Research Article

Oscillation Results for a Class of Nonlinear Fractional Order Difference Equations with Damping Term

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The paper studies the oscillation of a class of nonlinear fractional order difference equations with damping term of the form $\Delta[\psi(\lambda)z^\eta(\lambda)] + p(\lambda)z^\eta(\lambda) + q(\lambda)F(\sum_{s=\lambda_0}^{\lambda-1+\mu} (\lambda-s-1)^{(-\mu)}y(s)) = 0$, where $z(\lambda) = a(\lambda) + b(\lambda)\Delta^\mu y(\lambda)$, Δ^μ stands for the fractional difference operator in Riemann-Liouville settings and of order μ , $0 < \mu \leq 1$, and $\eta \geq 1$ is a quotient of odd positive integers and $\lambda \in \mathbb{N}_{\lambda_0+1-\mu}$. New oscillation results are established by the help of certain inequalities, features of fractional operators, and the generalized Riccati technique. We verify the theoretical outcomes by presenting two numerical examples.

1. Introduction and Background

The objective of this paper is to provide oscillation theorems for the equation

$$\Delta[\psi(\lambda)z^\eta(\lambda)] + p(\lambda)z^\eta(\lambda) + q(\lambda)F\left(\sum_{s=\lambda_0}^{\lambda-1+\mu} (\lambda-s-1)^{(-\mu)}y(s)\right) = 0, \quad (1)$$

where $\lambda \in \mathbb{N}_{\lambda_0+1-\mu}$, $\eta > 0$ is a quotient of odd positive integers, Δ^μ is the fractional difference operator in the sense of Riemann-Liouville (RL) and of order μ , $0 < \mu \leq 1$, and

$$z(\lambda) = a(\lambda) + b(\lambda)\Delta^\mu y(\lambda). \quad (2)$$

The μ^{th} fractional sum for $\mu > 0$, (see [1]) is defined by

$$\Delta_a^{-\mu} f(\lambda) = \frac{1}{\Gamma(\mu)} \sum_{s=a}^{\lambda-\mu} (\lambda-s-1)^{(\mu-1)} f(s), \quad \lambda \in \mathbb{N}_{a+\mu}, \quad (3)$$

where the fractional sum $\Delta_a^{-\mu}$ is defined from $\mathbb{N}_a$ to $\mathbb{N}_{a+\mu}$, $f(s)$ is defined for $s \equiv a \pmod{1}$ and $\Delta_a^{-\mu} f(\lambda)$ is defined for $\lambda \equiv (a + \mu) \pmod{1}$. The falling function is

$$\lambda^{(\mu)} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\mu)}. \quad (4)$$

where Γ is the Gamma function, given by

$$\Gamma(t) := \int_0^\infty s^{t-1} e^{-s} ds, \quad (5)$$

for $t \in \mathbb{R}_+ := (0, \infty)$.

Let μ and N be positive integers such that $N = \lceil \mu \rceil$, namely, $\mu \in (N-1, N)$. Set $\nu = N - \mu$. Then, μ^{th} fractional difference (see [2]) is defined as

$$\Delta^\mu f(\lambda) = \Delta^{N-\nu} f(\lambda) = \Delta^N \Delta^{-\nu} f(\lambda). \quad (6)$$

The following conditions are assumed to hold throughout this work:

- (i) (H_1) $\psi(\lambda) \in C([t_0, \infty))$, $p(\lambda) \in C([t_0, \infty))$, and $q(\lambda) \in C([t_0, \infty))$ are positive sequences with $\psi(\lambda) > p(\lambda)$
- (ii) (H_2)

$$\Delta \left[\frac{a(\lambda)}{b(\lambda)} \right] \neq 0 \quad (7)$$

for $\lambda \in [\lambda_0, \infty)$ and

$$\lim_{\lambda \rightarrow \infty} \sum_{k=\lambda_0}^{\lambda-1} \left[\frac{a(k)}{b(k)} \right] < \infty \quad (8)$$

- (iii) (H_3) $a(\lambda)$ is a nonnegative sequence on $[\lambda_0, \infty)$ for some $\lambda_0 > 0$. There exists a $n_0 > 0$ such that $b(\lambda) \geq n_0$ for $\lambda \in [\lambda_0, \infty)$
- (iv) (H_4) F is a monotone decreasing function satisfying $\Psi F(\Psi) > 0$ such that

$$\frac{F(\Psi)}{\Psi^\eta} \geq L > 0 \text{ for } \Psi \neq 0 \quad (9)$$

A nontrivial solution of Eq. (1) is called oscillatory if it has arbitrarily large zeros. Otherwise, it is called nonoscillatory. Eq. (1) is called oscillatory if all its solutions are oscillatory.

In the literature, Wang et al. [3] extended some oscillation results from integer-order differential equation to the fractional-order differential equation

$$\mathfrak{D}_a^\alpha x(t) + q(t)f(x(t)) = 0; \quad t \in [a, +\infty), a > 0, \quad (10)$$

where $\mathfrak{D}_a^\alpha$ denotes the standard RL differential operator of order α with $0 < \alpha \leq 1$, $q(t)$ is a positive real-valued function, $f(\cdot): [0, \infty) \rightarrow [0, \infty)$ is a continuous function satisfying

$$\frac{f(u)}{u} I^{\alpha-2} \geq K > 0, \quad (11)$$

and $I^{2-\alpha}$ denotes RL integral operator. Xiang et al. [4] investigated conditions for oscillation of fractional-order differential equation

$$[a(t)w^\gamma(t)] - b(\lambda)f\left(\int_t^\infty (s-t)^{-\alpha}x(s)ds\right) = 0; \quad t \geq t_0 > 0, \quad (12)$$

where $w(t) = p(t) + q(t)(\mathfrak{D}_-^\alpha x)(t)$, $\alpha \in (0, 1)$ is a fractional-order, $\gamma > 0$ is a quotient of odd positive integers, and $(\mathfrak{D}_-^\alpha x)$ is the RL right-sided fractional derivative of order α of x defined by

$$(\mathfrak{D}_-^\alpha x)(t) := \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^\infty (s-t)^{-\alpha} x(s) ds, \quad (13)$$

for $t \in \mathbb{R}_+$.

In this paper and motivated by the above work, we intend to carry forward the oscillation results from fractional differential Eq. (12) to the fractional difference Eq. (1). Moreover, we consider Eq. (1) under a damping term.

Fractional difference calculus is evolving as a powerful tool for studying problems in many fields such as biology, mechanics, control systems, ecology, electrical networks, electrochemical of corrosion, chemical physics, optics, and signal processing, economics and so forth (see [5–12]) and the references therein. Basic definitions and properties of fractional difference calculus were presented by Goodrich and Peterson [1]. In addition, there are other research works dealing with fractional difference equations which have helped to build up some of the basic theory in this area (see [2, 13–15]).

Fractional difference equations are applied to model physical processes which vary with time and space and its nonlocal property enables to model systems with memory effect. The study of the qualitative analysis of these equations has gained momentum in recent years and thus numerous publications have been reviled [16–20].

Of late, the investigation of the oscillation of solutions for fractional order difference equations has accelerated with several articles (see [21–29]). In what follows, we state some lemmas and preliminaries that will contribute in proving our main results.

Definition 1 (see [2]). For any $\mu \geq 0$, the falling factorial is known as

$$s^{(\mu)} = \frac{\Gamma(s+1)}{\Gamma(s+1-\mu)}, \quad (14)$$

where Γ is as in (5).

Lemma 2 (see [23]). Let $y(\lambda)$ be a solution of (1) and

$$\Psi(\lambda) = \sum_{s=\lambda_0}^{\lambda-I+\mu} (\lambda-s-I)^{(-\mu)} y(s), \quad (15)$$

then

$$\Delta \Psi(\lambda) = \Gamma(1-\mu) \Delta^\mu y(\lambda). \quad (16)$$

Lemma 3 (see [30]). The product and quotient rules of the difference operator Δ are defined to be

$$\Delta[x(\lambda)y(\lambda)] = \Delta x(\lambda)y(\lambda+I) + x(\lambda)\Delta y(\lambda), \quad (17)$$

and

$$\Delta \left[\frac{x(\lambda)}{y(\lambda)} \right] = \frac{\Delta x(\lambda)y(\lambda) - x(\lambda)\Delta y(\lambda)}{y(\lambda)y(\lambda+I)}, \quad (18)$$

where $\Delta x(\lambda) = x(\lambda+I) - x(\lambda)$.

Lemma 4 (see [31]). Let $\eta \geq 1$ be a quotient of two positive odd integers. If

$$\Psi(\lambda + 1) > \Psi(\lambda) > 0 (\Psi(\lambda + 1) < \Psi(\lambda) < 0), \quad (19)$$

then

$$\Delta \Psi^\eta(\lambda) \geq (\Delta \Psi(\lambda))^\eta (\Delta \Psi^\eta(\lambda) \leq (\Delta \Psi(\lambda))^\eta). \quad (20)$$

Lemma 5 (see [32]). Let $\xi, \zeta \in \mathbb{R}$ with $\zeta > 0$. Then, the inequality

$$\xi X - \zeta X^2 \leq \frac{\xi^2}{4\zeta}, \quad (21)$$

holds for all $X \in \mathbb{R}$.

2. Main Results

Herein, new oscillation theorems for Eq. (1) are established by using mathematical inequalities, the properties of RL sum and difference operators, and the generalized Riccati technique.

Define the sequence

$$u(\lambda) = \prod_{s=\lambda_0}^{\lambda-1} \frac{\psi(s)}{\psi(s) - p(s)}. \quad (22)$$

Then $u(\lambda) > 0$,

$$u(\lambda + 1) = \frac{\psi(\lambda)}{\psi(\lambda) - p(\lambda)} u(\lambda), \quad (23)$$

and hence

$$\Delta u(\lambda) = u(\lambda + 1) - u(\lambda) = \frac{p(\lambda)}{\psi(\lambda) - p(\lambda)} u(\lambda). \quad (24)$$

Theorem 6. Assume that $(H_1) - (H_4)$ hold. If

$$\lim_{\lambda \rightarrow \infty} \sum_{v=\lambda_0}^{\lambda-1} \left[\frac{1}{u(v)\psi(v)} \right]^{1/\eta} = \infty, \quad (25)$$

and

$$\lim_{\lambda \rightarrow \infty} \sum_{v=\lambda_2}^{\lambda-1} \left[Lu(v)q(v) - \frac{(\Delta u(v))^2 \psi(v+1)N^\eta}{4u(v+1)} \right] = \infty, \quad (26)$$

then Eq. (1) is oscillatory.

Proof. Suppose that $y(\lambda)$ is a nonoscillatory solution of Eq. (1). Without loss of generality, we may assume that $y(\lambda)$ is an eventually positive solution of Eq. (1). Then, there exists $\lambda_1 \in [\lambda_0, \infty)$ such that $y(\lambda) > 0$, $\Psi(\lambda) > 0$ for $\lambda \in [\lambda_1, \infty)$,

where $\Psi(\lambda)$ is defined in Lemma 2. Considering the assumption (H_4) and using Eq. (1), we get

$$\Delta[\psi(\lambda)z^\eta(\lambda)] + p(\lambda)z^\eta(\lambda) = -q(\lambda)F(\Psi(\lambda)), \quad (27)$$

or

$$\Delta[\psi(\lambda)z^\eta(\lambda)] + p(\lambda)z^\eta(\lambda) \leq -Lq(\lambda)\Psi^\eta(\lambda), \quad (28)$$

where $z(\lambda)$ is defined in (2). Hence, we proceed from (22) and (17) to

$$\begin{aligned} \Delta[u(\lambda)\psi(\lambda)z^\eta(\lambda)] &= u(\lambda+1)\Delta[\psi(\lambda)z^\eta(\lambda)] \\ &\quad + \Delta u(\lambda)\psi(\lambda)z^\eta(\lambda) \\ &= u(\lambda+1)\Delta[\psi(\lambda)z^\eta(\lambda)] \\ &\quad + u(\lambda)\frac{p(\lambda)}{\psi(\lambda)-p(\lambda)}\psi(\lambda)z^\eta(\lambda) \\ &= u(\lambda+1)\Delta[\psi(\lambda)z^\eta(\lambda)] \\ &\quad + u(\lambda+1)\frac{\psi(\lambda)-p(\lambda)}{\psi(\lambda)} \\ &\quad \cdot \frac{p(\lambda)}{\psi(\lambda)-p(\lambda)}\psi(\lambda)z^\eta(\lambda) \\ &= u(\lambda+1)[\Delta[\psi(\lambda)z^\eta(\lambda)] + p(\lambda)z^\eta(\lambda)] \\ &\leq -Lu(\lambda+1)q(\lambda)\Psi^\eta(\lambda) < 0. \end{aligned} \quad (29)$$

Then, $u(\lambda)\psi(\lambda)z^\eta(\lambda)$ is strictly decreasing on $[\lambda_1, \infty)$ and is eventually of constant sign. Since $u(\lambda) > 0$, $\psi(\lambda) > 0$, and $\eta > 0$ is a quotient of odd positive integers, we observe that $z(\lambda)$ is eventually of constant sign.

First, we show that

$$z(\lambda) > 0 \text{ for } \lambda \in [\lambda_1, \infty). \quad (30)$$

If not, then there exists $\lambda_2 \geq \lambda_1$ such that $z(\lambda_2) < 0$, and we obtain

$$u(\lambda)\psi(\lambda)z^\eta(\lambda) < u(\lambda_2)\psi(\lambda_2)z^\eta(\lambda_2) = d < 0, \quad (31)$$

which implies

$$z^\eta(\lambda) < \frac{d}{u(\lambda)\psi(\lambda)} < 0, \quad (32)$$

for $\lambda \in [\lambda_2, \infty)$ that is

$$z(\lambda) < \left[\frac{d}{u(\lambda)\psi(\lambda)} \right]^{1/\eta} < 0. \quad (33)$$

So we arrive at that $z(\lambda) < 0$ on $[\lambda_2, \infty)$. Hence for $\lambda \in [\lambda_2, \infty)$, we get

$$\begin{aligned} \frac{a(\lambda)}{b(\lambda)} + \Delta^\mu y(\lambda) &< \frac{z(\lambda)}{b(\lambda)} < \frac{1}{b(\lambda)} \left[\frac{d}{u(\lambda)\psi(\lambda)} \right]^{1/\eta} \\ &< \frac{1}{n_0} \left[\frac{d}{u(\lambda)\psi(\lambda)} \right]^{1/\eta}. \end{aligned} \quad (34)$$

From (16), we have

$$\frac{a(\lambda)}{b(\lambda)} + \frac{\Delta\Psi(\lambda)}{\Gamma(1-\mu)} = \frac{a(\lambda)}{b(\lambda)} + \Delta^\alpha y(\lambda) < \frac{1}{n_0} \left[\frac{d}{u(\lambda)\psi(\lambda)} \right]^{1/\eta}, \quad (35)$$

and summing from λ_2 to $\lambda - 1$, we obtain

$$\sum_{v=\lambda_2}^{\lambda-1} \Delta\Psi(v) < \Gamma(1-\mu) \sum_{v=\lambda_2}^{\lambda-1} \left[\frac{1}{n_0} \left[\frac{d}{u(v)\psi(v)} \right]^{1/\eta} - \frac{a(v)}{b(v)} \right], \quad (36)$$

and hence we have

$$\Psi(\lambda) < \Psi(\lambda_2) + \Gamma(1-\mu) \sum_{v=\lambda_2}^{\lambda-1} \left[\frac{1}{n_0} \left[\frac{d}{u(v)\psi(v)} \right]^{1/\eta} - \frac{a(v)}{b(v)} \right]. \quad (37)$$

Now, by letting $\lambda \rightarrow \infty$, we get

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} \Psi(\lambda) &< \lim_{\lambda \rightarrow \infty} \left\{ \Psi(\lambda_2) + \Gamma(1-\mu) \sum_{v=\lambda_2}^{\lambda-1} \left[\frac{1}{n_0} \left[\frac{d}{u(v)\psi(v)} \right]^{1/\eta} - \frac{a(v)}{b(v)} \right] \right\} \\ &< \Psi(\lambda_2) + \Gamma(1-\mu) \sum_{v=\lambda_2}^{\infty} \left[\frac{1}{n_0} \left[\frac{d}{u(v)\psi(v)} \right]^{1/\eta} - \frac{a(v)}{b(v)} \right] = -\infty, \end{aligned} \quad (38)$$

which contradicts that $\Psi(\lambda) > 0$, $\lambda \in [\lambda_1, \infty)$. Therefore $z(\lambda) > 0$ for $\lambda \in [\lambda_2, \infty)$.

Now, since

$$\frac{z(\lambda)}{b(\lambda)} = \frac{a(\lambda)}{b(\lambda)} + \Delta^\mu y(\lambda) = \frac{a(\lambda)}{b(\lambda)} + \frac{\Delta\Psi(\lambda)}{\Gamma(1-\mu)}, \quad (39)$$

and we have

$$\frac{\Delta\Psi(\lambda)}{\Gamma(1-\mu)} = \frac{z(\lambda) - a(\lambda)}{b(\lambda)} < \frac{z(\lambda) - a(\lambda)}{n_0} < \frac{z(\lambda)}{n_0}, \quad (40)$$

from (16), we obtain

$$\Delta\Psi(\lambda) < \frac{z(\lambda)}{n_0}. \quad (41)$$

Define the generalized Riccati function

$$w(\lambda) = u(\lambda) \frac{\psi(\lambda)z^\eta(\lambda)}{\Psi^\eta(\lambda)}; \quad \lambda \in [\lambda_1, \infty). \quad (42)$$

It is clear that $w(\lambda) > 0$. Using (17) and (18) for $\lambda \in [\lambda_0, \infty)$, we have

$$\begin{aligned} \Delta w(\lambda) &= \Delta u(\lambda) \frac{\psi(\lambda+1)z^\eta(\lambda+1)}{\Psi^\eta(\lambda+1)} \\ &\quad + u(\lambda) \Delta \left[\frac{\psi(\lambda)z^\eta(\lambda)}{\Psi^\eta(\lambda)} \right] = \Delta u(\lambda) \frac{w(\lambda+1)}{u(\lambda+1)} \\ &\quad + u(\lambda) \left[\frac{\Delta\{\psi(\lambda)z^\eta(\lambda)\}\Psi^\eta(\lambda) - \psi(\lambda)z^\eta(\lambda)\Delta\Psi^\eta(\lambda)}{\Psi^\eta(\lambda)\Psi^\eta(\lambda+1)} \right] \\ &= \Delta u(\lambda) \frac{w(\lambda+1)}{u(\lambda+1)} + u(\lambda) \frac{\Delta\{\psi(\lambda)z^\eta(\lambda)\}}{\Psi^\eta(\lambda+1)} \\ &\quad - u(\lambda) \frac{\psi(\lambda)z^\eta(\lambda)}{\Psi^\eta(\lambda)\Psi^\eta(\lambda+1)} \Delta\Psi^\eta(\lambda) = \Delta u(\lambda) \frac{w(\lambda+1)}{u(\lambda+1)} \\ &\quad - u(\lambda) \left[\frac{p(\lambda)z^\eta(\lambda) + q(\lambda)f(\Psi(\lambda))}{\Psi^\eta(\lambda+1)} \right] \\ &\quad - \frac{u(\lambda)\psi(\lambda)z^\eta(\lambda)}{\Psi^\eta(\lambda)\Psi^\eta(\lambda+1)} \Delta\Psi^\eta(\lambda) = \Delta u(\lambda) \frac{w(\lambda+1)}{u(\lambda+1)} \\ &\quad - \frac{u(\lambda)p(\lambda)z^\eta(\lambda)}{\Psi^\eta(\lambda+1)} - \frac{u(\lambda)q(\lambda)f(\Psi(\lambda))}{\Psi^\eta(\lambda+1)} \\ &\quad - u(\lambda) \frac{\psi(\lambda)z^\eta(\lambda)}{\Psi^\eta(\lambda)\Psi^\eta(\lambda+1)} \Delta\Psi^\eta(\lambda). \end{aligned} \quad (43)$$

Since

$$\frac{u(\lambda)\psi(\lambda)z^\eta(\lambda)}{\Psi^\eta(\lambda)\Psi^\eta(\lambda+1)} > 0, \quad (44)$$

and $\Delta\Psi(\lambda) > 0$, then $\Psi(\lambda+1) - \Psi(\lambda) > 0$; it follows from (H_4) that

$$\frac{f(\Psi(\lambda))}{\Psi^\eta(\lambda+1)} > \frac{f(\Psi(\lambda+1))}{\Psi^\eta(\lambda+1)} \geq L, \quad (45)$$

and $\Delta\Psi^\eta(\lambda) > (\Delta\Psi(\lambda))^\eta$ by Lemma 4.

Using the fact that $u(\lambda)\psi(\lambda)z^\eta(\lambda)$ is strictly decreasing, we arrive at

$$\frac{u(\lambda)\psi(\lambda)z^\eta(\lambda)}{\Psi^\eta(\lambda)\Psi^\eta(\lambda+1)} \Delta\Psi^\eta(\lambda) \geq \frac{u(\lambda)\psi(\lambda)z^\eta(\lambda)}{\Psi^\eta(\lambda)\Psi^\eta(\lambda+1)} (\Delta\Psi(\lambda))^\eta. \quad (46)$$

Also we have

$$(\Delta\Psi(\lambda))^\eta < \left[\frac{z(\lambda)}{n_0} \right]^\eta, \quad (47)$$

from (41).

Now substituting (45), (46), and (47) in (43), we get

$$\begin{aligned}
\Delta w(\lambda) &< \Delta u(\lambda) \frac{w(\lambda+1)}{u(\lambda+1)} - Lu(\lambda)q(\lambda) \\
&\quad - \frac{u(\lambda)\psi(\lambda)z''(\lambda)}{\Psi''(\lambda)\Psi''(\lambda+1)} \frac{z''(\lambda)}{n_0''} \\
&< \Delta u(\lambda) \frac{w(\lambda+1)}{u(\lambda+1)} - Lu(\lambda)q(\lambda) \\
&\quad - \frac{u(\lambda+1)\psi(\lambda+1)z''(\lambda+1)}{\Psi''(\lambda+1)\Psi''(\lambda+1)} \frac{z''(\lambda+1)}{n_0''} \\
&< \Delta u(\lambda) \frac{w(\lambda+1)}{u(\lambda+1)} - Lu(\lambda)q(\lambda) \\
&\quad - \frac{[u(\lambda+1)\psi(\lambda+1)z''(\lambda+1)]^2}{u(\lambda+1)\psi(\lambda+1)[\Psi''(\lambda+1)]^2 n_0''} \\
&< \Delta u(\lambda) \frac{w(\lambda+1)}{u(\lambda+1)} - Lu(\lambda)q(\lambda) \\
&\quad - \left[\frac{u(\lambda+1)\psi(\lambda+1)z''(\lambda+1)}{\Psi''(\lambda+1)} \right]^2 \\
&\quad \times \frac{1}{u(\lambda+1)\psi(\lambda+1)n_0''} < \Delta u(\lambda) \frac{w(\lambda+1)}{u(\lambda+1)} \\
&\quad - Lu(\lambda)q(\lambda) - \frac{1}{u(\lambda+1)\psi(\lambda+1)n_0''} w^2(\lambda+1).
\end{aligned} \tag{48}$$

Using (48) and (21) with $X = w(\lambda+1)$, $a = \Delta u(\lambda)/u(\lambda+1)$, and

$$b = \frac{1}{u(\lambda+1)\psi(\lambda+1)n_0''}, \tag{49}$$

we obtain

$$\begin{aligned}
\Delta w(\lambda) &< -Lu(\lambda)q(\lambda) + \frac{(\Delta u(\lambda))^2 \psi(\lambda+1)u(\lambda+1)n_0''}{4(u(\lambda+1))^2} \\
&= -Lu(\lambda)q(\lambda) + \frac{(\Delta u(\lambda))^2 \psi(\lambda+1)n_0''}{4u(\lambda+1)},
\end{aligned} \tag{50}$$

and hence

$$\left[Lu(\lambda)q(\lambda) - \frac{(\Delta u(\lambda))^2 \psi(\lambda+1)n_0''}{4u(\lambda+1)} \right] < -\Delta w(\lambda). \tag{51}$$

Summing up (51) from λ_2 to $\lambda-1$, we have

$$\begin{aligned}
\sum_{v=\lambda_2}^{\lambda-1} \left[Lu(v)q(v) - \frac{(\Delta u(v))^2 \psi(v+1)n_0''}{4u(v+1)} \right] &< - \sum_{v=\lambda_2}^{\lambda-1} \Delta w(v) \\
&= -[w(\lambda) - w(\lambda_2)] = w(\lambda_2) - w(\lambda),
\end{aligned} \tag{52}$$

and hence

$$\sum_{v=\lambda_2}^{\lambda-1} \left[Lu(v)q(v) - \frac{(\Delta u(v))^2 \psi(v+1)n_0''}{4u(v+1)} \right] < w(\lambda_2). \tag{53}$$

Letting $\lambda \rightarrow \infty$,

$$\lim_{\lambda \rightarrow \infty} \sum_{v=\lambda_2}^{\lambda-1} \left[Lu(v)q(v) - \frac{(\Delta u(v))^2 \psi(v+1)n_0''}{4u(v+1)} \right] < w(\lambda_2) < \infty, \tag{54}$$

which contradicts with (26).

Theorem 7. Assume that (25) and (26) hold and there exists a positive sequence $H(t, s)$ such that

- (1) $H(\lambda, \lambda) = 0$ for $\lambda \geq \lambda_0$; $H(\lambda, v) > 0$ for $\lambda > v \geq \lambda_0$
- (2) $\Delta_2 H(\lambda, v) = H(\lambda, v+1) - H(\lambda, v) < 0$ for $\lambda > v \geq \lambda_0$

If

$$\lim_{\lambda \rightarrow \infty} \frac{1}{H(\lambda, \lambda_0)} \sum_{v=\lambda_0}^{\lambda-1} \left[Lu(v)q(v)H(\lambda, v) - \frac{h_+^2(\lambda, v)u(v+1)\psi(v+1)n_0''}{4H(\lambda, v)} \right] = \infty, \tag{55}$$

then Eq. (1) is oscillatory, where $u(\lambda)$ is defined in Theorem 6 and

$$h_+(\lambda, v) = \Delta_2 H(\lambda, v) + \frac{\Delta u(v)H(\lambda, v)}{u(v+1)}. \tag{56}$$

Proof. Suppose that $y(\lambda)$ is a nonoscillatory solution of (1). Without loss of generality, we may assume that $y(\lambda)$ is an eventually positive solution of (1). Then, there exists $\lambda_1 \in [\lambda_0, \infty)$ such that $y(\lambda) > 0$, $\Psi(\lambda) > 0$ for $\lambda \in [\lambda_1, \infty)$. Proceeding as in proof of Theorem 6 we arrive at (48). Multiplying (48) by $H(\lambda, v)$ and summing up from λ_2 to $\lambda-1$, we obtain

$$\begin{aligned}
\sum_{v=\lambda_2}^{\lambda-1} \Delta w(v)H(\lambda, v) &< \sum_{v=\lambda_2}^{\lambda-1} \frac{\Delta u(v)w(v+1)H(\lambda, v)}{u(v+1)} \\
&\quad - \sum_{v=\lambda_2}^{\lambda-1} Lu(v)q(v)H(\lambda, v) \\
&\quad - \sum_{v=\lambda_2}^{\lambda-1} \frac{H(\lambda, v)w^2(v+1)}{u(v+1)\psi(v+1)n_0''}.
\end{aligned} \tag{57}$$

Using summation by parts formula, we get

$$\begin{aligned}
 -\sum_{v=\lambda_2}^{\lambda-1} \Delta w(v)H(\lambda, v) &= -H(\lambda, v)w(\lambda) \Big|_{v=\lambda_2}^{v=\lambda-1} + \sum_{v=\lambda_2}^{\lambda-1} w(v+1)\Delta_2 H(\lambda, v) \\
 &= H(\lambda, \lambda_2)w(\lambda_2) + \sum_{v=\lambda_2}^{\lambda-1} w(v+1)\Delta_2 H(\lambda, v).
 \end{aligned} \tag{58}$$

Hence, from (57)

$$\begin{aligned}
 \sum_{v=\lambda_2}^{\lambda-1} Lu(v)q(v)H(\lambda, v) &< H(\lambda, \lambda_2)w(\lambda_2) \\
 &+ \sum_{v=\lambda_2}^{\lambda-1} w(v+1)\Delta_2 H(\lambda, v) \\
 &+ \sum_{v=\lambda_2}^{\lambda-1} \frac{\Delta u(v)w(v+1)H(\lambda, v)}{u(v+1)} \\
 &- \sum_{v=\lambda_2}^{\lambda-1} \frac{H(\lambda, v)w^2(v+1)}{u(v+1)\psi(v+1)n_0^\eta} \\
 &< \sum_{v=\lambda_2}^{\lambda-1} \left[\Delta_2 H(\lambda, v) + \frac{\Delta u(v)H(\lambda, v)}{u(v+1)} \right] w(v+1) \\
 &- \sum_{v=\lambda_2}^{\lambda-1} \frac{H(\lambda, v)w^2(v+1)}{u(v+1)\psi(v+1)n_0^\eta} + H(\lambda, \lambda_2)w(\lambda_2) \\
 &< H(\lambda, \lambda_2)w(\lambda_2) + \sum_{v=\lambda_2}^{\lambda-1} \left[h_+(\lambda, v)w(v+1) \right. \\
 &\quad \left. - \frac{H(\lambda, v)w^2(v+1)}{u(v+1)\psi(v+1)n_0^\eta} \right].
 \end{aligned} \tag{59}$$

Now by using (21) with $X = w(v+1)$, $\xi = h_+(\lambda, v)$ and

$$\zeta = \frac{H(\lambda, v)}{u(v+1)\psi(v+1)n_0^\eta}, \tag{60}$$

we obtain

$$\begin{aligned}
 \sum_{v=\lambda_2}^{\lambda-1} \left[Lu(v)q(v)H(\lambda, v) - \frac{h_+^2(\lambda, v)u(v+1)\psi(v+1)n_0^\eta}{4H(\lambda, v)} \right] \\
 < H(\lambda, \lambda_2)w(\lambda_2) < H(\lambda, \lambda_0)w(\lambda_2),
 \end{aligned} \tag{61}$$

for $\lambda > \lambda_2 > \lambda_1 > \lambda_0$. Then

$$\begin{aligned}
 &\sum_{v=\lambda_0}^{\lambda-1} \left[Lu(v)q(v)H(\lambda, v) - \frac{h_+^2(\lambda, v)u(v+1)\psi(v+1)n_0^\eta}{4H(\lambda, v)} \right] \\
 &= \sum_{v=\lambda_0}^{\lambda_2-1} \left[Lu(v)q(v)H(\lambda, v) - \frac{h_+^2(\lambda, v)u(v+1)\psi(v+1)n_0^\eta}{4H(\lambda, v)} \right] \\
 &\quad + \sum_{v=\lambda_2}^{\lambda-1} \left[Lu(v)q(v)H(\lambda, v) - \frac{h_+^2(\lambda, v)u(v+1)\psi(v+1)n_0^\eta}{4H(\lambda, v)} \right] \\
 &< \sum_{v=\lambda_0}^{\lambda_2-1} \left[Lu(v)q(v)H(\lambda, v) - \frac{h_+^2(\lambda, v)u(v+1)\psi(v+1)n_0^\eta}{4H(\lambda, v)} \right] \\
 &\quad + H(\lambda, \lambda_0)w(\lambda_2) < H(\lambda, \lambda_0) \sum_{v=\lambda_0}^{\lambda_2-1} Lu(v)q(v) \\
 &\quad + H(\lambda, \lambda_0)w(\lambda_2),
 \end{aligned} \tag{62}$$

which yields

$$\begin{aligned}
 &\frac{1}{H(\lambda, \lambda_0)} \sum_{v=\lambda_0}^{\lambda-1} \left[Lu(v)q(v)H(\lambda, v) - \frac{h_+^2(\lambda, v)u(v+1)\psi(v+1)n_0^\eta}{4H(\lambda, v)} \right] \\
 &< \sum_{v=\lambda_0}^{\lambda_2-1} Lu(v)q(v) + w(\lambda_2).
 \end{aligned} \tag{63}$$

Taking limit as $\lambda \rightarrow \infty$, we get

$$\begin{aligned}
 &\lim_{\lambda \rightarrow \infty} \frac{1}{H(\lambda, \lambda_0)} \sum_{v=\lambda_0}^{\lambda-1} \left[Lu(v)q(v)H(\lambda, v) - \frac{h_+^2(\lambda, v)u(v+1)\psi(v+1)n_0^\eta}{4H(\lambda, v)} \right] \\
 &< \sum_{v=\lambda_0}^{\lambda_2-1} Lu(v)q(v) + w(\lambda_2) < \infty,
 \end{aligned} \tag{64}$$

which contradicts with (55).

Next, we consider the condition

$$\lim_{\lambda \rightarrow \infty} \sum_{v=\lambda_0}^{\lambda-1} \left[\frac{1}{u(v)\psi(v)} \right]^{1/\eta} < \infty, \tag{65}$$

which implies that (25) does not hold. Under this condition, we have the following result.

Theorem 8. Assume that $(H_1) - (H_4)$, (26), and (65) hold. If

$$\lim_{\lambda \rightarrow \infty} \sum_{s=\lambda_2}^{\lambda-1} \left[\left[\frac{1}{u(s)\psi(s)} \sum_{v=s}^{\infty} u(v+1)q(v) \right]^{1/\eta} - \frac{a(s)}{b(s)} \right] = \infty, \tag{66}$$

then every solution $y(\lambda)$ is oscillatory or satisfies $\lim_{\lambda \rightarrow \infty} y(\lambda) = 0$, where $\Psi(\lambda)$ is defined in Lemma 2.

Proof. Assume that $y(\lambda)$ is a nonoscillatory solution of (1). Without loss of generality, assume that $y(\lambda)$ is eventually a positive solution of (1). Proceeding like in the proof of Theorem 6, we get that (28) holds. Then, there are two signs of $z(\lambda)$. When $z(\lambda) > 0$ is eventually positive, we conclude from the proof of Theorem 6 that equation (1) is oscillatory.

Next, assume that $z(\lambda)$ is eventually negative, then there exists $\lambda_2 > \lambda_0$ such that $z(\lambda) < 0$ for $\lambda \geq \lambda_2$. Since $z(\lambda) = a(\lambda) + b(\lambda)\Delta^\mu y(\lambda)$, we have

$$a(\lambda) + b(\lambda) \frac{\Delta\Psi(\lambda)}{\Gamma(1-\mu)} = z(\lambda) < 0, \quad (67)$$

and hence

$$\Delta\Psi(\lambda) < -\Gamma(1-\mu) \frac{a(\lambda)}{b(\lambda)}. \quad (68)$$

On the other hand, Since (H_2) holds, we get

$$\lim_{\lambda \rightarrow \infty} \frac{a(\lambda)}{b(\lambda)} = 0. \quad (69)$$

Now, taking the limit of the both sides of (68) as λ tends to ∞ , we get

$$\lim_{\lambda \rightarrow \infty} \Delta\Psi(\lambda) \leq 0. \quad (70)$$

Since $\Psi(\lambda) > 0$ for $\lambda \in [\lambda_1, \infty)$, we have

$$\lim_{\lambda \rightarrow \infty} \Psi(\lambda) = \beta \geq 0. \quad (71)$$

Claim that $\beta = 0$. If not, then $\Psi(\lambda) \geq \beta$ for $\lambda \in [\lambda_2, \infty)$. Now we have

$$\begin{aligned} \Delta[u(\lambda)\psi(\lambda)z^\eta(\lambda)] &\leq -Lu(\lambda+1)q(\lambda)\Psi^\eta(\lambda), \\ &\leq -Lu(\lambda+1)q(\lambda)\beta^\eta, \end{aligned} \quad (72)$$

by (29). Summing up from λ to ∞ , we have

$$\begin{aligned} \sum_{v=\lambda}^{\infty} \Delta[u(v)\psi(v)z^\eta(v)] &= \lim_{v \rightarrow \infty} u(v)\psi(v)z^\eta(v) - u(\lambda)\psi(\lambda)z^\eta(\lambda) \\ &\leq -\beta^\eta L \sum_{v=\lambda}^{\infty} u(v+1)q(v), \end{aligned} \quad (73)$$

which yields

$$\begin{aligned} u(\lambda)\psi(\lambda)z^\eta(\lambda) &\geq \lim_{v \rightarrow \infty} u(v)\psi(v)z^\eta(v) + \beta^\eta L \sum_{v=\lambda}^{\infty} u(v+1)q(v) \\ &\geq \beta^\eta L \sum_{v=\lambda}^{\infty} u(v+1)q(v), \end{aligned} \quad (74)$$

and hence

$$\frac{a(\lambda)}{b(\lambda)} + \frac{\Delta\Psi(\lambda)}{\Gamma(1-\mu)} = z(\lambda) > L^{1/\eta} \beta \left[\frac{1}{u(\lambda)\psi(\lambda)} \sum_{v=\lambda}^{\infty} u(v+1)q(v) \right]^{1/\eta}. \quad (75)$$

The last inequality above implies that

$$\Delta\Psi(\lambda) > \Gamma(1-\mu) \left[L^{1/\eta} \beta \left[\frac{1}{u(\lambda)\psi(\lambda)} \sum_{v=\lambda}^{\infty} u(v+1)q(v) \right]^{1/\eta} - \frac{a(\lambda)}{b(\lambda)} \right]. \quad (76)$$

Summing up from λ_2 to $\lambda-1$, we get

$$\begin{aligned} \Psi(\lambda) &> \Psi(\lambda_2) + \Gamma(1-\mu) \\ &\cdot \left[L^{1/\eta} \beta \sum_{s=\lambda_2}^{\lambda-1} \left[\left[\frac{1}{u(s)\psi(s)} \sum_{v=s}^{\infty} u(v+1)q(v) \right]^{1/\eta} - \frac{a(s)}{b(s)} \right] \right]. \end{aligned} \quad (77)$$

Taking the limit of the both sides of the above inequality as λ tends to ∞ , we end up with

$$\lim_{\lambda \rightarrow \infty} \Psi(\lambda) = \infty, \quad (78)$$

which contradicts to $\Psi(\lambda) > 0$ for $\lambda \in [\lambda_1, \infty)$. Therefore, we obtain $\beta = 0$, that is

$$\lim_{\lambda \rightarrow \infty} \Psi(\lambda) = 0. \quad (79)$$

3. Applications

Example 9. Consider the equation

$$\Delta[\lambda^2 x^\eta(\lambda)] + Lx^\eta(\lambda) + \frac{1}{\lambda^2} \left(\sum_{s=\lambda_0}^{\lambda-1+\mu} (\lambda-s-1)^{-\mu} y(s) \right)^{-3} = 0, \quad (80)$$

where $\mu \in (0, 1)$,

$$x(\lambda) = e^{-\lambda} + \frac{1}{\lambda} \Delta^\mu y(\lambda), \quad \lambda \in \mathbb{N}_2 = \{2, 3, 4, \dots\}, \quad (81)$$

and that $\eta > 0$ is a quotient of odd positive integers in $\mathbb{N}_2$.

Comparing with Eq. (1), we get $\psi(\lambda) = \lambda^2$, $a(\lambda) = e^{-\lambda}$, $b(\lambda) = 1/\lambda$, $p(\lambda) = \lambda$, $F(y) = y^{-3}$, $q(\lambda) = 1/\lambda^2$, $\eta = 3$, $\lambda_0 = 2$, $n_0 = 4$, $L = 1/3$, and that

$$\frac{f(y)}{y^\eta} = \frac{1}{y^6} > \varepsilon = L > 0 \text{ for } y \neq 0, \quad (82)$$

where ε is a certain positive number. It is clear that assumptions $(H_1) - (H_4)$ hold.

Further from (22), we have

$$\begin{aligned} u(\lambda) &= \prod_{v=\lambda_0}^{\lambda-1} \frac{\psi(v)}{\psi(v)-p(v)} = \prod_{v=2}^{\lambda-1} \frac{v^2}{v^2-v} = \lambda - 1; \\ \Delta u(\lambda) &= u(\lambda+1) - u(\lambda) = 1; \\ \Delta \left[\frac{a(\lambda)}{b(\lambda)} \right] &= \Delta \left[\lambda e^{-\lambda} \right] = (\lambda+1)e^{-(\lambda+1)}(1-e) + e^{-\lambda} \neq 0; \\ \lim_{\lambda \rightarrow \infty} \sum_{v=\lambda_0}^{\lambda-1} \frac{a(v)}{b(v)} &= \lim_{\lambda \rightarrow \infty} \sum_{v=1}^{\lambda-1} v e^{-v} < \infty; \\ \lim_{\lambda \rightarrow \infty} \sum_{v=\lambda_0}^{\lambda-1} \left[\frac{1}{u(v)\psi(v)} \right]^{1/\eta} &= \lim_{\lambda \rightarrow \infty} \sum_{v=2}^{\lambda-1} \left[\frac{1}{v^2(v-1)} \right]^{1/3} \\ &> \sum_{v=2}^{\infty} \frac{1}{v} = \infty; \\ \lim_{\lambda \rightarrow \infty} \sum_{v=\lambda_2}^{\lambda-1} \left[Lu(v)q(v) - \frac{(\Delta u(v))^2 \psi(v+1)n_0^\eta}{4u(v+1)} \right] \\ &= \lim_{\lambda \rightarrow \infty} \sum_{v=2}^{\lambda-1} \left[\frac{(v-1)}{3v^2} - \frac{16(v+1)^2}{v} \right] = \infty. \end{aligned} \quad (83)$$

Thus, conditions (25) and (26) are satisfied. Therefore, all solutions of (80) are oscillatory by Theorem 6.

Example 10. Consider the equation

$$\Delta \left[\frac{1}{\lambda} z^\eta(\lambda) \right] + \frac{1}{\lambda^2} z^\eta(\lambda) + \frac{1}{\lambda-1} \left(\sum_{s=\lambda_0}^{\lambda-1+\mu} (\lambda-s-1)^{-\mu} y(s) \right)^{-1} = 0, \quad (84)$$

where $\mu \in (0, 1)$, $z(\lambda) = \lambda^2 + \lambda \Delta^\mu y(\lambda)$, $\lambda \in \mathbb{N}_2$, and that $\eta > 0$ is a quotient of odd positive integers in $\mathbb{N}_2$. Comparing with (1), we have $\psi(\lambda) = 1/\lambda$, $a(\lambda) = \lambda^2$, $b(\lambda) = \lambda$, $p(\lambda) = 1/\lambda^2$, $F(y) = y^{-1}$, $q(\lambda) = 1/(\lambda-1)$, $\eta = 1$, $\lambda_0 = 2$, $n_0 = 2$, $L = 1/2$, and that

$$\frac{f(y)}{y^\eta} = \frac{1}{y^2} > \varepsilon = L > 0 \text{ for } y \neq 0, \quad (85)$$

where ε is a certain positive number. It is clear that assumptions $(H_1) - (H_4)$ hold.

Further from (22), we have

$$\begin{aligned} u(\lambda) &= \prod_{v=\lambda_0}^{\lambda-1} \frac{\psi(v)}{\psi(v)-p(v)} = \prod_{v=2}^{\lambda-1} \frac{v}{v-1} = \lambda - 1; \\ \Delta u(\lambda) &= u(\lambda+1) - u(\lambda) = 1; \\ \Delta \left[\frac{a(\lambda)}{b(\lambda)} \right] &= \Delta \left[\frac{\lambda^2}{\lambda} \right] = \Delta[\lambda] \neq 0; \\ \lim_{\lambda \rightarrow \infty} \sum_{v=\lambda_0}^{\lambda-1} \frac{a(v)}{b(v)} &= \lim_{\lambda \rightarrow \infty} \sum_{v=1}^{\lambda-1} \frac{v^2}{v} = \lim_{\lambda \rightarrow \infty} \sum_{v=\lambda_0}^{\lambda-1} v < \infty; \\ \lim_{\lambda \rightarrow \infty} \sum_{v=\lambda_0}^{\lambda-1} \left[\frac{1}{u(v)\psi(v)} \right]^{1/\eta} &= \lim_{\lambda \rightarrow \infty} \sum_{v=2}^{\lambda-1} \left[\frac{v}{(v-1)} \right] = \infty. \end{aligned} \quad (86)$$

We define the double sequence $H(\lambda, v)$ as follows:

- (1) $H(\lambda, v) = (2\lambda - v)^2 > 0$ for $\lambda > v > 2$
- (2) $H(\lambda, 2) = (\lambda - 2)^2 > 0$ for $\lambda > s = 2$
- (3) $\Delta_2 H(\lambda, v) = H(\lambda, v+1) - H(\lambda, v) = [2\lambda - (v+1)]^2 - (2\lambda - v)^2$
- (4) $\Delta_2 H(\lambda, v) = 2v - 4\lambda + 1 < 0$ for $\lambda > v \geq 2$

and

$$\begin{aligned} h_+(\lambda, v) &= \Delta_2 H(\lambda, v) + \frac{\Delta u(v)H(\lambda, v)}{u(v+1)} \\ &= \frac{(3v^2 - 8lv + v + 4\lambda^2)}{v}. \end{aligned} \quad (87)$$

Then

$$\begin{aligned} \frac{1}{H(\lambda, \lambda_0)} \sum_{v=\lambda_2}^{\lambda-1} \left[Lu(v)q(v)H(\lambda, v) - \frac{h_+^2(\lambda, v)u(v+1)\psi(v+1)n_0^\eta}{4H(\lambda, v)} \right] \\ = \frac{1}{(\lambda-2)^2} \sum_{v=\lambda_2}^{\lambda-1} \left[\frac{(2\lambda-v)^2}{2} - \frac{(3v^2 - 8lv + v + 4\lambda^2)^2}{2(v+1)(2\lambda-v)^2} \right], \end{aligned} \quad (88)$$

for $\lambda_2 > \lambda_1$, and hence

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} \frac{1}{H(\lambda, \lambda_0)} \sum_{v=\lambda_2}^{\lambda-1} \left[Lu(v)q(v)H(\lambda, v) - \frac{h_+^2(\lambda, v)u(v+1)\psi(v+1)n_0^\eta}{4H(\lambda, v)} \right] \\ = \lim_{\lambda \rightarrow \infty} \frac{1}{(\lambda-2)^2} \sum_{v=\lambda_2}^{\lambda-1} \left[\frac{(2\lambda-v)^2}{2} - \frac{(3v^2 - 8lv + v + 4\lambda^2)^2}{2(v+1)(2\lambda-v)^2} \right] = \infty. \end{aligned} \quad (89)$$

Therefore, condition (55) is satisfied. We deduce that all solutions of Eq. (84) are oscillatory by Theorem 7.

4. A Concluding Remark

In this paper, we obtained new oscillation theorems for a class of fractional difference equation. The main outcomes are proved via the means of mathematical inequalities, properties of fractional operators, and generalized Riccati technique. We claim that the concluded results have merit and considered as an extension for the corresponding fractional differential equations. Particular examples that are consistent to the main results are demonstrated at the end of the paper.

The consideration of Eq. (1) with forcing term of the form

$$\Delta[\psi(\lambda)z^{\eta}(\lambda)] + p(\lambda)z^{\eta}(\lambda) + q(\lambda)F \cdot \left(\sum_{s=\lambda_0}^{\lambda-1+\mu} (\lambda-s-1)^{(-\mu)} y(s) \right) = g(\lambda), \quad (90)$$

could be further investigated. We leave this for future consideration.

Data Availability

Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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