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Research Article

Subnormality of Powers of Multivariable Weighted Shifts

Sang Hoon Lee,[1] Woo Young Lee ,[2] and Jasang Yoon[3]

[1]*Department of Mathematics, Chungnam National University, Daejeon 34134, Republic of Korea*
[2]*Department of Mathematics and RIM, Seoul National University, Seoul 08826, Republic of Korea*
[3]*School of Mathematical and Statistical Sciences, The University of Texas Rio Grande Valley, Edinburg, Texas 78539, USA*

Correspondence should be addressed to Woo Young Lee; wylee@snu.ac.kr

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Given a pair $\mathbf{T} \equiv (T_1, T_2)$ of commuting subnormal Hilbert space operators, the Lifting Problem for Commuting Subnormals (LPCS) asks for necessary and sufficient conditions for the existence of a commuting pair $\mathbf{N} \equiv (N_1, N_2)$ of normal extensions of T_1 and T_2; in other words, $\mathbf{T}$ is a subnormal pair. The LPCS is a longstanding open problem in the operator theory. In this paper, we consider the LPCS of a class of powers of 2-variable weighted shifts. Our main theorem states that if a "corner" of a 2-variable weighted shift $\mathbf{T} = W_{(\alpha,\beta)} := (T_1, T_2)$ is subnormal, then $\mathbf{T}$ is subnormal if and only if a power $\mathbf{T}^{(m,n)} := (T_1^m, T_2^n)$ is subnormal for some $m, n \geq 1$. As a corollary, we have that if $\mathbf{T}$ is a 2-variable weighted shift having a tensor core or a diagonal core, then $\mathbf{T}$ is subnormal if and only if a power of $\mathbf{T}$ is subnormal.

1. Introduction

For a Hilbert space operator, a subnormal operator means an operator admitting a normal extension, i.e., an extension which is a normal operator. As a lifting problem of operators, many researchers of operator theory have considered necessary and sufficient conditions for a pair of subnormal operators on a Hilbert space to admit commuting normal extensions: more concretely, given a pair $\mathbf{T} = (T_1, T_2)$ of commuting subnormal operators T_1, T_2 on a Hilbert space, find a necessary and sufficient condition for the existence of commuting normal extensions N_1 and N_2 of T_1 and T_2, respectively. This problem is referred to as the Lifting Problem for Commuting Subnormals (LPCS). A pair of subnormal operators admitting commuting normal extensions is called a *subnormal* pair.

For a bounded linear operator T on a complex Hilbert space $\mathscr{H}$, it is well known that the subnormality of T implies the subnormality of powers $T^m (m \geq 2)$. However, its converse is not true in general; in fact, Stampfli [1, p. 378] showed that the subnormality of all powers $T^m (m \geq 2)$ does not necessarily imply the subnormality of T, even if $T \equiv W_a$ is a unilateral weighted shift. It is also well known that the hyponormality (i.e., $[T^*, T] \equiv T^*T - TT^*$ is positive semidefinite) of T does not imply the hyponormality of T^2 [2]. However, for a unilateral weighted shift W_a, the hyponormality of W_a (detected by the condition $a_k \leq a_{k+1}$ for all $k \geq 0$ when $a = \{a_n\}_{n=0}^\infty$) clearly implies the hyponormality of all powers $W_a^m (m \geq 1)$.

On the other hand, Franks [3] showed that given a pair $\mathbf{T} = (T_1, T_2)$ of commuting subnormal operators, if $p(\mathbf{T})$ is subnormal for all 2-variable polynomials $p \in \mathbb{C}[z_1, z_2]$ with $\deg p \leq 5$, then $\mathbf{T}$ is a subnormal pair. Clearly, if $\mathbf{T} = (T_1, T_2)$ is a subnormal pair and if $m, n \geq 1$, then $\mathbf{T}^{(m,n)} := (T_1^m, T_2^n)$ is also a subnormal pair. Motivated by Stampfli's work [1], it is natural to ask whether the subnormality of $\mathbf{T}^{(m,n)} := (T_1^m, T_2^n)$ for each $(m, n) > (1, 1)$ implies the subnormality of $\mathbf{T}$. For the 2-variable weighted shifts, we may consider these analogous results. The standard assumption on a pair $\mathbf{T} \equiv (T_1, T_2)$ is that each component T_i is subnormal ($i = 1, 2$). With this in mind, these analogous results are highly nontrivial. In the works [4–8], it was considered whether there is a 2-variable weighted shift $\mathbf{T} = W_{(\alpha,\beta)} := (T_1, T_2)$ such that $\mathbf{T}^{(m,n)}$ is subnormal for some $(m, n) > (1, 1)$, but $\mathbf{T}$ is not subnormal.

For $a \equiv \{a_n\}_{n=0}^\infty$, a bounded sequence of positive real numbers (called *weights*), a *weighted shift* $W_a : \ell^2(\mathbb{Z}_+) \longrightarrow$

$\ell^2(\mathbb{Z}_+)$ is defined by $W_a e_n := a_n e_{n+1}$ (all $n \geq 0$), where $\{e_n\}_{n=0}^{\infty}$ is the canonical orthonormal basis in $\ell^2(\mathbb{Z}_+)$. In this case, we write

$$W_a = \text{shift}(a_0, a_1, a_2, \cdots). \quad (1)$$

Now, for $0 < x < y < 1$, consider the weighted shift

$$W_a = \text{shift}(x, y, 1, 1, \cdots). \quad (2)$$

Then, W_a is hyponormal (detected by the condition $a_n \leq a_{n+1}$ for all n) but not subnormal. However, all powers $W_a^m (m \geq 2)$ are subnormal. If $y = 1$ in $W_a = \text{shift}(x, y, 1, 1, \cdots)$, then the following statements are equivalent:

(a) W_a is subnormal

(b) W_a^m is subnormal for all $m \geq 1$

(c) W_a^m is subnormal for some $m \geq 1$

In [5, 6], we have examined the above results for the class of 2-variable weighted shifts $\mathbf{T} = W_{(\alpha,\beta)}$. More concretely, for the class of 2-variable weighted shifts $\mathbf{T} = W_{(\alpha,\beta)}$ with a core of tensor form, denoted $\mathcal{TC}$ [5], or with a core of diagonal form, denoted $\mathcal{DC}$ [6], we have shown that if $\mathbf{T} = W_{(\alpha,\beta)} \in \mathcal{TC} \cup \mathcal{DC}$, then the following statements are equivalent:

(a) $\mathbf{T}$ is subnormal

(b) $\mathbf{T}^{(m,n)}$ is subnormal for all $m, n \geq 1$

(c) $\mathbf{T}^{(m,n)}$ is subnormal for some $m, n \geq 1$

In spite of the above facts for 1 or 2-variable weighted shifts and consideration of the recent works ([4–8]), we have guessed that there exists a class of 2-variable weighted shifts $W_{(\alpha,\beta)}$ such that $W_{(\alpha,\beta)}$ is not subnormal but $W_{(\alpha,\beta)}^{(m,n)}$ is subnormal for all $(m, n) > (1, 1)$, under a more general condition that a "corner" of $W_{(\alpha,\beta)}$ is subnormal. In this paper, we show that this guess is not right and that the above three statements are equivalent whenever a corner of $W_{(\alpha,\beta)}$ is subnormal. In the below, we will notice that $\mathcal{TC} \cup \mathcal{DC}$ is a very special corner of $W_{(\alpha,\beta)}$.

On the other hand, the reason why we take 2-variable weighted shifts for examining the subnormality of powers for pairs of operators is that 2-variable weighted shifts play an important role in detecting properties such as subnormality, via the Lambert-Lubin Criterion ([9, 10]): a commuting pair (T_1, T_2) of injective operators acting on a Hilbert space $\mathcal{H}$ admits a commuting normal extension if and only if for every nonzero vector $x \in \mathcal{H}$, the 2-variable weighted shift with weights

$$\alpha_{(i,j)} := \frac{\left\| T_1^{i+1} T_2^j x \right\|}{\left\| T_1^i T_2^j x \right\|} \text{ and } \beta_{(i,j)} := \frac{\left\| T_1^i T_2^{j+1} x \right\|}{\left\| T_1^i T_2^j x \right\|}, \quad (3)$$

has a normal extension.

The organization of this paper is as follows. In Section 2, we give preliminary notions and state the main theorem. In Section 3, we provide a proof of the main theorem.

2. Preliminaries and the Main Theorem

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded linear operators on $\mathcal{H}$. For $S, T \in \mathcal{B}(\mathcal{H})$ let $[S, T] := ST - TS$. We say that an n-tuple $\mathbf{T} = (T_1, \cdots, T_n)$ of operators on $\mathcal{H}$ is (jointly) *hyponormal* if the operator matrix

$$[\mathbf{T}^*, \mathbf{T}] := \begin{pmatrix} [T_1^*, T_1] & [T_2^*, T_1] & \cdots & [T_n^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] & \cdots & [T_n^*, T_2] \\ \vdots & \vdots & \ddots & \vdots \\ [T_1^*, T_n] & [T_2^*, T_n] & \cdots & [T_n^*, T_n] \end{pmatrix}, \quad (4)$$

is positive semidefinite on the direct sum of n copies of $\mathcal{H}$ (cf. [11, 12]). The n-tuple $\mathbf{T}$ is said to be *normal* if $\mathbf{T}$ is commuting and each T_i is normal, and $\mathbf{T}$ is *subnormal* if $\mathbf{T}$ is the restriction of a normal n-tuple to a common invariant subspace. For $k \geq 1$, a commuting pair $\mathbf{T} \equiv (T_1, T_2)$ is said to be *k -hyponormal* ([13]) if

$$\mathbf{T}(k) := \left(T_1, T_2, T_1^2, T_2 T_1, T_2^2, \cdots, T_1^k, T_2 T_1^{k-1}, \cdots, T_2^k \right), \quad (5)$$

is hyponormal, or equivalently

$$[\mathbf{T}(k)^*, \mathbf{T}(k)] = \left(\left[\left(T_2^q T_1^p \right)^*, T_2^m T_1^n \right] \right)_{1 \leq p+q \leq k}^{1 \leq n+m \leq k} \geq 0. \quad (6)$$

Clearly, normal $\Rightarrow$ subnormal $\Rightarrow$ k-hyponormal for n-tuples of operators. The Bram-Halmos criterion states that an operator $T \in \mathcal{B}(\mathcal{H})$ is subnormal if and only if the k-tuple $(T, T^2, \cdots, T^k)$ is hyponormal for all $k \geq 1$.

Let W_a be a weighted shift with weights $a \equiv \{a_n\}_{n=0}^{\infty}$. The moments of a are given as

$$\gamma_k \equiv \gamma_k(a) := \begin{cases} 1, & \text{if } k = 0 \\ a_0^2 \cdots a_{k-1}^2, & \text{if } k \geq 1 \end{cases}. \quad (7)$$

It is easy to see that W_a is never normal and that it is hyponormal if and only if $a_0 \leq a_1 \leq \cdots$. Similarly, consider double-indexed positive bounded sequences $\alpha \equiv \{\alpha_{\mathbf{k}}\}, \beta \equiv \{\beta_{\mathbf{k}}\} \in \ell^\infty(\mathbb{Z}_+^2)$ and $\mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2$ and let $\ell^2(\mathbb{Z}_+^2)$ be the Hilbert space of square-summable complex sequences indexed by $\mathbb{Z}_+^2$. Recall that $\ell^2(\mathbb{Z}_+^2)$ is canonically isometrically isomorphic to $\ell^2(\mathbb{Z}_+) \otimes \ell^2(\mathbb{Z}_+)$. We define the *2-variable weighted shift* $W_{(\alpha,\beta)} \equiv (T_1, T_2)$ by a pair of operators acting on the Hilbert space $\mathcal{H} \equiv \ell^2(\mathbb{Z}_+^2)$ given by

$$\begin{aligned} T_1 e_{(k_1,k_2)} &:= \alpha_{(k_1,k_2)} e_{(k_1+1,k_2)}, \\ T_2 e_{(k_1,k_2)} &:= \beta_{(k_1,k_2)} e_{(k_1,k_2+1)}, \end{aligned} \quad (8)$$

for each $(k_1, k_2) \in \mathbb{Z}_+^2$. For all $\mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2$, we clearly have

$$T_1T_2 = T_2T_1 \Leftrightarrow \beta_{(k_1+1,k_2)}\alpha_{(k_1,k_2)} = \alpha_{(k_1,k_2+1)}\beta_{(k_1,k_2)}. \quad (9)$$

For a commuting 2-variable weighted shift $W_{(\alpha,\beta)}$, the moment of $W_{(\alpha,\beta)}$ of order $\mathbf{k} \in \mathbb{Z}_+^2$ is

$$\gamma_{\mathbf{k}} \equiv \gamma_{\mathbf{k}}\left(W_{(\alpha,\beta)}\right) := \begin{cases} 1, & \text{if } k_1 = 0 \text{ and } k_2 = 0 \\ \alpha_{(0,0)}^2 \cdots \alpha_{(k_1-1,0)}^2, & \text{if } k_1 \geq 1 \text{ and } k_2 = 0 \\ \beta_{(0,0)}^2 \cdots \beta_{(0,k_2-1)}^2, & \text{if } k_1 = 0 \text{ and } k_2 \geq 1 \\ \alpha_{(0,0)}^2 \cdots \alpha_{(k_1-1,0)}^2 \beta_{(k_1,0)}^2 \cdots \beta_{(k_1,k_2-1)}^2, & \text{if } k_1 \geq 1 \text{ and } k_2 \geq 1. \end{cases} \quad (10)$$

We remark that, due to the commutativity condition (9), $\gamma_{\mathbf{k}}$ can be computed using any nondecreasing path from (0, 0) to (k_1, k_2). For a detailed discussion of the 2-variable weighted shifts, the reader may refer to ([4, 13, 14] [8]).

We now recall a well-known characterization of subnormality for multivariable weighted shifts [15], due to C. Berger (cf. [2, III.8.16]) and independently established by Gellar and Wallen [16]) in the single variable case: $W_{(\alpha,\beta)}$ admits a commuting normal extension if and only if there is a probability measure μ, called the *Berger measure* of $W_{(\alpha,\beta)}$, defined on the 2-dimensional rectangle $R = [0, c_1] \times 0, c_2]$ (where $c_i := \|T_i\|^2$) such that

$$\gamma_{\mathbf{k}} = \int_R \mathbf{t}^{\mathbf{k}} d\mu(\mathbf{t}) := \int_R t_1^{k_1} t_2^{k_2} d\mu(t_1, t_2), \text{for all } \mathbf{k} \in \mathbb{Z}_+^2. \quad (11)$$

Observe that $U_+ \equiv \text{shift}\,(1, 1, 1, \cdots)$ and $S_c \equiv \text{shift}(c, 1, 1, \cdots)(c \leq 1)$ are subnormal, with Berger measuresδ_1 and $(1-c^2)\delta_0 + c^2\delta_1$, respectively, where δ_p denotes the point-mass probability measure with support from the singleton set $\{p\}$.

Throughout this paper, we write $\mathscr{H} \equiv \ell^2(\mathbb{Z}_+^2) \equiv \vee \{e_{\mathbf{k}}\}_{\mathbf{k} \in \mathbb{Z}_+^2}$ and

$$\mathfrak{H}_0 := \text{the set of all commuting pairs of subnormal operators on } \mathscr{H}, \quad (12)$$

$$\mathscr{M}_p(\mathscr{H}) := \vee\{e_{\mathbf{k}} : \mathbf{k} = (k_1, k_2) \text{ with } k_1 \geq 0, k_2 \geq p\}, \quad (13)$$

$$\mathscr{N}_q(\mathscr{H}) := \vee\{e_{\mathbf{k}} : \mathbf{k} = (k_1, k_2) \text{ with } k_1 \geq q, k_2 \geq 0\}. \quad (14)$$

for $p, q \geq 0$. For a 2-variable weighted shift $W_{(\alpha,\beta)}$, a *corner* of $W_{(\alpha,\beta)}$ is defined by

$$W_{(\alpha,\beta)}\Big|_{\mathscr{M}_p(\mathscr{H}) \cap \mathscr{N}_q(\mathscr{H})} \text{ for some } p, q \geq 0, \quad (15)$$

which is a restriction of $W_{(\alpha,\beta)}$ to the invariant subspace $\mathscr{M}_p(\mathscr{H}) \cap \mathscr{N}_q(\mathscr{H})$. The *core* of $W_{(\alpha,\beta)}$, denoted by $c(W_{(\alpha,\beta)})$, is defined by a corner with $p = q = 1$, i.e.,

$$c\left(W_{(\alpha,\beta)}\right) := W_{(\alpha,\beta)}\Big|_{\mathscr{M}_1(\mathscr{H}) \cap \mathscr{N}_1(\mathscr{H})}. \quad (16)$$

Thus, for 2-variable weighted shifts, the core is a special form of a corner. A 2-variable weighted shift $W_{(\alpha,\beta)}$ is said to be of tensor form if it is of the form $(I \otimes W_a, W_b \otimes I)$. If a tensor form $W_{(\alpha,\beta)}$ is subnormal, then the corresponding Berger measure is given by a Cartesian product $\xi \times \eta$ where ξ and η are the Berger measure of W_a and W_b, respectively (cf. [5]). Also, for strictly increasing weight sequences $a \equiv \{a_n\}_{n=0}^{\infty}$, consider a 2-variable weighted shift $W_{(\alpha,\beta)}$ on $\ell^2(\mathbb{Z}_+^2)$ given by the double-indexed weight sequences

$$\alpha_{\mathbf{k}} = \beta_{\mathbf{k}} := a_{k_1+k_2} \text{for } \mathbf{k} = (k_1, k_2) \text{ with } k_1, k_2 \geq 0. \quad (17)$$

This 2-variable weighted shift $W_{(\alpha,\beta)}$ induced by a 1-variable weighted shift W_a is said to be of the diagonal form. If a diagonal form $W_{(\alpha,\beta)}$ is subnormal, then the corresponding Berger measure is given by a measure supported in the diagonal $\{(s, s) \in \mathbb{R}^2 : s \geq 0\}$ (cf. [6]). The class of all 2-variable weighted shifts $W_{(\alpha,\beta)} \in \mathfrak{H}_0$ whose core is of the tensor form will be denoted by $\mathscr{TC}$; in symbols,

$\mathscr{TC} := \{W_{(\alpha,\beta)} \in \mathfrak{H}_0 : c(W_{(\alpha,\beta)}) \text{ is of tensor form}\}$ (see Figure 1(i)).

Also, the class of all 2-variable weighted shifts $W_{(\alpha,\beta)} \in \mathfrak{H}_0$ whose core is of the diagonal form will be denoted by $\mathscr{DC}$; in symbols,

$\mathscr{DC} := \{W_{(\alpha,\beta)} \in \mathfrak{H}_0 : c(W_{(\alpha,\beta)}) \text{ is of diagonal form}\}$ (see Figure 1(ii)).

In [5, 6], it was shown that if $\mathbf{T} = W_{(\alpha,\beta)} \in \mathscr{TC} \cup \mathscr{DC}$, then $\mathbf{T}^{(m_0,n_0)}$ is subnormal for some $m_0, n_0 \geq 1$ if and only if $\mathbf{T}$ is subnormal. Now, it is natural to consider that given a 2-variable weighted shift $\mathbf{T} = W_{(\alpha,\beta)} \in \mathfrak{H}_0$, whether or not $\mathbf{T}^{(m,n)}$ is subnormal if and only if $\mathbf{T}$ is subnormal. In other words, we ask the following:

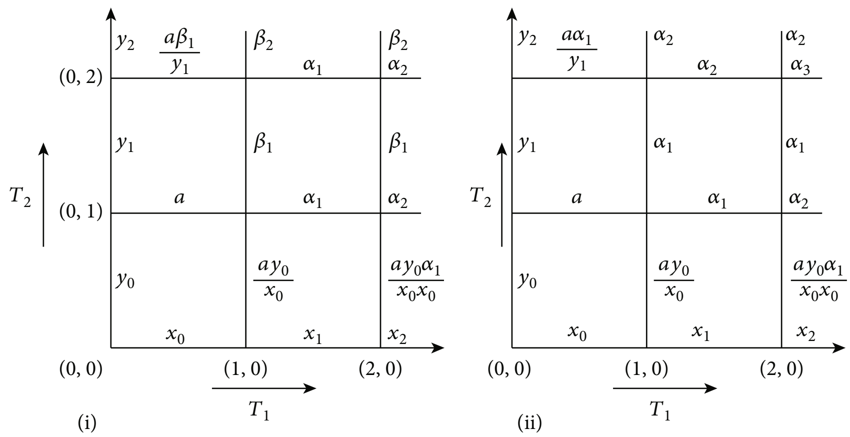

FIGURE 1: Weight diagram of the 2-variable weighted shift $\mathbf{T} \in \mathcal{TC}$ and weight diagram of the 2-variable weighted shift $\mathbf{T} \in \mathcal{DC}$, respectively.

Problem 1 ([6, 8]). Given a 2-variable weighted shift $\mathbf{T} = W_{(\alpha,\beta)} \in \mathfrak{H}_0$, assume that $\mathbf{T}^{(m,n)}$ is subnormal for all $(m, n) > (1, 1)$. Does it follow that $\mathbf{T}$ is subnormal?

For the class of 2-variable weighted shifts $W_{(\alpha,\beta)}$, it is often the case that the powers are less complex than the initial pair; thus, it becomes especially significant to unravel the invariance of subnormality under the action $W_{(\alpha,\beta)} \mapsto W_{(\alpha,\beta)}^{(m,n)} (m, n \geq 1)$. The aim of this paper is to shed new light on some of the intricacies associated with LPCS and powers of commuting subnormals in $\mathfrak{H}_0$.

Our main theorem now states:

Theorem 2. *Let* $\mathbf{T} \equiv W_{(\alpha,\beta)} \in \mathfrak{H}_0$. *If a corner of* $\mathbf{T}$ *is subnormal, i.e.,* $\mathbf{T}|_{\mathcal{M}_p(\mathcal{H})\cap\mathcal{N}_q(\mathcal{H})}$ *is subnormal for some* $p, q \in \mathbb{Z}_+$, *then the following are equivalent:*

(a) $\mathbf{T}$ *is subnormal*

(b) $\mathbf{T}^{(m,n)}$ *is subnormal for all* $m, n \geq 1$

(c) $\mathbf{T}^{(m,n)}$ *is subnormal for some* $m, n \geq 1$

As we observed before, $\mathcal{TC} \cup \mathcal{DC}$ is a special corner of $W_{(\alpha,\beta)}$. Indeed, if $\mathbf{T} = W_{(\alpha,\beta)} \in \mathcal{TC} \cup \mathcal{DC}$, then $\mathbf{T}|_{\mathcal{M}_1(\mathcal{H})\cap\mathcal{N}_1(\mathcal{H})}$ is subnormal, so that $\mathbf{T}$ satisfies the condition of Theorem 2. Therefore, if $\mathbf{T} = W_{(\alpha,\beta)} \in \mathcal{TC} \cup \mathcal{DC}$, then three conditions of Theorem 2 are equivalent. Thus, as immediate corollaries of Theorem 2, we can recapture the both main results of [5, Theorem 7.1] and [6, Theorem 3.2].

Briefly stated, our key idea to prove the main results is as follows: (i) we split the ambient space $\mathcal{H} \equiv \ell^2(\mathbb{Z}_+^2)$ as an orthogonal direct sum $\mathcal{H} = \oplus_{p=0}^{m-1} \oplus_{q=0}^{n-1} \mathcal{H}_{(p,q)}^{(m,n)}$; (ii) when $\mathbf{T}|_{\mathcal{M}_1(\mathcal{H})}$ is subnormal, we show that for some $m \geq 1, \mathbf{T}^{(m,1)}$ is subnormal if and only if $\mathbf{T}$ is subnormal by using the backward extension of subnormality; (iii) when $\mathbf{T}|_{\mathcal{M}_1(\mathcal{H})}$ is subnormal, we show that for some $m, n \geq 1 (\mathbf{T}^{(1,n)})^{(m,1)}$ is subnormal if and only if $\mathbf{T}^{(m,1)}$ is subnormal by using (i) and the backward extension of subnormality; and (iv) by combining (ii) and (iii), we have that if $\mathbf{T}|_{\mathcal{M}_p(\mathcal{H})\cap\mathcal{N}_q(\mathcal{H})}$ is subnormal for some $p, q \in \mathbb{Z}_+$, then $\mathbf{T}^{(m,n)} = (\mathbf{T}^{(m,1)})^{(1,n)}$ is subnormal for some $m, n \geq 1$ if and only if $\mathbf{T}$ is subnormal.

3. The Proof of the Main Theorem

We will first establish several auxiliary lemmas and then prove the main theorem (Theorem 2).

To study subnormality for powers of multivariable weighted shifts, we recall that, in one variable, the m-th power of a weighted shift is unitarily equivalent to the direct sum of m weighted shifts. First, we need some terminology. Let $\ell^2(\mathbb{Z}_+) = \vee_{j=0}^{\infty}\{e_j\}$. Given integers i and $m (m \geq 1, 0 \leq i \leq m-1)$, define $\mathcal{H}_{m,i} := \vee_{j=0}^{\infty}\{e_{mj+i}\}$; clearly, $\ell^2(\mathbb{Z}_+) = \oplus_{i=0}^{m-1} \mathcal{H}_{m,i}$. Following the notation in [17], for a weight sequence $a \equiv \{a_n\}_{n=0}^{\infty}$, we let

$$W_{a(m:i)} := \text{shift}\left(\Pi_{n=0}^{m-1} a_{mj+i+n}\right)_{j=0}^{\infty}, \tag{18}$$

that is, $W_{a(m:i)}$ denotes the shift with the weight sequence given by the products of weights in adjacent packets of size m, beginning with $a_i \cdots a_{i+m-1}$. For example, given a weight sequence $a \equiv \{a_n\}_{n=0}^{\infty}$, we have $W_{a(2:0)} = \text{shift}(a_0a_1, a_2a_3, \cdots)$, $W_{a(2:1)} = \text{shift}(a_1a_2, a_3a_4, \cdots)$, $W_{a(3:2)} = \text{shift}(a_2a_3a_4, a_5a_6a_7, \cdots)$, etc. For $m \geq 1$ and $0 \leq i \leq m-1$, we note that $W_{a(m:i)}$ is unitarily equivalent to $W_a^m|_{\mathcal{H}_{m,i}}$. Therefore, W_a^m is unitarily equivalent to $\oplus_{i=0}^{m-1} W_{a(m:i)}$. Thus, we have (cf. [17]) that if W_a is subnormal with the Berger measure μ, then $W_{a(m,i)}$ is subnormal with the Berger measure $\mu_{(m,i)}$, where

$$d\mu_{(m,i)}(s) = \frac{s^{i/m}}{\gamma_i} d\mu\left(s^{1/m}\right) \text{ for } 0 \leq i \leq m-1, \tag{19}$$

and furthermore,

$$W_a^m \text{ is subnormal} \Leftarrow W_{a(m:i)} \text{ is subnormal for } 0 \le i \le m-1. \quad (20)$$

For $h \ge 1$, we let $\mathscr{L}_h := \vee\{e_n : \text{n} \ge h\}$ denote the invariant subspace obtained by removing the first h vectors in the canonical orthonormal basis of $\ell^2(\mathbb{Z}_+)$. Thus, if $W_a \equiv \text{shift}(a_0, a_1, a_2, \cdots)$ is subnormal with Berger measure σ, then $W_a|_{\mathscr{L}_h}$ is subnormal for each $h \ge 1$, and the Berger measure $(\sigma)_h$ of $W_a|_{\mathscr{L}_h}$ is given by

$$d(\sigma)_h(s) = \frac{s^h}{\gamma_h} d\sigma(s). \quad (21)$$

Something similar happens in two variables, as we will see it below. For a 2-variable weighted shift $\mathbf{T} = W_{(\alpha,\beta)} = (T_1, T_2) \in \mathfrak{H}_0$ on $\mathscr{H} \equiv \ell^2(\mathbb{Z}_+^2)$, we observe a new direct sum decomposition for powers of 2-variable weighted shifts which parallels the decomposition used in [17] to analyze the subnormality for powers of (one-variable) weighted shifts. Specially, we split the ambient space $\mathscr{H} \equiv \ell^2(\mathbb{Z}_+^2) \equiv \vee \{e_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}_+^2}$ as an orthogonal direct sum $\mathscr{H} \equiv \oplus_{p=0}^{m-1} \oplus_{\text{q}=0}^{n-1} \mathscr{H}_{(p,q)}^{(m,n)}$, where for $p = 0, 1, \cdots, m-1$, and $q = 0, 1, \cdots, n-1$,

$$\mathscr{H}_{(p,q)}^{(m,n)}(\mathbf{T}) := \vee\left\{e_{(mi+p,nj+q)} : i = 0, 1, 2, \cdots, j = 0, 1, 2, \cdots,\right\}, \quad (22)$$

where $\mathbf{T} = W_{(\alpha,\beta)}$ is a 2-variable weighted shift on $\mathscr{H} \equiv \ell^2(\mathbb{Z}_+^2)$. Then, each of $\mathscr{H}_{(p,q)}^{(m,n)}(\mathbf{T})$ reduces T_1^m and T_2^n. Also, $\mathbf{T}^{(m,n)}$ is subnormal if and only if each $\mathbf{T}^{(m,n)}|_{\mathscr{H}_{(p,q)}^{(m,n)}(\mathbf{T})}$ is subnormal. Similarly, for $h, l \ge 0$, consider $\mathbf{T}|_{\mathscr{M}_h(\mathscr{H})\cap\mathscr{M}_l(\mathscr{H})}$ on $\mathscr{M}_h(\mathscr{H}) \cap \mathscr{M}_l(\mathscr{H})$ and let

$$\begin{aligned} &\mathscr{H}_{(p,q)}^{(m,n)}\left(\mathbf{T}|_{\mathscr{M}_h(\mathscr{H})\cap\mathscr{M}_l(\mathscr{H})}\right) \\ &\quad := \vee\left\{e_{(mi+p+h,nj+q+l)} : i = 0, 1, 2, \cdots, j = 0, 1, 2, \cdots\right\}. \end{aligned} \quad (23)$$

In a similar fashion to (13) and (14), we can define

$$\begin{aligned} \mathscr{M}_h\left(\mathscr{H}_{(p,q)}^{(m,n)}(\mathbf{T})\right) &:= \vee\left\{e_{\mathbf{k}} \in \mathscr{H}_{(p,q)}^{(m,n)}(\mathbf{T}) : i \ge 0 \text{ and } j \ge h\right\}, \\ \mathscr{N}_l\left(\mathscr{H}_{(p,q)}^{(m,n)}(\mathbf{T})\right) &:= \vee\left\{e_{\mathbf{k}} \in \mathscr{H}_{(p,q)}^{(m,n)}(\mathbf{T}) : i \ge l \text{ and } j \ge 0\right\}. \end{aligned} \quad (24)$$

We thus have:

Lemma 3. *Let $\mathbf{T} = W_{(\alpha,\beta)}$ be a 2-variable weighted shift. Then, for a fixed $m \ge 1$ and $0 \le p, q \le m-1$, we have*

$$\begin{aligned} &\left(\mathbf{T}^{(m,1)}\Big|_{\mathscr{H}_{(0,0)}^{(m,1)}(\mathbf{T})}\right)\Big|_{\mathscr{M}_p\left(\mathscr{H}_{(0,0)}^{(m,1)}(\mathbf{T})\right)} \\ &\quad\cong \left(\mathbf{T}_{\mathscr{M}_p(\mathscr{H})}\right)^{(m,1)}\Big|_{\mathscr{H}_{(0,0)}^{(m,1)}\left(\mathbf{T}|_{\mathscr{M}_p(\mathscr{H})}\right)}, \\ &\left(\mathbf{T}^{(m,1)}\Big|_{\mathscr{H}_{(0,0)}^{(m,1)}(\mathbf{T})}\right)\Big|_{\mathscr{N}_q\left(\mathscr{H}_{(0,0)}^{(m,1)}(\mathbf{T})\right)} \\ &\quad\cong \left(\mathbf{T}_{\mathscr{N}_{mq}(\mathscr{H})}\right)^{(m,1)}\Big|_{\mathscr{H}_{(0,0)}^{(m,1)}\left(\mathbf{T}|_{\mathscr{M}_{mq}(\mathscr{H})}\right)} \end{aligned} \quad (25)$$

and for $m, n \ge 1$, $0 \le p \le m-1$, $0 \le q \le n-1$, we have

$$\begin{aligned} &\left(\mathbf{T}|_{\mathscr{M}_1(\mathscr{H})\cap\mathscr{N}_1(\mathscr{H})}\right)^{(m,n)}\Big|_{\mathscr{H}_{(p,n-1)}^{(m,n)}\left(\mathbf{T}_{\mathscr{M}_1(\mathscr{H})\cap\mathscr{N}_1(\mathscr{H})}\right)} \\ &\quad\cong \left(\mathbf{T}^{(m,n)}{}_{\mathscr{H}_{(p,0)}^{(m,n)}(\mathbf{T})}\right)\Big|_{\mathscr{M}_1(\mathscr{H})\cap\mathscr{N}_1(\mathscr{H})\left(\mathscr{H}_{(p,0)}^{(m,n)}(\mathbf{T})\right)}; \\ &\left(\mathbf{T}|_{\mathscr{M}_q(\mathscr{H})}\right)^{(m,n)}\Big|_{\mathscr{H}_{(0,0)}^{(m,n)}\left(\mathbf{T}_{\mathscr{M}_q(\mathscr{H})}\right)} \cong \mathbf{T}^{(m,n)}\Big|_{\mathscr{H}_{(0,q)}^{(m,n)}(\mathbf{T})}; \\ &\left(\mathbf{T}|_{\mathscr{N}_p(\mathscr{H})}\right)^{(m,n)}\Big|_{\mathscr{H}_{(0,0)}^{(m,n)}\left(\mathbf{T}_{\mathscr{N}_p(\mathscr{H})}\right)} \cong \mathbf{T}^{(m,n)}\Big|_{\mathscr{H}_{(p,0)}^{(m,n)}(\mathbf{T})}; \\ &\left(\mathbf{T}|_{\mathscr{M}_1(\mathscr{H})}\right)^{(m,n)}\Big|_{\mathscr{H}_{(0,n-1)}^{(m,n)}\left(\mathbf{T}_{\mathscr{M}_1(\mathscr{H})}\right)} \\ &\quad\cong \left(\mathbf{T}^{(m,n)}{}_{\mathscr{H}_{(0,0)}^{(m,n)}(\mathbf{T})}\right)\Big|_{\mathscr{M}_1\left(\mathscr{H}_{(0,0)}^{(m,n)}(\mathbf{T})\right)}; \\ &\left(\mathbf{T}|_{\mathscr{N}_1(\mathscr{H})}\right)^{(m,n)}\Big|_{\mathscr{H}_{(p,n-1)}^{(m,n)}\left(\mathbf{T}_{\mathscr{N}_1(\mathscr{H})}\right)} \\ &\quad\cong \left(\mathbf{T}^{(m,n)}{}_{\mathscr{H}_{(p,0)}^{(m,n)}(\mathbf{T})}\right)\Big|_{\mathscr{N}_1\left(\mathscr{H}_{(p,0)}^{(m,n)}(\mathbf{T})\right)}, \end{aligned} \quad (26)$$

where $\cong$ means a unitary equivalence of two operators.

Lemma 4. *Let $\mathbf{T} = W_{(\alpha,\beta)}$ be a 2-variable weighted shift in $\mathfrak{H}_0$. If $\mathbf{T}$ is subnormal with Berger measure μ, then $\mathbf{T}^{(m,n)}$ is subnormal for all $m, n \ge 1$. Furthermore, for each $0 \le p \le m-1$, $0 \le q \le n-1$, the Berger measure of $\mathbf{T}^{(m,n)}|_{\mathscr{H}_{(p,q)}^{(m,n)}(\mathbf{T})}$ is given by*

$$d\mu_{(p,q)}^{(m,n)}(s,t) = \frac{s^{p/m} t^{q/n}}{\gamma_{(p,q)}} d\mu\left(s^{1/m}, t^{1/n}\right). \quad (27)$$

Proof. Observe first that if $\mathbf{T} = W_{(\alpha,\beta)}$ is subnormal then $\mathbf{T}^{(m,n)} \cong \oplus_{p=0}^{m-1} \oplus_{q=0}^{n-1} \mathbf{T}^{(m,n)}|_{\mathscr{H}_{(p,q)}^{(m,n)}(\mathbf{T})}$, where each direct summand is a subnormal 2-variable weighted shift.

For the second assertion, observe

$$\gamma_{(i,j)}\left(\mu^{(m,n)}_{(p,q)}\right) = \frac{\gamma_{(mi+p,nj+q)}(\mu)}{\gamma_{(p,q)}(\mu)} \text{ for } p, q \geq 0. \tag{28}$$

Thus, for $i, j \geq 0$,

$$\begin{aligned}\int s^i t^j d\mu^{(m,n)}_{(p,q)}(s,t) &= \gamma_{(i,j)}\left(\mu^{(m,n)}_{(p,q)}\right) = \frac{\gamma_{(mi+p,nj+q)}}{\gamma_{(p,q)}} \\ &= \frac{1}{\gamma_{(p,q)}}\int s^{mi}t^{nj+q} d\mu(s,t) \\ &= \frac{1}{\gamma_{(p,q)}}\int s^i t^j s^{p/m} t^{q/n} d\mu\left(s^{1/m}, t^{1/n}\right),\end{aligned} \tag{29}$$

so that

$$d\mu^{(m,n)}_{(p,q)}(s,t) = \frac{s^{p/m}t^{q/n}}{\gamma_{(p,q)}} d\mu\left(s^{1/m}, t^{1/n}\right), \tag{30}$$

which gives the result.

To detect the subnormality of 2-variable weighted shifts, we introduce some definitions.

(i) For a regular Borel measure μ on $\mathbb{R}_+$, we say that μ is *positive* if $\mu(E) \geq 0$ for all Borel subset $E \subseteq \mathbb{R}_+$, or equivalently, $\mu \geq 0$ if and only if $\int f d\mu \geq 0$ for all $f \in C(\mathbb{R}_+)$ such that $f \geq 0$ on $\mathbb{R}_+$. Similarly, we say that $d\mu(s)$ is positive (denoted by $d\mu(s) \geq 0$) if $\int f(s) d\mu(s) \geq 0$ for all $f \in C(\mathbb{R}_+)$ such that $f \geq 0$ on $\mathbb{R}_+$. For positive two measures μ and ν on $\mathbb{R}_+$, we say that $\mu \geq \nu$ on $\mathbb{R}_+$ if $\mu - \nu$ is positive

(ii) Let μ be a probability measure on $X \times Y$, and assume that $1/t \in L^1(\mu)$. The *extremal measure* μ_{ext} (which is also a probability measure) on $X \times Y$ is given by

$$d\mu_{\text{ext}}(s,t) := (1 - \delta_0(t))\frac{1}{t\|1/t\|_{L^1(\mu)}} d\mu(s,t). \tag{31}$$

(iii) Given a measure μ on $X \times Y$, the *marginal measure* μ^X is given by $\mu^X := \mu \circ \pi_X^{-1}$, where $\pi_X : X \times Y \longrightarrow X$ is the canonical projection onto X. Thus, $\mu^X(E) = \mu(E \times Y)$ for every $E \subseteq X$, or equivalently, $d\mu^X(s) = \int_Y d\mu(s,t)$.

Lemma 5 [14, Proposition 3.10] (subnormal backward extension). *Let* $\mathbf{T} = W_{(\alpha,\beta)}$ *be a 2-variable weighted shift, and assume that* $\mathbf{T}|_{\mathcal{M}_1(\mathcal{H})}$ *is subnormal with associated Berger measure* η *and that* $W_0 := shift(\alpha_{00}, \alpha_{10}, \cdots)$ *is subnormal associated with Berger measure* ξ_{α_0}. *Then,* $\mathbf{T}$ *is subnormal if and only if*

$$\begin{aligned}&\frac{1}{t} \in L^1(\eta), \\ &\beta^2_{(0,0)} \leq \left(\left\|\frac{1}{t}\right\|_{L^1(\eta)}\right)^{-1}, \\ &\beta^2_{(0,0)}\left\|\frac{1}{t}\right\|_{L^1(\eta)}(\eta)^X_{\text{ext}} \leq \xi_{\alpha_0}.\end{aligned} \tag{32}$$

Moreover, if $\beta^2_{00}\|1/t\|_{L^1(\eta)} = 1$, *then* $(\eta)^X_{\text{ext}} = \xi_{\alpha_0}$. *In the case when* $\mathbf{T}$ *is subnormal, the Berger measure* μ *of* $\mathbf{T}$ *is given by*

$$\begin{aligned}d\mu(s,t) &= \beta^2_{(0,0)}\left\|\frac{1}{t}\right\|_{L^1(\eta)} d(\eta)_{\text{ext}}(s,t) \\ &+ \left(d\xi_{\alpha_0}(s) - \beta^2_{(0,0)}\left\|\frac{1}{t}\right\|_{L^1(\eta)} d(\eta)^X_{\text{ext}}(s)\right) d\delta_0(t).\end{aligned} \tag{33}$$

We also recall:

Lemma 6 [18]. *For a positive measure* μ *on* $Z \equiv X \times Y \equiv \mathbb{R}_+ \times \mathbb{R}_+$, *let* $1/t \in L^1(\mu)$. *Then,* $1/t \in L^1(\mu^Y)$ *and*

$$\left\|\frac{1}{t}\right\|_{L^1(\mu)} = \left\|\frac{1}{t}\right\|_{L^1(\mu^Y)}, \tag{34}$$

where $\mu^Y := \mu \circ \pi_Y^{-1}$ *and* $\pi_Y : Z \longrightarrow Y$ *is the canonical projection onto* Y.

Given a 2-variable weighted shift $\mathbf{T} = W_{(\alpha,\beta)} \in \mathfrak{H}_0$, and given $k_1, k_2 \geq 0$, we let

$$W_{k_2} := \text{shift}\left(\alpha_{(0,k_2)}, \alpha_{(1,k_2)}, \cdots\right), \tag{35}$$

be the k_2-th horizontal slice of T_1 with associated Berger measure $\xi_{\alpha_{k_2}}$; similarly, we let

$$V_{k_1} := \text{shift}\left(\beta_{(k_1,0)}, \beta_{(k_1,1)}, \cdots\right), \tag{36}$$

be the k_1-th vertical slice of T_2 with associated Berger measure $\eta_{\beta_{k_1}}$. Clearly, W_0 and V_0 are the unilateral weighted shifts associated with the 0th row and 0-column in the weight diagram for $\mathbf{T}$, respectively.

Then, we have:

Lemma 7. *Let* $T = W_{(\alpha,\beta)} \in \mathfrak{H}_0$. *If* $\mathbf{T}|_{\mathcal{M}_p(\mathcal{H})\cap\mathcal{N}_q(\mathcal{H})}$ *is subnormal for some* $p, q \in \mathbb{Z}_+$, *then* $\mathbf{T}|_{\mathcal{M}_1(\mathcal{H})\cap\mathcal{N}_1(\mathcal{H})}$ *is also subnormal.*

Proof. It is enough to show that if $\mathbf{T}|_{\mathcal{M}_2(\mathcal{H})\cap\mathcal{N}_1(\mathcal{H})}$ or $\mathbf{T}|_{\mathcal{M}_1(\mathcal{H})\cap\mathcal{N}_2(\mathcal{H})}$ is subnormal, then $\mathbf{T}|_{\mathcal{M}_1(\mathcal{H})\cap\mathcal{N}_1(\mathcal{H})}$ is subnormal. Without loss of generality, assume that $\mathbf{T}|_{\mathcal{M}_2(\mathcal{H})\cap\mathcal{N}_1(\mathcal{H})}$ is subnormal. Since $W_{(\alpha,\beta)} \in \mathfrak{H}_0$, we have that $V_{k_1} \equiv \text{shift}(\beta_{(k_1 0)}, \beta_{(k_1 1)}, \cdots)$ is subnormal for all $k_1 \geq 0$. By Lemma 5, we have $\beta^2_{(k_1,1)}\|1/t\|_{L^1(\eta_{\beta_{k_1}})} = 1 (k_1 \geq 0)$. Let ζ be the Berger measure of $\mathbf{T}|_{\mathcal{M}_2(\mathcal{H})\cap\mathcal{N}_1(\mathcal{H})}$. By Lemma 6, since $\|1/t\|_{L^1(\eta_{\beta_1})} = \|1/t\|_{L^1(\zeta)}$ and $\beta^2_{(1,1)}\|1/t\|_{L^1(\zeta)} = 1$, it follows from Lemma 5 that $(\zeta)^X_{\text{ext}} = \xi_{\alpha_1}$, that is,

$$(\zeta)^X_{\text{ext}} = \xi_{\alpha_1} \Rightarrow \beta^2_{(1,1)}\left\|\frac{1}{t}\right\|_{L^1(\zeta)}(\zeta)^X_{\text{ext}} \leq \xi_{\alpha_1}. \quad (37)$$

Moreover, $1/t \in L^1(\zeta)$ and $\beta^2_{(1,1)} \leq (\|1/t\|_{L^1(\zeta)})^{-1}$. Thus, by Lemma 5 again, we have that $\mathbf{T}|_{\mathcal{M}_1(\mathcal{H})\cap\mathcal{N}_1(\mathcal{H})}$ is subnormal, as desired.

Next, we have:

Lemma 8. *Assume $\mathbf{T} = W_{(\alpha,\beta)} \in \mathfrak{H}_0$ and $\mathbf{T}|_{\mathcal{M}_1(\mathcal{H})}$ is subnormal with Berger measure η. Assume $1/t \in L^1(\eta)$. Let ξ_{α_0} be the Berger measure of subnormal shift $W_0 \equiv (\alpha_{(0,0)}, \alpha_{(1,0)}, \cdots)$ which is in the zero level of $\mathbf{T}$. We also let*

$$l\psi := \xi_{\alpha_0} - \beta^2_{00}\left\|\frac{1}{t}\right\|_{L^1(\eta)}(\eta)^X_{\text{ext}}. \quad (38)$$

If $\mathbf{T}^{(m,1)}|_{\mathcal{H}^{(m,1)}_{(0,0)}(\mathbf{T})}$ is subnormal for some $m \geq 1$, then $\psi \geq 0$. Moreover, if $\mathbf{T}^{(m,1)}|_{\mathcal{H}^{(m,1)}_{(0,0)}(\mathbf{T})}$ is subnormal, then its Berger measure is

$$\beta^2_{00}\left\|\frac{1}{t}\right\|_{L^1(\eta)} d(\eta_{\text{ext}})\left(s^{1/m}, t\right) + d\psi\left(s^{1/m}\right)d\delta_0(t). \quad (39)$$

Proof. We first claim that

$$d\psi(s) \geq 0 \Leftrightarrow d\psi\left(s^{1/m}\right) \geq 0 \text{ for any } m \in \mathbb{N}. \quad (40)$$

To see (40), we note that the positivity of ψ depends on the densities of $(\eta)^X_{\text{ext}}$ and ξ_{α_0}. There are three cases to consider.

(i) If $(\eta)^X_{\text{ext}}$ and ξ_{α_0} are finite atomic measures, then it is clear

(ii) If $(\eta)^X_{\text{ext}}$ and ξ_{α_0} are continuous measures, then by a change of variables, letting $u = s^{1/m}$, which goes in both directions because $s^{1/m}$ is an invertible function on the positive axis. That is, one can change the name of the variable from $s^{1/m}$ to u and then relabel u as s

(iii) If $(\eta)^X_{\text{ext}}$ and ξ_{α_0} are any probability measures, then by the above arguments, we have the claim (40).

Now, suppose that $\mathbf{T}^{(m,1)}|_{\mathcal{H}^{(m,1)}_{(0,0)}(\mathbf{T})}$ is subnormal with the Berger measure ϖ. Since $\mathbf{T}^{(m,1)}|_{\mathcal{H}^{(m,1)}_{(0,0)}(\mathbf{T})}$ and $(\mathbf{T}^{(m,1)}{}_{\mathcal{H}^{(m,1)}_{(0,0)}(\mathbf{T})})|_{\mathcal{M}_1(\mathcal{H}^{(m,1)}_{(0,0)}(\mathbf{T}))}$ are subnormal, we thus reconstruct the subnormality of $\mathbf{T}^{(m,1)}|_{\mathcal{H}^{(m,1)}_{(0,0)}(\mathbf{T})}$ as a backward extension of

$$\left(\mathbf{T}^{(m,1)}{}_{\mathcal{H}^{(m,1)}_{(0,0)}(\mathbf{T})}\right)\Big|_{\mathcal{M}_1\left(\mathcal{H}^{(m,1)}_{(0,0)}(\mathbf{T})\right)} (\text{in the } t \text{ direction}), \quad (41)$$

by applying Lemma 5. We let ς be the Berger measure of

$$\left(\mathbf{T}^{(m,1)}{}_{\mathcal{H}^{(m,1)}_{(0,0)}(\mathbf{T})}\right)\Big|_{\mathcal{M}_1\left(\mathcal{H}^{(m,1)}_{(0,0)}(\mathbf{T})\right)}. \quad (42)$$

Since η is the Berger measure of $\mathbf{T}|_{\mathcal{M}_1(\mathcal{H})}$, by Lemma 4, we have

$$\begin{aligned} l\gamma_{(k_1,k_2)}(\varsigma) &= \frac{1}{\beta^2_{00}} \cdot \gamma_{(k_1,k_2+1)}(\varpi) \Rightarrow \int s^{k_1}t^{k_2}d\varsigma(s,t) \\ &= \frac{1}{\beta^2_{00}} \cdot \int s^{k_1}t^{k_2+1}d\varpi(s,t) \Rightarrow \int s^{k_1}t^{k_2}d\eta^{(m,1)}_{(0,0)}(s,t) \\ &= \int s^{k_1}t^{k_2}\left(\frac{t}{\beta^2_{00}}d\varpi(s,t)\right)(\text{for all}(k_1,k_2)). \end{aligned} \quad (43)$$

Thus, we obtain

$$d\eta^{(m,1)}_{(0,0)}(s,t) = \frac{t}{\beta^2_{00}}d\varpi(s,t). \quad (44)$$

Also, by Lemma 4 and (44), we have

$$d\eta\left(s^{1/m}, t\right) = \frac{t}{\beta^2_{00}}d\varpi(s,t). \quad (45)$$

Since ς is the Berger measure of

$$\left(\mathbf{T}^{(m,1)}{}_{\mathcal{H}^{(m,1)}_{(0,0)}(\mathbf{T})}\right)\Big|_{\mathcal{M}_1\left(\mathcal{H}^{(m,1)}_{(0,0)}(\mathbf{T})\right)}, \quad (46)$$

and ϖ is the Berger measure of $\mathbf{T}^{(m,1)}|_{\mathcal{H}^{(m,1)}_{(0,0)}(\mathbf{T})}$, by Lemma 4, (44) and (45), we have

$$d\varsigma(s,t) = \frac{t}{\beta^2_{00}}d\varpi(s,t) = d\eta^{(m,1)}_{(0,0)}(s,t) = d\eta\left(s^{1/m}, t\right). \quad (47)$$

We let η_{β_0} be the Berger measure of the following subnormal shift:

$$\text{shift}\left(\beta_{(0,0)}, \beta_{(0,1)}, \beta_{(0,2)}, \cdots\right). \quad (48)$$

Since ς is the Berger measure of $(\mathbf{T}^{(m,1)}{}_{\mathscr{H}_{(0,0)}^{(m,1)}(\mathbf{T})})|_{\mathscr{M}_1(\mathscr{H}_{(0,0)}^{(m,1)}(\mathbf{T}))}$, by Lemma 6, we have

$$\left\|\frac{1}{t}\right\|_{L^1(\eta)} = \left\|\frac{1}{t}\right\|_{L^1((\eta_{\beta_0})_1)} = \left\|\frac{1}{t}\right\|_{L^1(\varsigma)}, \quad d(\varsigma_{\text{ext}}(s,t))^X = d(\eta_{\text{ext}}(s^{1/m},t))^X. \tag{49}$$

If shift$(\alpha_{(0,0)},\alpha_{(1,0)},\cdots)$ in the zero level of $\mathbf{T}$ is subnormal with the Berger measure ξ_{α_0}, then

$$\text{shift}\left(\alpha_{(0,0)}\cdots\alpha_{(m-1,0)},\alpha_{(m,0)}\cdots\alpha_{(2m-1,0)},\cdots\right), \tag{50}$$

is subnormal with the Berger measure $d(\xi_{\alpha_0})_{(m,0)}(s) = d\xi_{\alpha_0}(s^{1/m})$ by (18).

Since $\mathbf{T} = W_{(\alpha,\beta)} \in \mathfrak{H}_0$ and $\mathbf{T}^{(m,1)}|_{\mathscr{H}_{(0,0)}^{(m,1)}(\mathbf{T})}$ is subnormal, it follows from Lemma 5, (22), (44), (47), and (49) that

$$\begin{aligned} &\mathbf{T}^{(m,1)}\Big|_{\mathscr{H}_{(0,0)}^{(m,1)}(\mathbf{T})} \text{ is subnormal} \overset{\text{Lemma 3.3}}{\Rightarrow} \beta_{00}^2\left\|\frac{1}{t}\right\|_{L^1(\varsigma)} d(\varsigma_{\text{ext}}(s,t))^X \\ &\le d\xi_{\alpha_0}(s^{1/m}) \overset{(3.18)}{\Rightarrow} \beta_{00}^2\left\|\frac{1}{t}\right\|_{L^1(\eta)} d(\eta_{\text{ext}}(s^{1/m},t))^X \\ &\le d\xi_{\alpha_0}(s^{1/m}) \Rightarrow d\psi(s^{1/m}) \ge 0 \overset{(3.14)}{\Leftrightarrow} d\psi(u) \\ &\ge 0 \ (\text{by letting } u = s^{1/m}) \Leftrightarrow \psi \ge 0. \end{aligned} \tag{51}$$

This proves the first assertion. The second assertion is clear from Lemma 4 and analyzing the moments of $\mathbf{T}^{(m,1)}|_{\mathscr{H}_{(0,0)}^{(m,1)}(\mathbf{T})}$.

The following corollary is an immediate result of Lemma 8.

Corollary 9. *Assume $\mathbf{T} = W_{(\alpha,\beta)} \in \mathfrak{H}_0$ and $\mathbf{T}|_{\mathscr{M}_1(\mathscr{H})}$ is subnormal with Berger measure η. Let $1/t \in L^1(\eta)$. Then*

(a) *$\mathbf{T}$ is subnormal $\Leftrightarrow \psi \ge 0$*

(b) *For any $m,n \ge 1$, $\mathbf{T}^{(m,1)}$ is subnormal $\Leftrightarrow \mathbf{T}^{(n,1)}$ is subnormal. Hence, in particular*

$$\mathbf{T}^{(m,1)} \textit{ is subnormal for some } m \ge 1 \Leftrightarrow \mathbf{T} \textit{ is subnormal}. \tag{52}$$

Proof. (a) This follows from Lemma 5. (b) If $\mathbf{T}^{(m,1)}$ is subnormal for some $m \ge 1$, then $\mathbf{T}^{(m,1)}|_{\mathscr{H}_{(0,0)}^{(m,1)}(\mathbf{T})}$ is subnormal. By Lemma 8, we have $\psi \ge 0$ and hence $\mathbf{T}$ is subnormal by (a). Clearly, if $\mathbf{T}$ is subnormal then $\mathbf{T}^{(n,1)}$ is subnormal for all $n \ge 1$.

Remark 10. We remark that if $\mathbf{T} = W_{(\alpha,\beta)} \in \mathfrak{H}_0$ and $\mathbf{T}|_{\mathscr{M}_1(\mathscr{H})}$ is subnormal with Berger measure η, then $1/t \in L^1(\eta)$ because $V_0 := \text{shift}(\beta_{(0,0)},\beta_{(0,1)},\cdots)$ is subnormal.

We next have:

Corollary 11. *Let $\mathbf{T} = W_{(\alpha,\beta)} \in \mathfrak{H}_0$. If there exists $p \in \mathbb{Z}_+$ such that $\mathbf{T}|_{\mathscr{M}_p(\mathscr{H})}$ is subnormal, then $\mathbf{T}^{(m,1)}$ is subnormal for some $m \ge 1$ if and only if $\mathbf{T}$ is subnormal.*

Proof. It suffices to consider the case of $p = 2$. In the case, if $\mathbf{T}^{(m,1)}$ is subnormal for some $m \ge 1$, then $(\mathbf{T}|_{\mathscr{M}_1(\mathscr{H})})^{(m,1)}$ is subnormal. Thus, by Corollary 9, $\mathbf{T}|_{\mathscr{M}_1(\mathscr{H})}$ is subnormal, and therefore, $\mathbf{T}$ is subnormal. The converse is clear.

We now have:

Lemma 12. *Let $\mathbf{T} = W_{(\alpha,\beta)} \in \mathfrak{H}_0$ and let $\mathbf{T}|_{\mathscr{M}_1(\mathscr{H})}$ be subnormal. Then,*

$$l\mathbf{T}^{(1,n)} \textit{ is subnormal for some } n \ge 1 \Leftrightarrow \mathbf{T} \textit{ is subnormal}. \tag{53}$$

Proof. ($\Rightarrow$) Let η be the Berger measure of $\mathbf{T}|_{\mathscr{M}_1(\mathscr{H})}$ and suppose that $\mathbf{T}^{(1,n)}$ is subnormal for fixed $n \ge 1$. Then, $(\mathbf{T}^{(1,n)}{}_{\mathscr{H}_{(0,0)}^{(1,n)}(\mathbf{T})})|_{\mathscr{M}_1(\mathscr{H}_{(0,0)}^{(1,n)}(\mathbf{T}))}$ is subnormal. Let τ be its Berger measure. By Lemma 3, we have

$$\left(\mathbf{T}|_{\mathscr{M}_1(\mathscr{H})}\right)^{(1,n)}\Big|_{\mathscr{H}_{(0,n-1)}^{(1,n)}(\mathbf{T}_{\mathscr{M}_1(\mathscr{H})})} \cong \left(\mathbf{T}^{(1,n)}{}_{\mathscr{H}_{(0,0)}^{(1,n)}(\mathbf{T})}\right)\Big|_{\mathscr{M}_1(\mathscr{H}_{(0,0)}^{(1,n)}(\mathbf{T}))}. \tag{54}$$

Hence, by Lemma 4 and (54), we have

$$d\tau(s,t) = \frac{t^{n-1/n}}{\gamma_{(0,n-1)}(\eta)}\,d\eta(s,t^{1/n}). \tag{55}$$

Observe

$$d\tau_{\text{ext}}(s,t) = \left\|\frac{1}{t}\right\|_{L^1(\tau)}^{-1} \frac{t^{-1/n}}{\gamma_{(0,n-1)}(\eta)}\,d\eta(s,t^{1/n}), \tag{56}$$

$$d(\tau_{\text{ext}}(s,t))^X = \frac{\|1/t\|_{L^1(\tau)}^{-1}}{\gamma_{(0,n-1)}(\eta)}\left(\int_0^{\|T_2\|^{2n}} t^{-1/n}\,d\eta(s,t^{1/n})\right). \tag{57}$$

We now characterize the subnormality of $\mathbf{T}^{(1,n)}|_{\mathscr{H}_{(0,0)}^{(1,n)}(\mathbf{T})}$ as a backward extension of

$$\left(\mathbf{T}^{(1,n)}{}_{\mathscr{H}_{(0,0)}^{(1,n)}(\mathbf{T})}\right)\Big|_{\mathscr{M}_1(\mathscr{H}_{(0,0)}^{(1,n)}(\mathbf{T}))} \text{ (in the } t \text{ direction)}, \tag{58}$$

after applying Lemma 5. To do so, let shift$(\alpha_{(0,0)}, \alpha_{(1,0)}, \cdots)$ in the zero level of $\mathbf{T}$ be subnormal with the Berger measure ξ_{α_0}. Then, by (57), we have that

$$\begin{aligned} &\mathbf{T}^{(1,n)}\Big|_{\mathscr{H}^{(1,n)}_{(0,0)}(\mathbf{T})} \text{ is subnormal} \overset{\text{Lemma 3.3}}{\Rightarrow} \gamma_{(0,n)}(\mathbf{T}) \left\|\frac{1}{t}\right\|_{L^1(\tau)} d(\tau_{\text{ext}}(s,t))^X \\ &\leq d\xi_{\alpha_0}(s) \overset{(3.22)}{\Rightarrow} \beta_{00}^2 \left(\int_0^{\|T_2\|^{2n}} t^{-1/n} d\eta(s, t^{1/n}) \right) \\ &\leq d\xi_{\alpha_0}(s) \Rightarrow \beta_{00}^2 \left(\int_0^{\|T_2\|^2} u^{-1} d\eta(s,u) \right) \\ &\leq d\xi_{\alpha_0}(s) (\text{by letting } u = t^{1/n}) \Rightarrow \beta_{00}^2 \left(\int_0^{\|T_2\|^2} t^{-1} d\eta(s,t) \right) \\ &\leq d\xi_{\alpha_0}(s) \Rightarrow \beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\eta)} (\eta)_{\text{ext}}^X \leq \xi_{\alpha_0}. \end{aligned} \tag{59}$$

By again Lemma 5 and (59), therefore, $\mathbf{T}$ is subnormal.

$$(\Leftarrow)\text{Clear.} \tag{60}$$

By Corollary 9 and Lemma 12, we have:

Corollary 13. *Let* $\mathbf{T} = W_{(\alpha,\beta)} \in \mathfrak{H}_0$. *If* $\mathbf{T}|_{\mathscr{M}_1(\mathscr{H})}$ *or* $\mathbf{T}|_{\mathscr{N}_1(\mathscr{H})}$ *is subnormal, then the following are equivalent:*

(a) $\mathbf{T}^{(m,1)}$ *is subnormal for some* $m \geq 1$

(b) $\mathbf{T}^{(m,1)}$ *is subnormal for all* $m \geq 1$

(c) $\mathbf{T}^{(1,n)}$ *is subnormal for some* $n \geq 1$

(d) $\mathbf{T}^{(1,n)}$ *is subnormal for all* $n \geq 1$

(e) ***T*** *is subnormal*

The following theorem is a core of our main result.

Theorem 14. *If* $\mathbf{T}|_{\mathscr{M}_1(\mathscr{H})}$ *is subnormal, then for some* $m, n \geq 1$

$$\left(\mathbf{T}^{(1,n)}\right)^{(m,1)} \text{is subnormal} \Leftrightarrow \mathbf{T}^{(m,1)} \text{is subnormal}. \tag{61}$$

Proof. ($\Rightarrow$). For fixed $m, n \geq 1$, suppose that $(\mathbf{T}^{(1,n)})^{(m,1)} = \mathbf{T}^{(m,n)}$ is subnormal. Then, $W := (\mathbf{T}^{(m,n)}{}_{\mathscr{H}^{(m,n)}_{(p,0)}(\mathbf{T})})|_{\mathscr{M}_1(\mathscr{H}^{(m,n)}_{(p,0)}(\mathbf{T}))}$ is also subnormal. Let η be the Berger measure of $\mathbf{T}|_{\mathscr{M}_1(\mathscr{H})}$ and let $\tau^{(m,n)}_{(p,0)}$ be the Berger measure of W. For each $0 \leq p \leq m-1$, using Lemma 5, we characterize the subnormality of $\mathbf{T}^{(m,n)}|_{\mathscr{H}^{(m,n)}_{(p,0)}(\mathbf{T})}$ as a backward extension of W (in the t direction).

First observe that from (26) in Lemma 3, for $0 \leq p \leq m-1$,

$$\left(\mathbf{T}|_{\mathscr{M}_1(\mathscr{H})}\right)^{(m,n)}\Big|_{\mathscr{H}^{(m,n)}_{(p,n-1)}(\mathbf{T}_{\mathscr{M}_1(\mathscr{H})})} \cong \left(\mathbf{T}^{(m,n)}{}_{\mathscr{H}^{(m,n)}_{(p,0)}(\mathbf{T})}\right)\Big|_{\mathscr{M}_1\left(\mathscr{H}^{(m,n)}_{(p,0)}(\mathbf{T})\right)}. \tag{62}$$

By Lemma 4 and (62), we have

$$d\tau^{(m,n)}_{(p,0)}(s,t) = \frac{s^{p/m} t^{n-1/n}}{\gamma_{(p,n-1)}(\eta)} d\eta(s^{1/m}, t^{1/n}). \tag{63}$$

Observe

$$d\left(\tau^{(m,n)}_{(p,0)}\right)_{\text{ext}}(s,t) = \left\|\frac{1}{t}\right\|^{-1}_{L^1\left(\tau^{(m,n)}_{(p,0)}\right)} \frac{s^{p/m} t^{-1/n}}{\gamma_{(p,n-1)}(\eta)} d\eta(s^{1/m}, t^{1/n}) \tag{64}$$

$$\begin{aligned} &d\left(\left(\tau^{(m,n)}_{(p,0)}\right)_{\text{ext}}(s,t)\right)^X \\ &= \frac{\|1/t\|^{-1}_{L^1\left(\tau^{(m,n)}_{(p,0)}\right)}}{\gamma_{(p,n-1)}(\eta)} \left(\int_0^{\|T_2\|^{2n}} s^{p/m} t^{-1/n} d\eta(s^{1/m}, t^{1/n}) \right). \end{aligned} \tag{65}$$

Let $W_{k_2} = \text{shift}(\alpha_{(0,k_2)}, \alpha_{(1,k_2)}, \cdots)$ be the k_2-th horizontal slice of $\mathbf{T}$ with Berger measure $\xi_{\alpha_{k_2}}$ for $k_2 \geq 0$. By (65) and a similar way to (59), for $0 \leq p \leq m-1$, we have that

$$\begin{aligned} &\mathbf{T}^{(m,n)}\Big|_{\mathscr{H}^{(m,n)}_{(p,0)}(\mathbf{T})} \text{is subnormal} \underset{(3.2)}{\overset{\text{Lemma 3.3}}{\Rightarrow}} \beta^2_{(p,0)} \cdots \beta^2_{(p,n-1)} \\ &\cdot \left\|\frac{1}{t}\right\|_{L^1\left(\tau^{(m,n)}_{(p,0)}\right)} d\left(\tau^{(m,n)}_{(p,0)}(s,t)\right)^X_{\text{ext}} \\ &\leq \frac{s^{p/m}}{\gamma_p(W_0)} d\xi_{\alpha_0}(s^{1/m}) \overset{\text{checking moments}}{\Rightarrow} \frac{\beta^2_{(p,0)} \cdot \gamma_{(p,n-1)}(\eta)}{\gamma_p(W_1)} \\ &\cdot \left\|\frac{1}{t}\right\|_{L^1\left(\tau^{(m,n)}_{(p,0)}\right)} d\left(\tau^{(m,n)}_{(p,0)}(s,t)\right)^X_{\text{ext}} \\ &\leq \frac{s^{p/m}}{\gamma_p(W_0)} d\xi_{\alpha_0}(s^{1/m}) \overset{(3.26)}{\Rightarrow} \frac{\beta^2_{(p,0)}}{\gamma_p(W_1)} \left\|\frac{1}{t}\right\|_{L^1\left(\tau^{(m,n)}_{(p,0)}\right)} \\ &\cdot \left\|\frac{1}{t}\right\|^{-1}_{L^1\left(\tau^{(m,n)}_{(p,0)}\right)} \left(\int_0^{\|T_2\|^{2n}} s^{p/m} t^{-1/n} d\eta(s^{1/m}, t^{1/n}) \right) \\ &\leq \frac{s^{p/m}}{\gamma_p(W_0)} d\xi_{\alpha_0}(s^{1/m}) \Rightarrow \beta^2_{(p,0)} \\ &\cdot \left(\int_0^{\|T_2\|^{2n}} \frac{s^{p/m} t^{-1/n}}{\gamma_p(W_1)} d\eta(s^{1/m}, t^{1/n}) \right) \\ &\leq \frac{s^{p/m}}{\gamma_p(W_0)} d\xi_{\alpha_0}(s^{1/m}) \Rightarrow \beta^2_{(p,0)} \\ &\cdot \left(\int_0^{\|T_2\|^2} u^{-1} \frac{s^{p/m}}{\gamma_p(W_1)} d\eta(s^{1/m}, u) \right) \end{aligned}$$

$$\le \frac{s^{p/m}}{\gamma_p(W_0)} d\xi_{\alpha_0}\left(s^{1/m}\right) \left(\text{by letting } u = t^{1/n}\right) \Rightarrow \beta^2_{(p,0)} \cdot \left(\int_0^{\|T_2\|^2} t^{-1} \frac{s^{p/m}}{\gamma_{(p,0)}(\eta)} d\eta\left(s^{1/m}, t\right)\right) \le \frac{s^{p/m}}{\gamma_p(W_0)} d\xi_{\alpha_0}\left(s^{1/m}\right) \overset{(3.26)}{\Rightarrow} \beta^2_{(p,0)} \left\|\frac{1}{t}\right\|_{L^1\left(\eta^{(m,1)}_{(p,0)}\right)} \cdot \left(\eta^{(m,1)}_{(p,0)}\right)^X_{\text{ext}} \le \left(\xi_{\alpha_0}\right)_{(m,p)}, \tag{66}$$

where $\left(\xi_{\alpha_0}\right)_{(m,p)}$ is the Berger measure of the subnormal shift:

$$\text{shift}\left(\alpha_{(p,0)} \cdots \alpha_{(p+m-1,0)}, \alpha_{(p+m,0)} \cdots \alpha_{(p+2m-1,0)}, \cdots\right). \tag{67}$$

By Lemma 5, (66), and a similar way to the proof of Lemma 12, for $0 \le p \le m-1$, we have

$$\left(\mathbf{T}^{(1,n)}\right)^{(m,1)} \text{is subnormal} \Rightarrow \mathbf{T}^{(m,n)}\Big|_{\mathscr{H}^{(m,n)}_{(p,0)}(\mathbf{T})} \text{ is subnormal} \Rightarrow \beta^2_{(p,0)} \left\|\frac{1}{t}\right\|_{L^1\left(\eta^{(m,1)}_{(p,0)}\right)} \left(\eta^{(m,1)}_{(p,0)}\right)^X_{\text{ext}} \le \left(\xi_{\alpha_0}\right)_{(m,p)} \Rightarrow \mathbf{T}^{(m,1)} \text{is subnormal}. \tag{68}$$

Therefore, by (68), we get the result.

($\Leftarrow$) Since $\left(\mathbf{T}^{(1,n)}\right)^{(m,1)} = \mathbf{T}^{(m,n)} = \left(\mathbf{T}^{(m,1)}\right)^{(1,n)}$, it follows that $\left(\mathbf{T}^{(1,n)}\right)^{(m,1)}$ is subnormal whenever $\mathbf{T}^{(m,1)}$ is subnormal.

Corollary 15. *Let* $\mathbf{T} = W_{(\alpha,\beta)} \in \mathfrak{H}_0$. *If* $\mathbf{T}|_{\mathscr{N}_1(\mathscr{H})}$ *is subnormal, then the following are equivalent:*

(a) $\left(\mathbf{T}^{(m,1)}\right)^{(1,n)}$ *is subnormal for some* $m, n \ge 1$

(b) $\mathbf{T}^{(1,n)}$ *is subnormal*

(c) $\mathbf{T}$ *is subnormal*

We are ready to give a proof of our main theorem (Theorem 2). For convenience, we restate Theorem 2:

Theorem 16. *Let* $\mathbf{T} \equiv W_{(\alpha,\beta)} \in \mathfrak{H}_0$. *If a corner of* $\mathbf{T}$ *is subnormal, i.e.,* $\mathbf{T}|_{\mathscr{M}_p(\mathscr{H})\cap\mathscr{N}_q(\mathscr{H})}$ *is subnormal for some* $p, q \in \mathbb{Z}_+$, *then the following are equivalent:*

(a) $\mathbf{T}$ *is subnormal*

(b) $\mathbf{T}^{(m,n)}$ *is subnormal for all* $m, n \ge 1$

(c) $\mathbf{T}^{(m,n)}$ *is subnormal for some* $m, n \ge 1$

Proof. (a) $\Rightarrow$ (b): This is clear from the functional calculus. (b) $\Rightarrow$ (c): Clear. (c) $\Rightarrow$ (a): Suppose $\mathbf{T}|_{\mathscr{M}_p(\mathscr{H})\cap\mathscr{N}_q(\mathscr{H})}$ is subnormal for some $p, q \in \mathbb{Z}_+$ and $\mathbf{T}^{(m,n)}$ is subnormal for some $m, n \ge 1$. By Lemma 7, we have $\mathbf{T}|_{\mathscr{M}_1(\mathscr{H})\cap\mathscr{N}_1(\mathscr{H})}$ is subnormal. Also, $\left(\mathbf{T}|_{\mathscr{N}_1(\mathscr{H})}\right)^{(m,n)}$ is subnormal. We thus have

$$\left(\mathbf{T}_{\mathscr{N}_1(\mathscr{H})}\right)^{(m,n)} = \left(\left(\mathbf{T}_{\mathscr{N}_1(\mathscr{H})}\right)^{(1,n)}\right)^{(m,1)} \text{is subnormal} \overset{\text{Theorem3.12}}{\Rightarrow} \left(\mathbf{T}_{\mathscr{N}_1(\mathscr{H})}\right)^{(m,1)} \text{is subnormal} \overset{\text{Corollary3.123.11}}{\Rightarrow} \mathbf{T}\Big|_{\mathscr{N}_1(\mathscr{H})} \text{ is subnormal}. \tag{69}$$

Therefore, by Corollary 15, we can see that $\mathbf{T}$ is subnormal.

We conclude by revealing examples to illustrate Theorem 2. Their proofs are given from a straightforward calculation. We will omit their proofs.

Example 1. Let $\mathbf{T} = (T_1, T_2) \equiv W_{(\alpha,\beta)} \in \mathfrak{H}_0$ and let $\mathbf{T}|_{\mathscr{M}_1(\mathscr{H})}$ be subnormal with Berger measure

$$d\eta(s,t) = t\,ds\,dt + \frac{1}{6} d\delta_{(0,1/3)}(s,t) + \frac{1}{6} d\delta_{(1/2,1)}(s,t) + \frac{1}{6} d\delta_{(1,1/2)}(s,t). \tag{70}$$

Also, let

$$d\xi_{\alpha_0}(s) := \frac{1}{2} ds + \frac{1}{6} d\delta_0(s) + \frac{1}{6} d\delta_{1/2}(s) + \frac{1}{6} d\delta_1(s), \tag{71}$$

be the Berger measure of $W_0 \equiv \text{shift}(\alpha_{(0,0)}, \alpha_{(1,0)}, \cdots)$. We then have:

(a) $\mathbf{T}$ is subnormal

(b) $\mathbf{T}^{(2,1)}$ is subnormal

(c) $\mathbf{T}^{(1,2)}$ is subnormal

$$0 < \beta_{(0,0)} \le \sqrt{\frac{1}{3}}. \tag{72}$$

Example 2. Let $\mathbf{T} = (T_1, T_2) \equiv W_{(\alpha,\beta)} \in \mathfrak{H}_0$ and let $\mathbf{T}|_{\mathscr{M}_1(\mathscr{H})\cap\mathscr{N}_1(\mathscr{H})}$ be subnormal with Berger measure

$$d\mu_{\text{core}}(s,t) = 2st\,ds\,dt + \frac{1}{4} d\delta_{(1,1/2)}(s,t) + \frac{1}{4} d\delta_{(1/2,1)}(s,t). \tag{73}$$

Also, let

$$d\xi_{\alpha_0}(s) := \frac{1}{2} ds + \frac{1}{6} d\delta_0(s) + \frac{1}{6} d\delta_{1/2}(s) + \frac{1}{6} d\delta_1(s), \tag{74}$$

be the Berger measures on $[0, 1]$ of $W_0 \equiv \text{shift}(\alpha_{(0,0)}, \alpha_{(1,0)}, \cdots)$ and

$$d\eta_{\beta_0}(t) := y^2 dt + \left(1 - \frac{11y^2}{6}\right) d\delta_0(t) + \frac{2y^2}{3} d\delta_{1/2}(t) + \frac{y^2}{6} d\delta_1(t), \tag{75}$$

be the Berger measures on $[0, 1]$ of $V_0 \equiv \text{shift}(\beta_{(0,0)}, \beta_{(0,1)}, \cdots)$. We then have:

(a) $\mathbf{T}$ is subnormal

(b) $\mathbf{T}^{(2,2)}$ is subnormal

$$\left(\alpha_{(0,1)}, \beta_{(0,0)}\right) \in \left(0, \sqrt{\frac{1}{3}}\right] \times \left(0, \sqrt{\frac{1}{11}}\right]. \tag{76}$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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References

[1] J. Stampfli, "Which weighted shifts are subnormal?," *Pacific Journal of Mathematics*, vol. 17, no. 2, pp. 367–379, 1966.

[2] P. R. Halmos, *A Hilbert Space Problem Book*, vol. 19 of Graduate Texts in Mathematics, Springer New York, New York, NY, 1982.

[3] E. Franks, "Polynomially subnormal operator tuples," *Journal of Operator Theory*, vol. 31, pp. 219–228, 1994.

[4] R. Curto, S. H. Lee, and J. Yoon, "Hyponormality and subnormality for powers of commuting pairs of subnormal operators," *Journal of Functional Analysis*, vol. 245, no. 2, pp. 390–412, 2007.

[5] R. Curto, S. H. Lee, and J. Yoon, "Subnormality for arbitrary powers of 2-variable weighted shifts whose restrictions to a large invariant subspace are tensor products," *Journal of Functional Analysis*, vol. 262, no. 2, pp. 569–583, 2012.

[6] R. Curto, S. H. Lee, and J. Yoon, "Subnormality of 2-variable weighted shifts with diagonal coreSous-normalité de shifts pondérés à deux variables avec cœur diagonal," *Comptes Rendus Mathematique*, vol. 351, no. 5-6, pp. 203–207, 2013.

[7] R. Curto, S. H. Lee, and J. Yoon, "Quasinormality of powers of commuting pairs of bounded operators," *Journal of Functional Analysis*, vol. 278, no. 3, article 108342, 2020.

[8] R. Curto and J. Yoon, "When is hyponormality for 2-variable weighted shifts invariant under powers ?," *Indiana University Mathematics Journal*, vol. 60, no. 3, pp. 997–1032, 2011.

[9] A. Lambert, "Subnormality and weighted shifts," *Journal of the London Mathematical Society*, vol. s2-14, no. 3, pp. 476–480, 1976.

[10] A. R. Lubin, "Weighted shifts and commuting normal extension," *Journal of the Australian Mathematical Society*, vol. 27, no. 1, pp. 17–26, 1979.

[11] A. Athavale, "On joint hyponormality of operators," *Proceedings of the American Mathematical Society*, vol. 103, no. 2, pp. 417–423, 1988.

[12] R. Curto, P. Muhly, and J. Xia, "Hyponormal pairs of commuting operators," *Contributions to Operator Theory and its Applications*, vol. 35, pp. 1–22, 1988.

[13] R. Curto, S. H. Lee, and J. Yoon, "*k*-hyponormality of multivariable weighted shifts," *Journal of Functional Analysis*, vol. 229, no. 2, pp. 462–480, 2005.

[14] R. Curto and J. Yoon, "Jointly hyponormal pairs of commuting subnormal operators need not be jointly subnormal," *Transactions of the American Mathematical Society*, vol. 358, no. 11, pp. 5139–5159, 2006.

[15] N. P. Jewell and A. R. Lubin, "Commuting weighted shifts and analytic function theory in several variables," *Journal of Operator Theory*, vol. 1, pp. 207–223, 1979.

[16] R. Gellar and L. J. Wallen, "Subnormal weighted shifts and the Halmos-Bram criterion," *Proceedings. Japan Academy*, vol. 46, no. 4, pp. 375–378, 1970.

[17] R. Curto and S. Park, "*k*-hyponormality of powers of weighted shifts," *Proceedings of the American Mathematical Society*, vol. 131, no. 9, pp. 2761–2769, 2003.

[18] S. H. Lee, W. Y. Lee, and J. Yoon, "An answer to a question of A. Lubin: the lifting problem for commuting subnormals," *Israel Journal of Mathematics*, vol. 222, no. 1, pp. 201–222, 2017.