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Research Article

Positive Solutions for a Weakly Singular Hadamard-Type Fractional Differential Equation with Changing-Sign Nonlinearity

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In this paper, we focus on the existence of positive solutions for a class of weakly singular Hadamard-type fractional mixed periodic boundary value problems with a changing-sign singular perturbation. By using nonlinear analysis methods combining with some numerical techniques, we further discuss the effect of the perturbed term for the existence of solutions of the problem under the positive, negative, and changing-sign cases. The interesting points are that the nonlinearity can be singular at the second and third variables and be changing-sign.

1. Introduction

In this paper, we study the existence of positive solutions for the following Hadamard-type fractional differential equation with mixed periodic boundary conditions:

$$\begin{cases} -\left(\mathscr{D}_{\mathbf{t}}{}^{\beta} z(t)\right) - b(t)\mathscr{D}_{\mathbf{t}}{}^{\beta} z(t) = f\left(t, z(t), -\mathscr{D}_{\mathbf{t}}{}^{\beta} z(t)\right) + \chi(t), \quad 1 < t < e, \\ z(1) = z'(1) = z'(e) = 0,\ \mathscr{D}_{\mathbf{t}}{}^{\beta} z(1) = \mathscr{D}_{\mathbf{t}}{}^{\beta} z(e),\ \mathscr{D}_{\mathbf{t}}{}^{\beta+1} z(1) = \mathscr{D}_{\mathbf{t}}{}^{\beta+1} z(e), \end{cases} \tag{1}$$

where $2 < \beta \le 3$, $\mathscr{D}_{\mathbf{t}}{}^{\beta}$ is the Hadamard fractional derivative of order α, β, $b \in L^p(1, e)$ and $f : [1, e] \times \mathbb{R} \setminus \{0\} \times \mathbb{R}\{0\} \to (0, \infty)$ is a continuous function with singularity at the second and third space variables; $\chi \in L^p([1, e], \mathbb{R})$, $p \ge 1$ is a variable sign function.

Nonlocal characteristics are the most important property of the fractional differential operator; because of this, the fractional differential equations can describe many viscoelasticities and memory phenomena of natural science. For example, some viscoelastic materials, such as the silicone gel with the property of weak frequency dependency, often involve a complicated strain-stress relationship; if let $\delta(t)$ and $\rho(t)$ be the stress and the strain, respectively, then the stress decays after a shear jump is governed by the following fractional order viscoelasticity Kelvin-Voigt equation [1, 2]:

$$\delta(t) = \omega\tau^{\alpha}\mathscr{D}_{\mathbf{t}}{}^{\alpha}\rho(t) + \omega\tau^{\beta}\mathscr{D}_{\mathbf{t}}{}^{\beta}\rho(t), \tag{2}$$

where $\alpha > \beta > 0$, ω, τ are constants and $\mathscr{D}_{\mathbf{t}}{}^{\alpha}, \mathscr{D}_{\mathbf{t}}{}^{\beta}$ are Riemann-Liouville fractional derivatives. In practice, the study of the qualitative properties of solutions for the corresponding fractional models such as existence, uniqueness, multiplicity, and stability is necessary to analyze and control the model under consideration [3–21]. In [3], Zhang et al. considered a singular fractional differential equation with signed measure

$$\begin{cases} (-\mathscr{D}_{\mathbf{t}}{}^{\alpha}x)(t) = f\left(t, x(t), \mathscr{D}_{\mathbf{t}}{}^{\beta}x(t)\right), & t \in (0,1), \\ \mathscr{D}_{\mathbf{t}}{}^{\beta}x(0) = 0, \mathscr{D}_{\mathbf{t}}{}^{\beta}x(1) = \int_0^1 \mathscr{D}_{\mathbf{t}}{}^{\beta}x(s)dA(s), \end{cases} \tag{3}$$

where $\mathscr{D}_{\mathbf{t}}{}^{\alpha}, \mathscr{D}_{\mathbf{t}}{}^{\beta}$ are the standard Riemann-Liouville derivatives, $\int_0^1 x(s)dA(s)$ is denoted by a Riemann-Stieltjes integral and $0 < \beta \le 1 < \alpha \le 2$, $\alpha - \beta > 1, A$ is a function of bounded variation and dA can be a signed measure; the nonlinearity $f(t,x,y)$ may be singular at both $t = 0, 1$ and $x = y = 0$. By using the spectral analysis of the relevant linear operator and Gelfand's formula combining the calculation of a fixed point index of the nonlinear operator, some sufficient conditions for the existence of positive solutions were established. Recently, by using the fixed point index theory, Wang [10] established the existence and multiplicity of positive solutions for a class of singular fractional differential equations with nonlocal boundary value conditions, where the nonlinearity may be singular at some time and space variables.

In the recent years, to improve and develop the fractional calculus, there are several kinds of fractional derivatives and integral operators with different kernels such as Hadamard, Erdelyi-Kober, Caputo-Fabrizio derivatives, Hilfer derivatives, and integrals to be given to enrich the application of the fractional calculus such as the Rubella disease and human liver model [22–25]. In particular, the Hadamard derivative is a nonlocal fractional derivative with singular logarithmic kernel. So the study of Hadamard fractional differential equations is relatively difficult; see [26–32].

In this paper, we are interested in the existence of solutions for the Hadamard-type fractional differential equation (1) which involves a singular perturbed term χ; we will further discuss the effect of the perturbed term for the existence of solutions when χ is positive, negative, and changing-sign. Our main tools rely on nonlinear analysis methods as well as some numerical techniques. Thus, in order to make our work be more self-contained, a brief overview for these methods and techniques should be necessary. In recent work [33–41], some fixed point theorems were employed to study the qualitative properties of solutions for various types of differential equations. For obtaining numerical and analytical results, many authors [42–59] also developed iterative techniques to solve some nonlinear problems with practical applications. In addition, variational methods [60–78] and upper and lower solution methods [2, 25, 79, 80] also offered wonderful tools for dealing with various nonlinear ordinary and partial differential equations arising from natural science fields. These analysis and techniques not only improved and perfected the relative theoretical framework of differential equations but also gave some new understand for the corresponding natural phenomena.

Our work has some new features. Firstly, the equation contains a Hadamard-type fractional derivative which has a singular logarithmic kernel. Secondly, the nonlinearity involves a perturbed term which can be positive, negative, or changing-sign. Thirdly, the nonlinearity is allowed to have weakly singular at space variables, which is a class of interesting natural phenomena. In the end, the effect of the perturbed term for the existence of solutions of the equation is discussed, and the criteria on the existence of positive solutions are established for all cases of the perturbed term including positive, negative, and changing-sign cases. The rest of this paper is organized as follows. In Section 2, we firstly introduce the concept of the Hadamard fractional integral and differential operators and then give the logarithmic kernel and Green function of periodic boundary value problem and their properties. Our main results are summarized in Section 3 which includes three theorems for three different situations. An example is given in Section 4.

2. Basic Definitions and Preliminaries

Before starting our main results, we firstly recall the definition of the Hadamard-type fractional integrals and derivatives; for detail, see [81].

Let $\alpha \in \mathbb{C}$, $\mathbf{R}e(\alpha) > 0$, $n = [\mathbf{R}e(\alpha)]$, and (a,b) be a finite or infinite interval of $\mathbb{R}^+$. The α-order left Hadamard fractional integral is defined by

$$(I_a^{\alpha}x)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{x(s)}{s}ds, \quad t \in (a,b), \tag{4}$$

and the α left Hadamard fractional derivative is defined by

$$(\mathscr{D}_{\mathbf{t}}{}^{\alpha})x(t) = \frac{1}{\Gamma(n-\alpha)}\left(t\frac{d}{dt}\right)^n \int_a^t \left(\ln\frac{t}{s}\right)^{n-\alpha-1}\frac{x(s)}{s}ds, \quad t \in (a,b). \tag{5}$$

In what follows, we firstly consider the linear auxiliary problem

$$\begin{cases} -\mathscr{D}_{\mathbf{t}}{}^{\beta}z(t) = x(t), & 1 < t < e, \\ z(1) = z'(1) = z'(e) = 0. \end{cases} \tag{6}$$

It follows from [26] that the problem (6) has a unique solution:

$$z(t) = \int_1^e H(t,s)x(s)\frac{ds}{s}, \tag{7}$$

where

$$H(t,s) = \frac{1}{\Gamma(\beta)}\begin{cases} (\ln t)^{\beta-1}(1-\ln s)^{\beta-2} - (\ln t - \ln s)^{\beta-1}, & 1 \le s \le t \le e, \\ (\ln t)^{\beta-1}(1-\ln s)^{\beta-2}, & 1 \le t \le s \le e, \end{cases} \tag{8}$$

is Green's function of (6). Now, let $x(t) = -\mathscr{D}_{\mathbf{t}}{}^{\beta}z(t)$, then the Hadamard-type fractional differential equation (1) reduces to the following second order integrodifferential equation:

$$\begin{cases} x''(t) + b(t)x(t) = f\left(t, \int_1^e H(t,s)x(s)\frac{ds}{s}, x(t)\right) + \chi(t), & t \in (1,e), \\ x(1) = x(e), \quad x'(1) = x'(e). \end{cases} \tag{9}$$

In order to obtain the solution of the second order integrodifferential equation (9), let us consider the linear periodic boundary value problem

$$\begin{cases} x'' + b(t)x = 0, \\ x(1) = x(e), \quad x'(1) = x'(e). \end{cases} \tag{10}$$

Clearly, by Fredholm's alternative, the nonhomogeneous equation $x'' + b(t)u = a(t)$ has a unique $(e-1)$-periodic solution:

$$x(t) = \int_1^e G(t,s)a(s)ds, \tag{11}$$

where $G(t,s)$ is the Green function of line equation (10) subject to periodic boundary conditions $x(1) = x(e)$, $x'(1) = x'(e)$.

Thus, it follows from (10) that the equation (9) is equivalent to the following integral equation:

$$x(t) = \int_1^e G(t,s)\left[f\left(s, \int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau}, x(s)\right) + \chi(s)\right]ds. \tag{12}$$

As a result, in order to obtain the solution of equation (1), it is sufficient to find the fixed point of the following operator:

$$Tx(t) = \int_1^e G(t,s)\left[f\left(s, \int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau}, x(s)\right) + \chi(s)\right]ds. \tag{13}$$

Lemma 1 (see [26]). *Green's function H has the following properties:*

(i) $H \in C([1,e]\times[1,e], \mathbb{R}^+)$

(ii) For all $t, s \in [1,e]$, the following inequality holds:

$$\frac{1}{\Gamma(\beta)}(\ln t)^{\beta-1}\ln s(1-\ln s)^{\beta-2} \le H(t,s) \le \frac{1}{\Gamma(\beta)}\ln s(1-\ln s)^{\beta-2} \tag{14}$$

In order to obtain the properties of the Green function G, define the best Sobolev constants as

$$Y(\zeta) = \begin{cases} \dfrac{2\pi}{\zeta(e-1)^{1+(1/\zeta)}}\left(\dfrac{2}{2+\zeta}\right)^{1-(2/\zeta)}\left(\dfrac{\Gamma(1/\zeta)}{\Gamma((1/2)+(1/\zeta))}\right)^2, & \text{if } 1 \le \zeta < \infty, \\ \dfrac{4}{e-1}, & \text{if } \zeta = \infty, \end{cases} \tag{15}$$

where Γ is the gamma function. Let $b \in L^p(1,e)$ and denote $b \succ 0$ means that $b(t) \ge 0$ for all $t \in [1,e]$ and $b(t) > 0$ for t in a subset of positive measure. Define

$$p^* = \frac{p}{p-1}, \quad \text{if } 1 \le p < \infty, \qquad p^* = 1, \quad \text{if } p = \infty. \tag{16}$$

Then, the following lemma is a direct consequence of Theorem 2.1 and Corollary 2.3 in [82]:

Lemma 2. *Let $1 \le p \le \infty$, assume $b \in L^p(0,T)$ and $b(t) \succ 0$. If $\|b\|_p \le Y(2p^*)$, then $G(t,s) > 0$ for all $(t,s) \in [1,e]\times[1,e]$.*

3. Main Results

In this section, we firstly give our work space $E = C[1,e]$ which equips the maximum norm $\|x\| = \max_{1\le t\le e}|x(t)|$. Let

$$P = \{x \in E : x(t) \ge 0, \quad t \in [1,e]\}, \tag{17}$$

then $(E, \|\cdot\|)$ is a real Banach space and P is a normal cone of E with normal constant 1. For convenience, we denote a set of functions as

$$\Lambda = \left\{b \in L^p(1,e) \colon b \succ 0, \|b\|_p \le Y(2p^*), \quad \text{for } 1 \le p \le \infty\right\}. \tag{18}$$

Now, let us list the hypotheses to be used in the rest the paper.

(F0) $b \in \Lambda$

(F1) There exist a constant $\lambda \in (0,1)$ and two functions $h_1, h_2 \in P$ with $h_1, h_2 \equiv 0$ on any subinterval of $(1,e)$, such that

$$h_1(t)(u+v)^{-\lambda} \le f(t,u,v) \le h_2(t)(u+v)^{-\lambda}, \quad (u,v) \in (0,\infty)\times(0,\infty), \quad t \in [1,e]. \tag{19}$$

Suppose $\mu(t)$ is the unique solution of the following equation:

$$\begin{cases} x'' + b(t)x = \chi(t), \quad t \in (1,e), \ x(1) = x(e), \\ x'(1) = x'(e), \end{cases} \tag{20}$$

then from (10), $\mu(t)$ can be written as

$$\mu(t) = \int_1^e G(t,s)\chi(s)ds. \tag{21}$$

Similar to (21), we denote $\kappa_1(t), \kappa_2(t)$ as

$$\kappa_1(t) = \left(1 + \frac{1}{(\beta-1)\Gamma(\beta)}\right)^{-\lambda}\int_1^e G(t,s)h_1(s)ds, \kappa_2(t) = \int_1^e G(t,s)h_2(s)ds. \tag{22}$$

Let

$$\begin{aligned}\mu_* &= \inf_{1\le t\le e}\mu(t), \quad \mu^* = \sup_{1\le t\le e}\mu(t),\\ \kappa_{1*} &= \min_{1\le t\le e}\kappa_1(t), \quad \kappa_1^* = \max_{1\le t\le e}\kappa_1(t),\\ \kappa_{2*} &= \min_{1\le t\le e}\kappa_2(t), \quad \kappa_2^* = \max_{1\le t\le e}\kappa_2(t).\end{aligned} \tag{23}$$

By Lemma 2 and (F0)–(F1), we know that $\kappa_i^*, \kappa_{i*} > 0$, $i = 1, 2$.

Now we shall divide χ into three cases to discuss its influence for the solution of equation (1).

3.1. Positive Case $\chi(t) \ge 0$. In this case, we have the following result:

Theorem 3. *Assume that (F0)–(F1) hold. If $\mu_* \ge 0$, then the Hadamard-type fractional differential equation (1) has at least one positive solution.*

Proof. Let l and L be fixed positive constants; we introduce a closed convex set of cone P:

$$B = \{x \in P : l \le x(t) \le L, \quad t \in [1, e]\}. \tag{24}$$

For any $x \in B$, it follows from Lemma 1 that

$$\frac{l(\ln s)^{\beta-1}}{(\beta-1)\Gamma(\beta)} \le \int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau} \le \frac{L}{(\beta-1)\Gamma(\beta)}. \tag{25}$$

With the help of (25), for any $s \in [1, e]$, from (19), one gets

$$\begin{aligned}\frac{h_1(s)}{L^\lambda(1+(1/(\beta-1)\Gamma(\beta)))^\lambda} &\le f\left(s, \int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau}, x(s)\right)\\ &\le \frac{h_2(s)}{l^\lambda\left(1+\left((\ln s)^{\beta-1}/(\beta-1)\Gamma(\beta)\right)\right)^\lambda} \le \frac{h_2(s)}{l^\lambda}.\end{aligned} \tag{26}$$

It follows from (26) and (F1) that

$$\begin{aligned}(Tx)(t) &= \int_1^e G(t,s)f\left(s, \int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau}, x(s)\right)ds + \mu(t)\\ &\ge \frac{1}{L^\lambda(1+(1/(\beta-1)\Gamma(\beta)))^\lambda}\int_1^e G(t,s)h_1(s)ds \ge \frac{\kappa_{1*}}{L^\lambda},\end{aligned} \tag{27}$$

$$\begin{aligned}(Tx)(t) &= \int_1^e G(t,s)f\left(s, \int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau}, x(s)\right)ds + \mu(t)\\ &\le \frac{1}{l^\lambda}\int_1^e G(t,s)h_2(s)ds + \mu^* \le \frac{\kappa_2^*}{l^\lambda} + \mu^*.\end{aligned} \tag{28}$$

Thus, (27), (28), Lemma 1, (F1), and the Arzela-Ascoli theorem guarantee that the operator $T : B \to P$ is completely continuous.

In the following, we shall choose a suitable $L > 0$ such that T maps the closed convex set B into itself. To do this, we only need to choose $0 < l < L$ such that

$$\frac{\kappa_{1*}}{L^\lambda} \ge l, \ \frac{\kappa_2^*}{l^\lambda} + \mu^* \le L. \tag{29}$$

In fact, take $l = 1/L$, it follows from $\kappa_{1*} > 0$ and $0 < \lambda < 1$ that there exists a sufficiently large $L > 1$ such that

$$\kappa_{1*}L^{1-\lambda} \ge 1, L^\lambda\kappa_2^* + \mu^* \le L, \tag{30}$$

which imply that (29) holds and $l < L$.

Thus, from Schauder's fixed point theorem, T has a fixed point $x^* \in B$, and hence, the Hadamard-type fractional differential equation (1) has at least one positive solution:

$$z^*(t) = \int_1^e H(t,s)x^*(s)\frac{ds}{s}. \tag{31}$$

3.2. Negative Case $\chi(t) \le 0$. For this case, we have the following existence result:then the Hadamard-type fractional differential equation (1) has at least one positive solution.

Theorem 4. *Assume that (F0)–(F1) hold. If $\mu^* \le 0$, and*

$$\mu_* \ge \left(\frac{\kappa_{1*}\lambda^2}{\kappa_2^{*\lambda}}\right)^{1/(1-\lambda^2)}\left(1 - \frac{1}{\lambda^2}\right), \tag{32}$$

Proof. Firstly, from (26) and (F1), we have

$$\begin{aligned}(Tx)(t) &= \int_1^e G(t,s)f\left(s, \int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau}, x(s)\right)ds + \mu(t)\\ &\ge \frac{1}{L^\lambda(1+(1/(\beta-1)\Gamma(\beta)))^\lambda}\int_1^e G(t,s)h_1(s)ds + \mu_*\\ &\ge \frac{\kappa_{1*}}{L^\lambda} + \mu_*,\end{aligned} \tag{33}$$

$$\begin{aligned}(Tx)(t) &= \int_1^e G(t,s)f\left(s, \int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau}, x(s)\right)ds + \mu(t)\\ &\le \frac{1}{l^\lambda}\int_1^e G(t,s)h_2(s)ds \le \frac{\kappa_2^*}{l^\lambda}.\end{aligned} \tag{34}$$

Thus, by (33) and (34), in order to guarantee $T : B \to B$, it is sufficient to choose $0 < l < L$ such that

$$\frac{\kappa_{1*}}{L^\lambda} + \mu_* \ge l, \ \frac{\kappa_2^*}{l^\lambda} \le L. \tag{35}$$

Let us fix $L = \kappa_2^*/l^\lambda$; to ensure (35) holds, we only need to choose some $l > 0$ such that

$$0 < l < \kappa_2^{*\,1/(1+\lambda)}, \mu_* \geq l - \frac{\kappa_{1*}}{\kappa_2^{*\lambda}} l^{\lambda^2} = g(l). \quad (36)$$

Obviously, the function $g(l)$ in $(0, \infty)$ has a minimum at

$$\tilde{l} = \left(\frac{\lambda^2 \kappa_{1*}}{\kappa_2^{*\lambda}}\right)^{1/(1-\lambda^2)}. \quad (37)$$

Taking $l = \tilde{l}$, since $\kappa_{1*} \leq \kappa_2^*$, $0 < \lambda^2 < 1$, we have

$$0 < l = \tilde{l} = \left(\frac{\lambda^2 \kappa_{1*}}{\kappa_2^{*\lambda}}\right)^{1/(1-\lambda^2)} < \left(\frac{\kappa_2^*}{\kappa_2^{*\lambda}}\right)^{1/(1-\lambda^2)} = \kappa_2^{*\,1/(1+\lambda)}, \quad (38)$$

and from (32), we also have

$$\mu_* \geq \left(\frac{\kappa_{1*}\lambda^2}{\kappa_2^{*\lambda}}\right)^{1/(1-\lambda^2)} \left(1 - \frac{1}{\lambda^2}\right) = g\left(\tilde{l}\right) \geq g(l), \quad (39)$$

which implies that (36) holds if $l = \tilde{l}$, $L = \kappa_2^*/\tilde{l}^{\lambda}$. Consequently, we have $T : B \to B$.

Thus, from Schauder's fixed point theorem, T has a fixed point $x^* \in B$, and hence, the Hadamard-type fractional differential equation (1) has at least one positive solution:

$$z^*(t) = \int_1^e H(t,s) x^*(s) \frac{ds}{s}. \quad (40)$$

Remark 5. Note that the right side of inequality (32) is negative because of the weak force condition $0 < \lambda < 1$, so if $\mu_* = 0$, the inequality (32) is always valid. Thus, if $\mu_* = 0$, we can omit the assumption (32).

3.3. Changing-Sign Case for $\chi(t)$. In order to establish the existence result under the case where $\chi(t)$ is changing-sign, we need the following lemma.

Lemma 6. *If $0 < \lambda < 1$ and $x \in (0,\infty)$, the equation*

$$x^{1-\lambda^2}\left(\kappa_2^* + \mu^* x^{\lambda}\right)^{1+\lambda} = \lambda^2 \kappa_2^* \kappa_{1*}, \quad (41)$$

has a unique solution $\tilde{l}$ which satisfies

$$0 < \tilde{l} < \varepsilon := \left(\lambda^2 \kappa_2^* \kappa_{1*}\right)^{1/(2+2\lambda)}. \quad (42)$$

Proof. Let

$$\varphi(x) = \lambda^2 \kappa_2^* \kappa_{1*} - x^{1-\lambda^2}\left(\kappa_2^* + \mu^* x^{\lambda}\right)^{1+\lambda}, \quad (43)$$

then

$$\varphi(0) = \lim_{x\to 0} \varphi(x) = \lambda^2 \kappa_2^* \kappa_{1*} > 0. \quad (44)$$

On the other hand, since $0 < \lambda < 1$, $\kappa_2^* \geq \kappa_{1*}$, one has

$$0 < \varepsilon \leq \kappa_2^{*\,1/(1+\lambda)}. \quad (45)$$

Therefore,

$$\begin{aligned} \varphi(\varepsilon) &= \varepsilon^{2+2\lambda} - \varepsilon^{1-\lambda^2}\left(\kappa_2^* + \mu^* \varepsilon^{\lambda}\right)^{1+\lambda} \\ &= \varepsilon^{1-\lambda^2}\left[\varepsilon^{(1+\lambda)^2} - \left(\kappa_2^* + \mu^* \varepsilon^{\lambda}\right)^{1+\lambda}\right] \\ &< \varepsilon^{1-\lambda^2}\left[\varepsilon^{(1+\lambda)^2} - \kappa_2^{*\,1+\lambda}\right] < 0. \end{aligned} \quad (46)$$

Moreover,

$$\begin{aligned} \varphi'(x) &= -\left(1-\lambda^2\right) x^{-\lambda^2}\left(\kappa_2^* + \mu^* x^{\lambda}\right)^{1+\lambda} - \mu^* \lambda (1+\lambda) x^{1-\lambda^2} \\ &\quad \cdot \left(\kappa_2^* + \mu^* x^{\lambda}\right)^{\lambda} x^{\lambda-1} < 0, \quad x \in (0,\infty). \end{aligned} \quad (47)$$

Thus, it follows from (44) to (47) that equation (41) has a unique positive solution $\tilde{l} \in (0, \varepsilon)$.

Theorem 7. *Assume that (F0)–(F1) hold. Let $\tilde{l}$ be the unique solution of equation (41), if $\mu_* \leq 0$, $\mu^* \geq 0$ and satisfy*

$$\mu_* \geq \tilde{l} - \frac{\kappa_{1*} \tilde{l}^{2\lambda}}{\left(\kappa_2^* + \mu^* \tilde{l}^{\lambda}\right)^{\lambda}}, \quad (48)$$

then the Hadamard-type fractional differential equation (1) has at least one positive solution.

Proof. In this case, following the strategy and notations of (33) and (34), we have

$$\begin{aligned} (Tx)(t) &\geq \frac{1}{L^{\lambda}\left(1 + (1/(\beta-1)\Gamma(\beta))\right)^{\lambda}} \int_1^e G(t,s) h_1(s) ds + \mu_* \\ &\geq \frac{\kappa_{1*}}{L^{\lambda}} + \mu_*, (Tx)(t) \leq \frac{1}{l^{\lambda}} \int_1^e G(t,s) h_2(s) ds + \mu^* \\ &\leq \frac{\kappa_2^*}{l^{\lambda}} + \mu^*. \end{aligned} \quad (49)$$

Thus, to ensure $T : B \to B$, it is sufficient to choose $0 < l < L$ such that

$$\frac{\kappa_{1*}}{L^{\lambda}} + \mu_* \geq l, \ \frac{\kappa_2^*}{l^{\lambda}} + \mu^* \leq L. \quad (50)$$

To do this, we fix $L = \kappa_2^*/l^{\lambda} + \mu^*$. Clearly, if l satisfies

$$0 < l < L = \frac{\kappa_2^*}{l^{\lambda}} + \mu^*, \quad (51)$$

$$\mu_* \geq l - \frac{\kappa_{1*} l^{2\lambda}}{\left(\kappa_2^* + \mu^* l^\lambda\right)^\lambda}, \tag{52}$$

then the inequalities of (50) hold.

Let

$$\psi(x) = x - \frac{\kappa_{1*} x^{2\lambda}}{\left(\kappa_2^* + \mu^* x^\lambda\right)^\lambda}, \tag{53}$$

notice that

$$\begin{aligned}\psi'(x) &= 1 - \frac{\kappa_{1*}\left[\lambda^2 x^{\lambda^2-1}\left(\kappa_2^* + \lambda^2\mu^* x^\lambda\right)^\lambda - \lambda^* x^{\lambda^2}\left(\kappa_2^* + \mu^* x^\lambda\right)^{\lambda-1} x^{\lambda-1}\right]}{\left(\kappa_2^* + \mu^* x^\lambda\right)^{2\lambda}} \\ &= 1 - \frac{\kappa_{1*}\lambda^2 x^{\lambda^2-1}}{\left(\kappa_2^* + \mu^* x^\lambda\right)^\lambda}\left[1 - \frac{\mu^* x^\lambda}{\kappa_2^* + \mu^* x^\lambda}\right] = 1 - \frac{\kappa_{1*}\kappa_2^*\lambda^2 x^{\lambda^2-1}}{\left(\kappa_2^* + \mu^* x^\lambda\right)^{\lambda+1}},\end{aligned} \tag{54}$$

and $\psi'(0) = -\infty, \psi'(\infty) = 1$; consequently, there exists $\tilde{l}$ such that

$$\psi'\left(\tilde{l}\right) = 1 - \frac{\kappa_{1*}\kappa_2^*\lambda^2 \tilde{l}^{\lambda^2-1}}{\left(\kappa_2^* + \mu^* \tilde{l}^\lambda\right)^{\lambda+1}} = 0, \tag{55}$$

which implies that $\tilde{l}$ solves the equation

$$x^{1-\lambda^2}\left(\kappa_2^* + \mu^* x^\lambda\right)^{1+\lambda} = \lambda^2\kappa_2^*\kappa_{1*}, \tag{56}$$

i.e., $\tilde{l}$ is the unique solution of equation (41).

On the other hand, since

$$\psi''(x) = \frac{\left(1-\lambda^2\right)\lambda^2\kappa_{1*}\kappa_2^* x^{\lambda^2-2}}{\left(\kappa_2^* + \mu^* x^\lambda\right)^{\lambda+1}} + \frac{(1+\lambda)\lambda^3\mu^*\kappa_{1*}\kappa_2^* x^{\lambda^2+\lambda-2}}{\left(\kappa_2^* + \mu^* x^\lambda\right)^{\lambda+2}} > 0, \tag{57}$$

the function $\psi(x)$ gets the minimum at $\tilde{l}$, i.e.,

$$\psi\left(\tilde{l}\right) = \min_{x\in(0,\infty)} \psi(x). \tag{58}$$

Taking $l = \tilde{l}$, then the assumption (48) implies that (52) holds.

Thus, we only need to prove that the inequality $L > l$ is also satisfied. Notice that $0 < l = \tilde{l} < \varepsilon =: \left(\lambda^2\kappa_2^*\kappa_{1*}\right)^{1/(2+2\lambda)}$ solves (41), we have

$$l^{1-\lambda^2}\left(\kappa_2^* + \mu^* l^\lambda\right)^{1+\lambda} = \lambda^2\kappa_2^*\kappa_{1*}. \tag{59}$$

It follows from $L = \kappa_2^*/l^\lambda + \mu^*$ that

$$l^{1+\lambda}L^{1+\lambda} = \lambda^2\kappa_2^*\kappa_{1*}, \tag{60}$$

and then

$$L = \frac{1}{l}\left(\lambda^2\kappa_2^*\kappa_{1*}\right)^{1/(1+\lambda)}. \tag{61}$$

Thus, it follows from

$$l < \left(\lambda^2\kappa_2^*\kappa_{1*}\right)^{1/(2+2\lambda)}, \tag{62}$$

that

$$L = \frac{1}{l}\left(\lambda^2\kappa_2^*\kappa_{1*}\right)^{1/(1+\lambda)} > \left(\lambda^2\kappa_2^*\kappa_{1*}\right)^{1/(2+2\lambda)} > l. \tag{63}$$

That is, under the assumptions of Theorem 7, (51) and (52) all hold. Thus, according to Schauder's fixed point theorem, the Hadamard-type fractional differential equation (1) has at least one positive solution

$$z^*(t) = \int_1^e H(t,s)x^*(s)\frac{ds}{s}. \tag{64}$$

Remark 8. In this case, the nonlinearity $f(t,u,v)$ of equation (1) may be singular at $u = 0$ and $v = 0$; moreover, $\chi(t)$ can be changing-sign $L^p(1,e)$ function, which is allowed to be singular at some $t \in [1,e]$.

4. Examples

In this section, we give some examples with positive, negative, and changing-sign perturbations to demonstrate the application of our main results.

We consider the following Hadamard-type fractional differential equation with different perturbations:

$$\begin{cases} \mathscr{D}_{\mathbf{t}}^{9/2}z(t) + \frac{1}{4}\mathscr{D}_{\mathbf{t}}^{5/2}z(t) = \frac{39}{2\left(z(t) + \left|\mathscr{D}_{\mathbf{t}}^{5/2}z(t)\right|\right)^{1/2}} + \chi(t), & 1 < t < e, \\ z(1) = z'(1) = z'(e) = 0,\ \mathscr{D}_{\mathbf{t}}^{5/2}z(1) = \mathscr{D}_{\mathbf{t}}^{5/2}z(e),\ \mathscr{D}_{\mathbf{t}}^{9/2}z(1) = \mathscr{D}_{\mathbf{t}}^{9/2}z(e). \end{cases} \tag{65}$$

Example 9. Consider the case of equation (65) with positive force term $\chi(t) = 1$.

Conclusion. The Hadamard-type fractional differential equation (65) has at least one positive solution.

Proof. Take $\lambda = 1/2$, $h_1(t) = 19$, $h_2(t) = 20$ and

$$f(t,u,v) = \frac{39}{2(u+v)^{1/2}}, (u,v) \in (0,\infty)\times(0,\infty), \quad t \in (1,e), \tag{66}$$

then

$$\frac{19}{(u+v)^{1/2}} \leq f(t,u,v) \leq \frac{20}{(u+v)^{1/2}}, (u,v) \in (0,\infty)\times(0,\infty), \quad t \in (1,e). \tag{67}$$

It is easy to check that the Green function of the equation $x''(t) + (1/4)x(t) = a(t)$ with periodic boundary conditions $x(1) = x(e)$, $x'(1) = x'(e)$ is

$$G(t,s) = \begin{cases} \sin \dfrac{(e-1)-t+s}{2} \sin \dfrac{t-s}{2}, & 1 \le s \le t \le e, \\ \sin \dfrac{(e-1)-s+t}{2} \sin \dfrac{s-t}{2}, & 1 \le t \le s \le e. \end{cases} \tag{68}$$

Thus, $G(t,s) > 0$, $(t,s) \in [1,e] \times [1,e]$, and (F1) holds.

Now, let us compute μ^*, μ_* and κ_{1*}, κ_2^*. Note that

$$\mu(t) = \int_1^e G(t,s)\chi(s)ds = \sin\left(\frac{1}{2}e - \frac{1}{2}\right) + \frac{1}{2}\cos\left(\frac{1}{2}e - \frac{1}{2}\right) - \frac{1}{2}\cos\left(\frac{1}{2}e - \frac{1}{2}\right)e = 0.1935. \tag{69}$$

By simple computations, one has

$$\mu_* = \mu^* = 0.1935,\ \kappa_{1*} = \kappa_1^* = 3.0004,\ \kappa_{2*} = \kappa_2^* = 3.8700. \tag{70}$$

Now, take $L = 15.3615$ and let $l = 1/L = 0.0651$. Clearly, L satisfies the inequalities

$$3.0004L^{1/2} \ge 1,\ 3.87L^{1/2} + 0.1935 \le L, \tag{71}$$

thus, according to Theorem 3, the Hadamard-type fractional differential equation (65) has at least one positive solution.

Example 10. We consider the case of Hadamard-type fractional differential equation (65) with negative force term $\chi(t) = -3$.

Conclusion. The Hadamard-type fractional differential equation (65) has at least one positive solution.

Proof. We still take $\lambda = 1/2$, $h_1(t) = 19$, $h_2(t) = 20$, then as Example 9, we have $G(t,s) > 0$, $(t,s) \in [1,e] \times [1,e]$, and (F1) holds.

Now, let us compute μ^*, μ_* and κ_{1*}, κ_2^*. Note that

$$\mu(t) = \int_1^e G(t,s)\chi(s)ds = -3\left(\sin\left(\frac{1}{2}e - \frac{1}{2}\right) + \frac{1}{2}\cos\left(\frac{1}{2}e - \frac{1}{2}\right) - \frac{1}{2}\cos\left(\frac{1}{2}e - \frac{1}{2}\right)e\right) = -0.5805. \tag{72}$$

By simple computations, one has

$$\mu_* = \mu^* = -0.5805,\ \kappa_{1*} = \kappa_1^* = 3.0004,\ \kappa_{2*} = \kappa_2^* = 3.8700. \tag{73}$$

So $\mu^* = -0.5805 \le 0$, and

$$\mu_* = -0.5805 \ge \left(\frac{\kappa_{1*}\lambda^2}{\kappa_2^{*\lambda}}\right)^{1/(1-\lambda^2)}\left(1 - \frac{1}{\lambda^2}\right) = -0.8295, \tag{74}$$

thus, according to Theorem 4, the Hadamard-type fractional differential equation (65) has at least one positive solution.

Example 11. Consider the case of Hadamard-type fractional differential equation (65) with changing-sign force term

$$\chi(t) = \begin{cases} -1, & t \in [1,2), \\ \dfrac{10000}{729}, & t \in [2,e]. \end{cases} \tag{75}$$

Conclusion. The Hadamard-type fractional differential equation (65) has at least one positive solution.

Proof. It follows from Example 9 that $G(t,s) > 0$, $(t,s) \in [1,e] \times [1,e]$, and (F1) holds.

Now, let us compute μ^*, μ_* and κ_{1*}, κ_2^*. Note that

$$\mu(t) = \int_1^e G(t,s)\chi(s)ds = \begin{cases} -\left[\sin\dfrac{e-1}{2} - \dfrac{1}{2}\cos\dfrac{e-1}{2} - \sin\dfrac{e-2}{2}\cos\left(\dfrac{3}{2} - t\right)\right], & 1 \le t < 2, \\ \dfrac{10000}{729}\left[-\dfrac{1}{2}e\cos\dfrac{e-1}{2} + \cos\dfrac{e-1}{2} + \sin\dfrac{e-1}{2} - \sin\dfrac{1}{2}\cos\left(\dfrac{e+2}{2} - t\right)\right], & 2 \le t \le e, \end{cases}$$

$$\kappa_1(t) = \left(1 + \frac{1}{(\beta-1)\Gamma(\beta)}\right)^{-\lambda}\int_1^e G(t,s)h_1(s)ds = 15.5059\int_1^e G(t,s)ds, \kappa_2(t) = \int_1^e G(t,s)h_2(s)ds = 20\int_1^e G(t,s)ds. \tag{76}$$

By simple computations, we have

$$\mu_* = -0.0798,\ \mu^* = 1,\ \kappa_{1*} = \kappa_1^* = 3.0004,\ \kappa_{2*} = \kappa_2^* = 3.8700. \tag{77}$$

Consequently, equation (41) reduces to

$$x^{3/4}\left(3.8700 + x^{1/2}\right)^{3/2} = 2.9029, \tag{78}$$

which has a unique solution $\tilde{l} = 0.2200$. Thus, one has

$$\mu_* = -0.0798 \geq \tilde{l} - \frac{\kappa_{1*}\tilde{l}^{2\lambda}}{\left(\kappa_2^* + \mu^*\tilde{l}^{\lambda}\right)^{\lambda}} = -0.0969, \tag{79}$$

which implies that (48) holds.

According to Theorem 7, the Hadamard-type fractional differential equation (65) has at least one positive solution.

5. Conclusion

The force effect from outside in many complex processes often leads to the system governed equations possessing perturbations. In this work, we establish several criteria on the existence of positive solutions for a Hadamard-type fractional differential equation with singularity in all of the cases where the perturbed term is positive, negative, and changing-sign. The main advantage of our assumption is that it provides an effective bound for the perturbed term χ. This classification is valid and reasonable, and it is also easier to get the solution of the target equation by simple calculation.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Authors' Contributions

The study was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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