Since the multiplication operator induced by $\alpha = (1, 1,)$ is $I \in L((V(\gamma, p))_\varrho)$, therefore $I \notin L_c((V(\gamma, p))_\varrho)$.

Theorem 23. *Let $(V(\gamma, p))_\varrho$ be a prequasi Banach (sss), with $T_\alpha \in L((V(\gamma, p)))_\varrho)$ and $\varrho(x) = \sum_{i=0}^{\infty}(\sum_{j\in G_i}|x_j|/\gamma_i)^{p_i}$ for each $x \in V(\gamma, p)$. Hence, α is bounded away from zero on* ker $(\alpha)^c$ *if and only if T_α has closed range.*

Proof. Let the sufficient conditions be satisfied. Therefore, there is $\varepsilon > 0$ such that $|\alpha_j| \ge \varepsilon$, for every $j \in \ker(\alpha)^c$. To show that $R(T_\alpha)$ is closed, suppose q be a limit point of $R(T_\alpha)$. Hence, there is $T_\alpha x_j$ in $(V(\gamma, p))_\varrho$, for all $j \in \mathbb{N}$ such that $\lim_{j\to\infty} T_\alpha x_j = q$. Clearly, the sequence $T_\alpha x_j$ is a Cauchy sequence. Therefore, since ρ is nondecreasing, one can see

$$\begin{aligned}
&\varrho(T_\alpha x_n - T_\alpha x_m) \\
&= \varrho\left(\left(\alpha_j (x_n)_j - \alpha_j (x_m)_j\right)_{j=0}^{\infty}\right) \\
&= \sum_{i=0}^{\infty}\left(\frac{\sum_{j\in G_i}\left|\alpha_j (x_n)_j - \alpha_j (x_m)_j\right|}{\gamma_i}\right)^{p_i} \\
&= \sum_{i=0, i\in \ker(\alpha)^c}^{\infty}\left(\frac{\sum_{j\in G_i}\left|\alpha_j (x_n)_j - \alpha_j (x_m)_j\right|}{\gamma_i}\right)^{p_i} \\
&\quad + \sum_{i=0, i\notin \ker(\alpha)^c}^{\infty}\left(\frac{\sum_{j\in G_i}\left|\alpha_j (x_n)_j - \alpha_j (x_m)_j\right|}{\gamma_i}\right)^{p_i} \\
&\ge \sum_{i=0, i\in \ker(\alpha)^c}^{\infty}\left(\frac{\sum_{j\in G_i}\left|\alpha_j (x_n)_j - \alpha_j (x_m)_j\right|}{\gamma_i}\right)^{p_i} \\
&= \sum_{i=0}^{\infty}\left(\frac{\sum_{j\in G_i}\left|\alpha_j (y_n)_j - \alpha_j (y_m)_j\right|}{\gamma_i}\right)^{p_i} \\
&> \sum_{i=0}^{\infty}\left(\frac{\sum_{j\in G_i}\left|\varepsilon (y_n)_j - \varepsilon (y_m)_j\right|}{\gamma_i}\right)^{p_i} = \varrho(\varepsilon(y_n - y_m)),
\end{aligned} \tag{18}$$

where

$$(y_n)_j = \begin{cases} (x_n)_j, & j \in \ker(\alpha)^c, \\ 0, & j \notin \ker(\alpha)^c. \end{cases} \tag{19}$$

Hence, (y_n) is a Cauchy sequence in $(V(\gamma, p))_\varrho$ since $(V(\gamma, p))_\varrho$ is complete. So, there is $x \in (V(\gamma, p))_\varrho$ with $\lim_{n\to\infty} y_n = x$. Since T_α is continuous, hence $\lim_{n\to\infty} T_\alpha y_n = T_\alpha x$. But $\lim_{n\to\infty} T_\alpha x_n = \lim_{n\to\infty} T_\alpha y_n = q$. Therefore, $T_\alpha x = q$. Hence, $q \in R(T_\alpha)$. This gives $R(T_\alpha)$ is closed. Let the necessary condition be satisfied; this means $R(T_\alpha)$ be closed. Therefore, T_α is bounded away from zero on $((V(\gamma, p))_\rho)_{\ker(\alpha)^c}$, i.e., there is $\varepsilon > 0$ such that $\varrho(T_\alpha x) \ge \varrho(\varepsilon x)$, for all $x \in ((V(\gamma, p))_\varrho)_{\ker(\alpha)^c}$. Let $D = \{j \in \ker(\alpha)^c : |\alpha_j| < \varepsilon\}$. If $D \ne \phi$, then for $n_0 \in D$, since ϱ is nondecreasing, we have

$$\begin{aligned}
\varrho\left(T_\alpha e_{n_0}\right) &= \varrho\left(\left(\alpha_j (e_{n_0})_j\right)_{j=0}^{\infty}\right) = \varrho\left(\left(\left|\alpha_j\right|\left|(e_{n_0})_j\right|\right)_{j=0}^{\infty}\right) \\
&= \sum_{i=0}^{\infty}\left(\frac{\sum_{j\in G_i}\left|\alpha_j\right|\left|(e_{n_0})_j\right|}{\gamma_i}\right)^{p_i} < \sum_{i=0}^{\infty}\left(\frac{\sum_{j\in G_i}\varepsilon\left|(e_{n_0})_j\right|}{i+1}\right)^{p_i} \\
&= \varrho\left(\varepsilon e_{n_0}\right).
\end{aligned} \tag{20}$$

This gives a contradiction. Therefore, $D=\phi$ with $|\alpha_j| \geq \varepsilon$, for each $j \in \ker(\alpha)^c$. This completes the proof.

Theorem 24. *If $\alpha \in \mathbb{C}^{\mathbb{N}}$ and $(V(\gamma,p))_{\varrho}$ is a prequasi Banach (sss), with $\varrho(x)=\sum_{i=0}^{\infty}(\sum_{j\in G_i}|x_j|/\gamma_i)^{p_i}$ for each $x \in V(\gamma,p)$. Hence, there is $b>0$ and $B>0$ such that $b<\alpha_j<B$; for each $j \in \mathbb{N}$ if and only if $T_\alpha \in L((V(\gamma,p))_{\varrho})$ is invertible.*

Proof. Assume that the sufficient condition be verified. Define $\eta \in \mathbb{C}^{\mathbb{N}}$ by $\eta_j = 1/\alpha_j$. Then by Theorem 18, T_α and T_η are bounded linear operators. Also, $T_\alpha \cdot T_\eta = T_\eta \cdot T_\alpha = I$. Hence, T_η is the inverse of T_α. Conversely, suppose the necessary condition be satisfied. Therefore, $R(T_\alpha)=((V(\gamma,p))_\rho)_{\mathbb{N}}$. Hence, $R(T_\alpha)$ is closed. From Theorem 23, there is $b>0$ so that $|\alpha_j|\geq b$, for all $j \in \ker(\alpha)^c$. Now, $\ker(\alpha)=\phi$; otherwise, $\alpha_{n_0}=0$, for some $n_0 \in \mathbb{N}$, we have $e_{n_0} \in \ker(T_\alpha)$. Since $\ker(T_\alpha)$ is trivial, this gives a contradiction. Therefore, $|\alpha_j| \geq b$, for each $j \in \mathbb{N}$. From Theorem 18, since T_α is bounded, there is $B>0$ so that $|\alpha_j|\leq B$, for every $j\in\mathbb{N}$. Therefore, we have showed that $b \leq |\alpha_j| \leq B$, for all $j \in \mathbb{N}$.

Theorem 25. *If $(V(\gamma,p))_{\varrho}$ is a prequasi Banach (sss), with $T_\alpha \in L((V(\gamma,p))_{\varrho})$ and $\rho(y)=\sum_{i=0}^{\infty}(\sum_{j\in G_i}|y_j|/\gamma_i)^{p_i}$ for every $y \in V(\gamma,p)$. Hence, T_α be Fredholm operator if and only if (i) $\ker(\alpha)$ be a finite subset of $\mathbb{N}$. (ii) $|\alpha_j|\geq\varepsilon$, for each $j\in\ker(\alpha)^c$.*

Proof. Let T_α be Fredholm. Assume $ker(\alpha)$ be an infinite subset of $\mathbb{N}$, then $e_j \in ker(T_\alpha)$, for each $j \in \ker(\alpha)$. This gives $\dim(\ker(T_\alpha))=\infty$; hence, we have a contradiction. Therefore, $\ker(\alpha)$ be a finite subset of $\mathbb{N}$. From Theorem 23, condition (ii) follows. On the other hand, let the necessary condition be satisfied. From Theorem 23, condition (ii) shows that $R(T_\alpha)$ is closed and condition (i) gives that $\dim(\ker(T_\alpha))<\infty$ and $\dim(R(T_\alpha))^c<\infty$. This finishes the proof.

5. Closed Banach Prequasi Ideal

We study the sufficient conditions on $V(\gamma,p)$ so that the prequasi ideal $S_{V(\gamma,p)}$ are Banach and closed.

Theorem 26. *Let X and Y be Banach spaces, (a1) and (a2) be satisfied, then $(S_{(V(\gamma,p))_\rho}, g)$ is a prequasi Banach operator ideal, where $g(T)=\rho((s_n(T))_{n=0}^{\infty})$ and $\rho(x)=\sum_{i=0}^{\infty}(\sum_{j\in G_i}|x_j|/\gamma_i)^{p_i}$ for each $x\in V(\gamma,p)$.*

Proof. From Theorem 8, the space $(V(\gamma,p))_\rho$ is a premodular (sss). Hence, by using Theorem 11, we have the function $g(T)=\rho((s_n(T))_{n=0}^{\infty})$ that is a prequasi norm on $S_{(V(\gamma,p))_{\varrho}}$. Suppose (T_j) be a Cauchy sequence in $S_{(V(\gamma,p))_{\varrho}}(X,Y)$, then by part (vii) of Definition 7 there is $\xi>0$ and since $S_{(V(\gamma,p))_{\varrho}}(X,Y)\subseteq L(X,Y)$, we get

$$
\begin{aligned}
g(T_i-T_j) &= \sum_{r=0}^{\infty}\left(\frac{1}{\gamma_r}\sum_{k\in G_r}s_k(T_i-T_j)\right)^{p_r} \\
&\geq \sum_{r=0}^{\infty}\left(\frac{1}{\gamma_r}\|T_i-T_j\|\right)^{p_r} \geq \xi\|T_i-T_j\|\sum_{r=0}^{\infty}\left(\frac{1}{\gamma_r}\right)^{p_r}.
\end{aligned} \tag{21}
$$

Therefore, $(T_j)_{j\in\mathbb{N}}$ be a Cauchy sequence in $L(X,Y)$. Since $L(X,Y)$ is a Banach space, hence there is $T\in L(X,Y)$ with $\lim_{i\to\infty}\|T_i-T\|=0$. Since $(s_n(T_i))_{n=0}^{\infty}\in(V(\gamma,p))_{\varrho}$ for each $i\in\mathbb{N}$, from parts (ii), (iii), (iv), and (v) of Definition 7, we have

$$
\begin{aligned}
g(T) &= \sum_{r=0}^{\infty}\left(\frac{1}{\gamma_r}\sum_{k\in G_r}(s_k(T)\right)^{p_r} = \sum_{r=0}^{\infty}\left(\frac{1}{\gamma_r}\sum_{k\in G_r}(s_k(T-T_m+T_m)\right)^{p_r} \\
&\leq K\sum_{r=0}^{\infty}\left(\frac{1}{\gamma_r}\sum_{k\in G_r}s_{[\frac{k}{2}]}(T-T_m)\right)^{p_r} \\
&\quad + K\sum_{r=0}^{\infty}\left(\frac{1}{\gamma_r}\sum_{k\in G_r}s_{[\frac{k}{2}]}(T_m)\right)^{p_r} \\
&\leq K\sum_{r=0}^{\infty}\left(\frac{1}{\gamma_r}\sum_{k\in G_r}\|T_m-T\|\right)^{p_r} \\
&\quad + KK_0\sum_{r=0}^{\infty}\left(\frac{1}{\gamma_r}\sum_{k\in G_r}s_k(T_m)\right)^{p_r} < \varepsilon.
\end{aligned} \tag{22}
$$

Therefore, $(s_j(T))_{j=0}^{\infty}\in(V(\gamma,p))_\rho$; hence, $T\in S_{(V(\gamma,p))_{\varrho}}(X,Y)$.

Theorem 27. *Let X and Y be Banach spaces, (a1) and (a2) be satisfied, then $(S_{(V(\gamma,p))_{\varrho}}, g)$ is a prequasi closed operator ideal, where $g(T)=\rho((s_n(T))_{n=0}^{\infty})$ and $\rho(x)=\sum_{i=0}^{\infty}(\sum_{j\in G_i}|x_j|/\gamma_i)^{p_i}$ for each $x\in V(\gamma,p)$.*

Proof. From Theorem 8, the space $(V(\gamma,p))_\rho$ is a premodular (sss). Hence, by using Theorem 11, we have the function $g(T)=\rho((s_n(T))_{n=0}^{\infty})$ that is a prequasi norm on $S_{(V(\gamma,p))_{\varrho}}$. Suppose $T_j\in S_{(V(\gamma,p))_{\varrho}}(X,Y)$ for each $j\in\mathbb{N}$ and $\lim_{m\to\infty}g(T_m-T)=0$. Since $S_{(V(\gamma,p))_{\varrho}}(X,Y)\subseteq L(X,Y)$ and from part (vii) of Definition 7, we have

$$
\begin{aligned}
g(T-T_m) &= \sum_{r=0}^{\infty}\left(\frac{1}{\gamma_r}\sum_{k\in G_r}s_k(T-T_m)\right)^{p_r} \\
&\geq \sum_{r=0}^{\infty}\left(\frac{1}{\gamma_r}\|T-T_m\|\right)^{p_r} \geq \xi\|T-T_m\|\sum_{r=0}^{\infty}\left(\frac{1}{\gamma_r}\right)^{p_r}.
\end{aligned} \tag{23}
$$

Hence, $(T_m)_{m\in\mathbb{N}}$ is convergent in $L(X,Y)$. Since $(s_n(T_i))_{n=0}^{\infty}\in(V(\gamma,p))_{\varrho}$ for each $i\in\mathbb{N}$, from parts (ii), (iii), (iv), and (v) of Definition 7, we obtain

$$\begin{aligned} g(T)&=\sum_{r=0}^{\infty}\frac{1}{\gamma_r}\sum_{k\in G_r}(s_k(T))^{p_r}=\sum_{r=0}^{\infty}\frac{1}{\gamma_r}\sum_{k\in G_r}(s_k(T-T_m+T_m))^{p_r}\\ &\le K\sum_{r=0}^{\infty}\left(\frac{1}{\gamma_r}\sum_{k\in G_r}s_{[k/2]}(T-T_m)\right)^{p_r}\\ &\quad+K\sum_{r=0}^{\infty}\left(\frac{1}{\gamma_r}\sum_{k\in G_r}s_{[k/2]}(T_m)\right)^{p_r}\\ &\le K\sum_{r=0}^{\infty}\left(\frac{1}{\gamma_r}\sum_{k\in G_r}\|T_m-T\|\right)^{p_r}\\ &\quad+KK_0\sum_{r=0}^{\infty}\left(\frac{1}{\gamma_r}\sum_{k\in G_r}s_k(T_m)\right)^{p_r}<\varepsilon. \end{aligned} \tag{24}$$

Therefore, $(s_n(T))_{n=0}^{\infty}\in(V(\gamma,p))_{\rho}$. Hence, $T\in S_{(V(\gamma,p))_{\varrho}}(X,Y)$.

6. Small and Simple Prequasi Banach Operator Ideal

We introduce some inclusion relations concerning prequasi operator ideal constructed by the sequence of s-numbers and $V(\gamma,p)$.

Theorem 28. *If X, Y are infinite dimensional Banach spaces, $(p_r)\in\ell_{\infty}$, $(q_r)\in\ell_{\infty}$ with $1<p_r<q_r$, and $1\le\gamma_r\le\nu_r$ for every $r\in\mathbb{N}$, then*

$$S_{V(\gamma,p)}(X,Y)\subsetneqq S_{V(\nu,q)}(X,Y)\subsetneqq L(X,Y). \tag{25}$$

Proof. Assume the sufficient conditions be verified and $T\in S_{V(\gamma,p)}(X,Y)$. Therefore, $(s_k(T))\in V(\gamma,p)$. We have

$$\sum_{r=0}^{\infty}\left(\frac{1}{\nu_r}\sum_{k\in G_r}s_k(T)\right)^{q_r}<\sum_{r=0}^{\infty}\left(\frac{1}{\gamma_r}\sum_{k\in G_r}s_k(T)\right)^{p_r}<\infty. \tag{26}$$

Hence, $T\in S_{V(\nu,q)}(X,Y)$. By taking $(s_r(T))_{r=0}^{\infty}$ such that $\sum_{k\in G_r}s_k(T)=\gamma_r(r+1)^{-1/p_r}$, one can find $T\in L(X,Y)$ so that

$$\begin{aligned} \sum_{r=0}^{\infty}\left(\frac{1}{\gamma_r}\sum_{k\in G_r}s_k(T)\right)^{p_r}&=\sum_{r=0}^{\infty}\frac{1}{r+1}=\infty,\\ \sum_{r=0}^{\infty}\left(\frac{1}{\nu_r}\sum_{k\in G_r}s_k(T)\right)^{q_r}&\le\sum_{r=0}^{\infty}\left(\frac{1}{\gamma_r}\sum_{k\in G_r}s_k(T)\right)^{q_r}\\ &=\sum_{r=0}^{\infty}\left(\frac{1}{r+1}\right)^{q_r/p_r}<\infty. \end{aligned} \tag{27}$$

Hence, $T\in S_{V(\nu,q)}(X,Y)$ and $T\notin S_{V(\gamma,p)}(X,Y)$. Obviously, $S_{V(\gamma,p)}(X,Y)\subset L(X,Y)$. By choosing $(s_r(T))_{r=0}^{\infty}$ such that $\sum_{k\in G_r}s_k(T)=\nu_r(r+1)^{-1/q_r}$. We have $T\in L(X,Y)$ and $T\notin S_{V(\nu,q)}(X,Y)$. This finishes the proof.

Corollary 29. *If X, Y are infinite dimensional Banach spaces and $1<p<q<\infty$, it is true that*

$$S_{ces_p}(X,Y)\subsetneqq S_{ces_q}(X,Y)\subsetneqq L(X,Y). \tag{28}$$

We investigate the sufficient conditions so that the prequasi Banach operator ideal $S^{app}_{V(\gamma,p)}$ is small.

Theorem 30. *The prequasi Banach operator ideal $S^{app}_{V(\gamma,p)}$ is small, whenever X, Y are infinite dimensional Banach spaces and the conditions (a1) and (a2) are satisfied.*

Proof. If the sufficient conditions are verified, therefore the space $(S^{\rm app}_{V(\gamma,p)},g)$ is a prequasi Banach operator ideal, with $g(T)=1/\eta(\sum_{i=0}^{\infty}(\gamma_i^{-1}\sum_{j\in G_i}\alpha_j(T))^{p_i})^{1/h}$ and $\eta=(\sum_{i=0}^{\infty}\gamma_i^{-p_i})^{1/h}$. Suppose that $S^{\rm app}_{V(\gamma,p)}(X,Y)=L(X,Y)$, so there is $C>0$ so that $g(T)\le C\|T\|$ for each $T\in L(X,Y)$. From Dvoretzky's theorem [27] for all $m\in\mathbb{N}$, we obtain subspaces M_m of Y and quotient spaces X/N_m transform onto ℓ_2^m by isomorphisms A_m and H_m with $\|A_m\|\|A_m^{-1}\|\le 2$ and $\|H_m\|\|H_m^{-1}\|\le 2$. If J_m is the natural embedding map from M_m into Y and Q_m is the quotient map from X onto X/N_m, suppose ν_n denotes the Bernstein numbers [28], we have

$$\begin{aligned} 1=u_n(I_m)&=\nu_n\left(A_mA_m^{-1}I_mH_mH_m^{-1}\right)\\ &\le\|A_m\|\nu_n\left(A_m^{-1}I_mH_m\right)\left\|H_m^{-1}\right\|\\ &=\|A_m\|\nu_n\left(J_mA_m^{-1}I_mH_m\right)\left\|H_m^{-1}\right\|\\ &\le\|A_m\|d_n\left(J_mA_m^{-1}I_mH_m\right)\left\|H_m^{-1}\right\|\\ &=\|A_m\|d_n\left(J_mA_m^{-1}I_mH_mQ_m\right)\left\|H_m^{-1}\right\|\\ &\le\|A_m\|\alpha_n\left(J_mA_m^{-1}I_mH_mQ_m\right)\left\|H_m^{-1}\right\|, \end{aligned} \tag{29}$$

for $0\le i\le m$. Now

$$\begin{aligned} \sum_{j\in G_i}\left(\gamma_i^{-1}\right)&\le\sum_{j\in G_i}\|A_m\|\gamma_i^{-1}\alpha_j\left(J_mA_m^{-1}I_mH_mQ_m\right)\left\|H_m^{-1}\right\|\Rightarrow 1\\ &\le\|A_m\|\left(\gamma_i^{-1}\sum_{j\in G_i}\alpha_j\left(J_mA_m^{-1}I_mH_mQ_m\right)\right)\left\|H_m^{-1}\right\|\Rightarrow 1\\ &\le\left(\|A_m\|\left\|H_m^{-1}\right\|\right)^{p_i}\left(\gamma_i^{-1}\sum_{j\in G_i}\alpha_j\left(J_mA_m^{-1}I_mH_mQ_m\right)\right)^{p_i}. \end{aligned} \tag{30}$$

Therefore,

$$\begin{aligned}(m+1)^{1/h} &\le L\|A_m\|\|H_m^{-1}\| \cdot \left[\sum_{i=0}^{m}\left(\gamma_i^{-1}\sum_{j\in G_i}\alpha_j\left(J_mA_m^{-1}I_mH_mQ_m\right)\right)^{p_i}\right]^{1/h} \\ &\Rightarrow \frac{1}{\eta}(m+1)^{1/h} \le L\|A_m\|\|H_m^{-1}\|\frac{1}{\eta} \cdot \left[\sum_{i=0}^{m}\left(\gamma_i^{-1}\sum_{j\in G_i}\alpha_j\left(J_mA_m^{-1}I_mH_mQ_m\right)\right)^{p_i}\right]^{1/h} \\ &\Rightarrow \frac{1}{\eta}(m+1)^{1/h} \le L\|A_m\|\|H_m^{-1}\|g\left(J_mA_m^{-1}I_mH_mQ_m\right) \\ &\Rightarrow \frac{1}{\eta}(m+1)^{1/h} \le LC\|A_m\|\|H_m^{-1}\|\|J_mA_m^{-1}I_mH_mQ_m\| \\ &\Rightarrow \frac{1}{\eta}(m+1)^{1/h} \le LC\|A_m\|\|H_m^{-1}\|\|J_mA_m^{-1}\| \cdot \|I_m\|\|H_mQ_m\| = LC\|A_m\|\|H_m^{-1}\|\|A_m^{-1}\| \\ &\cdot \|I_m\|\|H_m\| \Rightarrow \frac{1}{\eta}(m+1)^{1/h} \le 4LC,\end{aligned} \tag{31}$$

for some $L \ge 1$. Since m is an arbitrary, this gives a contradiction. Therefore, dim $(X) < \infty$ and dim $(Y) < \infty$ when $S^{\text{app}}_{V(\gamma,p)}(X, Y) = L(X, Y)$. This finishes the proof.

We introduce the next theorem without verification; these can be set up utilizing standard procedure.

Theorem 31. *The prequasi Banach operator ideal $S^{Kol}_{V(\gamma,p)}$ is small, whenever X, Y are infinite dimensional Banach spaces and the conditions (a1) and (a2) are satisfied.*

Corollary 32. *The prequasi Banach operator ideal $S^{app}_{ces_p}$ is small, if X, Y are infinite dimensional Banach spaces and $p \in (1, \infty)$.*

Corollary 33. *The prequasi Banach operator ideal $S^{Kol}_{ces_p}$ is small, if X, Y are infinite dimensional Banach spaces and $p \in (1, \infty)$.*

We examine the sufficient conditions so that the prequasi Banach operator ideal $S_{V(\gamma,p)}$ is simple.

Theorem 34. *If X, Y are infinite dimensional Banach spaces, $(p_j) \in \ell_\infty$, $(q_j) \in \ell_\infty$ with $1 < p_j < q_j$, and $1 \le \gamma_j \le \nu_j$ for each $j \in \mathbb{N}$, it is true that*

$$\Psi\left(S_{V(\nu,q)}, S_{V(\gamma,p)}\right) = L\left(S_{V(\nu,q)}, S_{V(\gamma,p)}\right). \tag{32}$$

Proof. Let the sufficient conditions be given, $T \in L(S_{V(\nu,q)}, S_{V(\gamma,p)})$ and $T \notin \Psi(S_{V(\nu,q)}, S_{V(\gamma,p)})$. From Lemma 1, we have $A \in L(S_{V(\nu,q)})$ and $B \in L(S_{V(\gamma,p)})$ with $BTAI_k = I_k$. For each $k \in \mathbb{N}$, we obtain

$$\begin{aligned}\|I_k\|_{S_{V(\gamma,p)}} &= \sum_{n=0}^{\infty}\left(\frac{1}{\gamma_n}\sum_{i\in G_n}s_i(I_k)\right)^{p_n} \le \|BTA\|\|I_k\|_{S_{V(\nu,q)}} \\ &\le \sum_{n=0}^{\infty}\left(\frac{1}{\nu_n}\sum_{i\in G_n}s_i(I_k)\right)^{q_n}.\end{aligned} \tag{33}$$

In view of Theorem 28, we get a contradiction. Therefore, $T \in \Psi(S_{V(\nu,q)}, S_{V(\gamma,p)})$.

Corollary 35. *If X, Y are infinite dimensional Banach spaces, $(p_j) \in \ell_\infty$, $(q_j) \in \ell_\infty$ with $1 < p_j < q_j$, and $1 \le \gamma_j \le \nu_j$ for each $j \in \mathbb{N}$, it is true that*

$$L_C\left(S_{ces((q_n))}, S_{ces((p_n))}\right) = L\left(S_{V(\nu,q)}, S_{V(\gamma,p)}\right). \tag{34}$$

Theorem 36. *The prequasi Banach operator ideal $S_{V(\gamma,p)}$ is simple, if X, Y are infinite dimensional Banach spaces and the conditions (a1) and (a2) are satisfied.*

Proof. Let the sufficient conditions be given and there is $T \in L_C(S_{V(\gamma,p)})$ with $T \notin \Psi(S_{V(\gamma,p)})$. From Lemma 1, we have $A, B \in L(S_{V(\gamma,p)})$ so that $BTAI_k = I_k$. This means that $I_{S_{V(\gamma,p)}} \in L_C(S_{V(\gamma,p)})$. Consequently, $L(S_{V(\gamma,p)}) = L_C(S_{V(\gamma,p)})$. Therefore, $S_{V(\gamma,p)}$ is simple.

7. Eigenvalues of s-Type Operators

We investigate the strict inclusion relation between $S_{V(\gamma,p)}$ and S^λ_E (the class of all bounded linear operators whose sequence of eigenvalues belongs to $V(\gamma, p)$).

Notations 37. $S^\lambda_E := \{S^\lambda_E(X, Y)$; X and Y are Banach Spaces$\}$, where $S^\lambda_E(X, Y) := \{T \in L(X, Y)$: $(\lambda_i(T))_{n=0}^\infty \in E$ and $\|T - \lambda_n(T)I\|^{-1}$; does not exist for every $n \in \mathbb{N}\}$.

Theorem 38. *If X, Y are infinite dimensional Banach spaces and conditions (a1) and (a2) are satisfied, it is true that*

$$S_{V(\gamma,p)}(X, Y) \subsetneq S^\lambda_{V(\gamma,p)}(X, Y). \tag{35}$$

Proof. Let the sufficient conditions be verified and $T \in S_{V(\gamma,p)}(X, Y)$. Therefore, $(s_n(T))_{n=0}^\infty \in V(\gamma, p)$. Hence, we have

$$\sum_{r=0}^{\infty}\left(\frac{1}{\gamma_r}\sum_{k\in G_r}s_k(T)\right)^{p_r} \ge \sum_{r=0}^{\infty}[s_r(T)]^{p_r}. \tag{36}$$

Hence, $(s_r(T))_{r=0}^\infty \in \ell_{(p_r)}$, so $\lim_{r\to\infty}s_r(T) = 0$. Assume $\|T - s_r(T)I\|^{-1}$ exists for every $r \in \mathbb{N}$. Therefore, $\|T - s_r(T)I\|^{-1}$ exists and bounded for every $r \in \mathbb{N}$. This gives $\lim_{r\to\infty}\|T - s_r(T)I\|^{-1} = \|T\|^{-1}$ exists and bounded. From the prequasi operator ideal of $(S_{V(\gamma,p)}, g)$, we obtain

$$I = TT^{-1} \in S_{V(\gamma,p)}(X, Y) \Rightarrow (s_r(I))_{r=0}^{\infty} \in V(\gamma, p) \Rightarrow \lim_{r\to\infty} s_r(I) = 0. \quad (37)$$

We have a contradiction, since $\lim_{r\to\infty} s_r(I) = 1$. Therefore, $\|T - s_r(T)I\| = 0$ for every $r \in \mathbb{N}$. Hence, $(s_r(T))_{r=0}^{\infty}$ is the eigenvalues of T. Finally, if we choose $(s_r(T))_{r=0}^{\infty}$ such that $\sum_{k\in G_r} s_k(T) = \gamma_r(r+1)^{-1/p_r}$, we find $T \in L(X, Y)$ such that $\sum_{r=0}^{\infty}(1/\gamma_r \sum_{k\in G_r} s_k(T))^{p_r} = \sum_{r=0}^{\infty} 1/(r+1) = \infty$ and if we take $(\lambda_r(T))_{r=0}^{\infty}$ such that $\sum_{k\in G_r} \lambda_k(T) = \gamma_r/(r+1)$. Hence, $T \notin S_{V(\gamma,p)}(X, Y)$ and $T \in S^{\lambda}_{V(\gamma,p)}(X, Y)$. This finishes the proof.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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