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Research Article

Best Proximity Point Theorems for (G, D)-Proximal Geraghty Maps in JS-Metric Spaces

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We study (G, D)-proximal Geraghty contractions in a JS-metric space X endowed with graph G. We obtain some best proximity theorems for such contractions. An example and several consequences are given. As a consequence of our results, we also provide the best proximity point results in X endowed with a binary relation.

1. Introduction and Preliminaries

Let A and B be nonempty subsets of a metric space (X, d) and $T : A \longrightarrow B$ be a map. It is known that if T is a nonself map, the equation $Tx = x$ does not always have a solution, and it clearly has no solution when A and B are disjoint. However, it is possible to determine an approximate solution x^* such that the error is $d(x^*, Tx^*) = d(A, B)$. Such point x^* is called the best proximity point of T. In the case that T is a self-mapping, the best proximity point is simply a fixed point of T. The best proximity point theorem was first studied in [1]. Then, there has been a wide range of research in this framework. Many researchers have studied and generalized the result in many aspects, for example, see [2–22].

In 2011, Raj [23] introduced the notion of P-property and subsequently obtained a best proximity point result for a weakly contractive nonself map $T : A \longrightarrow B$. Best proximity point theorems for subsets of X having the P-property were also studied in great details in [24–26]. Zhang and Su [27] weakened the P-property, called the weak P-property, as well as improved the best proximity point theorem for Geraghty nonself contractions, see also [28].

Fixed-point theorems concerning a metric space endowed with graph G, which generalizes the Banach contraction principle, were proposed by Jachymski [29]. Klanarong and Suantai [30] recently presented the notion of a G-proximal generalized contraction. Several best proximity point results for these mappings were obtained. The concept of generalized metric spaces, also called JS-metric spaces, was introduced in [31]. It is a generalization of standard metric spaces covering many topological structures. Since then, many researchers have worked with these concepts and gave a large number of results, see [32–36], for example.

In this paper, we introduce a type of contractions called (G, D)-proximal, Geraghty mappings. These maps are defined on subsets A and B of a JS-metric space X which is endowed with graph G. Then, we establish a result on the existence and uniqueness of the best proximity point for these mappings. An example showing the validity of the main result is illustrated, and several corollaries are listed. Finally, by applying our main result, we obtain a best proximity point result in X endowed with a symmetric binary relation.

2. Preliminaries and Definitions

Let X be a nonempty set, and let $D : X \times X \longrightarrow [0,\infty]$ be a function. For each $x \in X$, define

$$C(D, X, x) = \left\{ \{x_n\} \subseteq X : \lim_{n\to\infty} D(x_n, x) = 0 \right\}. \qquad (1)$$

In 2015, Jleli and Samet [31] introduced a generalization of metric space as follows.

Definition 1 (see [31]). Let X be a nonempty set. A function $D : X \times X \longrightarrow [0,\infty]$ is said to be a generalized metric on X if the following conditions hold:

(D_1) For any $x, y \in X$, if $D(x, y) = 0$, then $x = y$

(D_2) For any $x, y \in X$, $D(x, y) = D(y, x)$

(D_3) There exists $C > 0$ such that for any $x, y \in X$

$$D(x, y) \le C \limsup_{n\to\infty} D(x_n, y), \tag{2}$$

where $\{x_n\} \in C(D, X, x)$.

In this case, we say that (X, D) is a generalized metric space, also known as a JS-metric space.

Later, Khemphet [37] modified the condition (D_3), which will be denoted by (D_3^*), as follows: "there exists $C > 0$ such that for any $x, y \in X$, $D(x, y) \le C \limsup_{n\to\infty} D(x_n, y_n)$, where $\{x_n\} \in C(D, X, x)$ and $\{y_n\} \in C(D, X, y)$."

Clearly, (D_3^*) is stronger than (D_3). For convenience, when (D_3) is replaced by (D_3^*), the JS-metric space (X, D) will be called a JS^*-metric space.

Now, let $X \coloneqq (X, D)$ be a JS-metric space if not otherwise specified. We are ready to discuss convergence and continuity in these spaces.

Definition 2 (see [31]). Let $\{x_n\}$ be a sequence in X. The sequence $\{x_n\}$ is said to D -converge to $x \in X$ if $\{x_n\} \in C(D, X, x)$. Moreover, $\{x_n\}$ is called a D-Cauchy sequence if $\lim_{m,n\to\infty} D(x_n, x_m) = 0$. Finally, (X, D) is said to be D-complete if each D -Cauchy sequence in X is a D -convergent sequence in X.

Any convergent sequence in a JS-metric space converges to a unique point.

Proposition 3 (see [31]). *Let $\{x_n\}$ be a sequence in X. For any $x, y \in X$, if $\{x_n\} \in C(D, X, x) \cap C(D, X, y)$, then $x = y$.*

Definition 4 (see [37]). A function $f : X \longrightarrow X$ is said to be D -continuous at a point $x_0 \in X$ if for any $\{x_n\} \in C(D, X, x_0)$, $\{f x_n\} \in C(D, X, f x_0)$. In addition, f is said to be D -continuous on X if it is D-continuous at each x in X.

Definition 5. A JS-metric space (X, D) is said to be endowed with a graph $G = (V(G), E(G))$; if the set of vertices (denoted by $V(G)$) is X, the set of edges (denoted by $E(G)$) contains the diagonal of $X \times X$ but parallel edges.

We say that G is transitive if for all $x, y, z \in X$, (x, z), and $(z, y) \in E(G) \Rightarrow (x, y) \in E(G)$.

Definition 6. (see [38]). Let (X, D) be endowed with a graph G. A function $f : (X, D) \longrightarrow (X, D)$ is said to be G -continuous at $x \in X$ if for each $\{x_n\} \in C(D, X, x)$ with $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, $\{f x_n\} \in C(D, X, f x)$.

Let A and B be nonempty subsets of X. We require the following notations:

$$\begin{aligned} D(A, B) &\coloneqq \inf \{D(a, b) \colon a \in A, b \in B\} ; \\ A_0 &\coloneqq \{a \in A : \text{there exists } b \in B \text{ such that } D(a, b) = D(A, B)\} ; \\ B_0 &\coloneqq \{b \in B : \text{there exists } a \in A \text{ such that } D(a, b) = D(A, B)\}. \end{aligned} \tag{3}$$

Definition 7 (see [39]). Let $T : A \longrightarrow B$ be a mapping. An element $x^* \in A$ is said to be a best proximity point of T if $D(x^*, Tx^*) = D(A, B)$. We denote the set of all best proximity points of T by $B(T)$.

Definition 8 (see [27]). Let A_0 be nonempty. Then, the pair (A, B) is said to have the weak P-property if

$$D(x_1, y_1) = D(x_2, y_2) = D(A, B) \Rightarrow D(x_1, x_2) \le D(y_1, y_2), \tag{4}$$

where $x_1, x_2 \in A$ and $y_1, y_2 \in B$.

Definition 9. Let $x \in X$. A mapping $T : A \longrightarrow B$ is said to be (G, D)-proximal if $(x_1, x_2) \in E(G)$ and $D(u_1, Tx_1) = D(u_2, Tx_2) = d(A, B) \Rightarrow (u_1, u_2) \in E(G)$ and $D(u_1, u_2) < \infty$ for all $x_1, x_2, u_1, u_2 \in A$.

3. Main Results

In this section, we assume that X is a JS-metric space (or JS^*-metric space when specified) endowed with a transitive graph $G = (V(G), E(G))$. Let A and B be nonempty subsets of X for which A_0 is nonempty.

The class of functions

$$\mathscr{B} \coloneqq \{\beta : [0,\infty) \longrightarrow [0, 1]\colon \beta(t_n) \longrightarrow 1 \text{ implies } t_n \longrightarrow 0\} \tag{5}$$

was used as an important tool in [24]. It is clearly a generalization of the well-known class of $[0, 1)$-valued functions introduced by Geraghty [40].

We now introduce a new type of Geraghty contractions.

Definition 10. A mapping $T : A \longrightarrow B$ is said to be a (G, D)-proximal Geraghty mapping if the following hold:

(i) T is (G, D)-proximal

(ii) For all $x, y \in A$ such that $(x, y) \in E(G)$, there exists $\beta \in \mathscr{B}$ such that

$$D(Tx, Ty) \le \beta(\mathrm{D}(x, y))D(x, y) \tag{6}$$

Lemma 11. *Let $T : A \longrightarrow B$ be a (G, D)-proximal Geraghty mapping and the pair (A, B) have the weak P-property. Then, for any $x, y \in B(T)$,* *(1) $D(x, x) = 0$*

(2) If $(x, y) \in E(G)$, then $x = y$

Proof.

(1) Let $x \in B(T)$. Then, $d(x, Tx) = d(A, B)$. Since $(x, x) \in E(G)$, we have

$$D(x, x) \leq D(Tx, Tx) \leq \beta(D(x, x))D(x, x) \leq D(x, x) < \infty \quad (7)$$

We can easily check that $D(x, x) = 0$.

(2) Let $x, y \in B(T)$ such that $(x, y) \in E(G)$. From (6), $D(x, x) = D(y, y) = 0$. By assumptions, $d(x, Tx) = d(y, Ty) = d(A, B)$ and

$$D(x, y) = D(Tx, Ty) \leq \beta(D(x, y))D(x, y) \leq D(x, y) < \infty \quad (8)$$

Similarly, $D(x, y) = 0$ and so $x = y$.

Theorem 12. *Let X be a JS^*-metric space and A_0 be D-complete. Let $T : A \longrightarrow B$ be a (G, D)-proximal Geraghty map. Suppose that the following conditions hold:*

(i) $T(A_0) \subseteq B_0$ and (A, B) has the weak P-property

(ii) There exist $x, y \in A_0$ such that $d(x, Ty) = d(A, B)$, $(y, x) \in E(G)$, and $D(y, x) < \infty$

(iii) T is G-continuous and for $C > 0$ such that $D(x, Tx) \leq C\limsup_{n\to\infty} D(x_n, Tx_{n-1})$, there exists $\lambda \geq 1$ such that $C\lambda \leq 1$

Then, there exists $x^* \in B(T)$. Moreover, T has a unique best proximity point if $(x^*, y^*) \in E(G)$ for all $x^*, y^* \in B(T)$.

Proof. From (ii), there exist $x_0, x_1 \in A_0$ such that

$$D(x_1, Tx_0) = D(A, B), (x_0, x_1) \in E(G), \quad D(x_0, x_1) < \infty. \quad (9)$$

Since $x_1 \in A_0, Tx_1 \in T(A_0) \subseteq B_0$. Then, there exists $x_2 \in A$ such that $d(x_2, Tx_1) = d(A, B)$ and so $x_2 \in A_0$.

Since T is (G, D)-proximal, we finally have that

$$D(x_2, Tx_1) = D(A, B), (x_1, x_2) \in E(G), \quad D(x_1, x_2) < \infty. \quad (10)$$

Continuing this process, we obtain a sequence $\{x_n\} \subseteq A_0$ such that

$$D(x_n, Tx_{n-1}) = D(A, B), (x_{n-1}, x_n) \in E(G), \quad D(x_{n-1}, x_n) < \infty \text{ for all } n \geq 1. \quad (11)$$

By using the weak P-property with (11), we have that

$$D(x_n, x_{n+1}) \leq D(Tx_{n-1}, Tx_n) \text{ for all } n \geq 1. \quad (12)$$

If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0-1}$, then

$$D(x_{n_0}, Tx_{n_0-1}) = D(x_{n_0}, Tx_{n_0}) = D(A, B). \quad (13)$$

Now, suppose that $x_n \neq x_{n-1}$ for all $n \geq 1$. We shall prove that $\{x_n\}$ is a Cauchy sequence. We first show that $\lim_{n\to\infty} D(x_{n-1}, x_n) = 0$.

Since $(x_{n-1}, x_n) \in E(G)$ and T is a (G, D)-proximal, Geraghty mapping, then there exists $\beta \in \mathscr{B}$ such that

$$D(x_n, x_{n+1}) \leq D(Tx_{n-1}, Tx_n) \leq \beta(D(x_{n-1}, x_n))D(x_{n-1}, x_n) \leq D(x_{n-1}, x_n) \quad \text{for all } n \geq 1. \quad (14)$$

Clearly, $D(x_{n-1}, x_n)$ is nonincreasing. Thus, $\lim_{n\to\infty} D(x_n, x_{n-1}) = r \geq 0$. Suppose that $r > 0$ and let $n \longrightarrow \infty$ in (14). Then,

$$1 \leq \lim_{n\to\infty} \beta(D(x_{n-1}, x_n)) \leq 1. \quad (15)$$

It follows that $\lim_{n\to\infty} \beta(D(x_{n-1}, x_n)) = 1$. By the definition of β, $\lim_{n\to\infty} D(x_n, x_{n-1}) = r = 0$ which is a contradiction. Thus, $\lim_{n\to\infty} D(x_{n-1}, x_n)$ must be 0.

Now, suppose that $\{x_n\}$ is not a Cauchy sequence. Then, there exists $\varepsilon > 0$ such that for all $k \in \mathbb{N}$, there are subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that $D(x_{n_k}, x_{m_k}) \geq \varepsilon$ for $k \in \mathbb{N}$.

Since G is transitive, $(x_{n_k}, x_{m_k}) \in E(G)$ for all $k \in \mathbb{N}$. Since T is a (G, D)-proximal Geraghty mapping and (12),

$$D(x_{n_k}, x_{m_k}) \leq D(Tx_{n_k-1}, Tx_{m_k-1}) \leq \beta(D(x_{n_k-1}, x_{m_k-1}))D(x_{n_k-1}, x_{m_k-1}). \quad (16)$$

Consequently,

$$D(x_{n_k}, x_{m_k}) \leq \prod_{i=1}^{n_k} \beta(D(x_{n_k-i}, x_{m_k-i}))D(x_0, x_{m_k-n_k}). \quad (17)$$

For $i_k \in \{1, 2, \cdots, n_k\}$, we have that

$$\beta(D(x_{n_k-i_k}, x_{m_k-i_k})) = \max \{\beta(D(x_{n_k-i}, x_{m_k-i})) : 1 \leq i \leq n_k\}. \quad (18)$$

Set $\eta = \limsup_{k\to\infty}\{\beta(D(x_{n_k-i_k}, x_{m_k-i_k}))\}$. If $\eta < 1$, then $\lim_{k\to\infty} D(x_{n_k}, x_{m_k}) = 0$ which is a contradiction. If $\eta = 1$, without loss of generality, we may assume that $\lim_{k\to\infty} \beta(D(x_{n_k-i_k}, x_{n_k+m_k-i_k})) = 1$. Then, we have that

$$\lim_{k\to\infty} D\left(x_{n_k-i_k}, x_{n_k+m_k-i_k}\right) = 0. \tag{19}$$

This implies that there exists $k_0 \in \mathbb{N}$ such that

$$D\left(x_{n_{k_0}-i_{k_0}}, x_{n_{k_0}+m_{k_0}-i_{k_0}}\right) < \frac{\varepsilon}{2}. \tag{20}$$

Thus,

$$\begin{aligned} \varepsilon &\le D\left(x_{n_{k_0}}, x_{n_{k_0}+m_{k_0}}\right) \\ &\le \prod_{j=1}^{i_{k_0}} \beta\left(D\left(x_{n_{k_0}-j}, x_{n_{k_0}+m_{k_0}-j}\right)\right) D\left(x_{n_{k_0}-i_{k_0}}, x_{n_{k_0}+m_{k_0}-i_{k_0}}\right) \\ &< \frac{\varepsilon}{2}, \end{aligned} \tag{21}$$

which is a contradiction. Therefore, $\{x_n\}$ is a D-Cauchy sequence in A_0.

Since A_0 is a D-complete, there exists $x^* \in A_0$ such that $\lim_{n\to\infty} D(x_n, x^*) = 0$ and so $\{x_n\} \in C(D, A_0, x^*)$. Since T is G-continuous, $\{Tx_n\} \in C(D, A_0, Tx^*)$.

By (D_3^*) and (iii), there exist $C > 0$ and $\lambda \ge 1$ such that

$$\begin{aligned} D(A,B) &\le \lambda D(A,B) \le \lambda D(x^*, Tx^*) \\ &\le C\lambda \limsup_{n\to\infty} D(x_n, Tx_{n-1}) \le D(A,B). \end{aligned} \tag{22}$$

It follows that $D(A,B) = D(x^*, Tx^*)$. Suppose that x^*, $y^* \in B(T)$ such that $(x^*, y^*) \in E(G)$. By Lemma 11, $x^* = y^*$. The proof is now completed.

Theorem 13. *Let A_0 be D-complete and $T : A \longrightarrow B$ be a (G, D)-proximal Geraghty map. Suppose that the following conditions hold:*

(i) $T(A_0) \subseteq B_0$ and (A, B) has the weak P-property

(ii) There exist $x, y \in A_0$ such that $d(x, Ty) = d(A, B)$, $(y, x) \in E(G)$, and $D(y, x) < \infty$

(iii) For $\{x_n\} \in C(D, A_0, x^*)$, if $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, then there exists a subsequence $\{x_{n_k}\}$ with $(x_{n_k}, x^*) \in E(G)$ for all $k \in \mathbb{N}$

Then, there exists $x^* \in B(T)$. Moreover, T has a unique best proximity point if $(x^*, y^*) \in E(G)$ for all $x^*, y^* \in B(T)$.

Proof. From the proof of Theorem 12, by using the assumptions (i)-(ii), we obtain a sequence $\{x_n\} \in A_0$ such that

$$\lim_{n\to\infty} D(x_n, x^*) = 0 \text{ for some } x^* \in A_0. \tag{23}$$

Equivalently, $\{x_n\} \in C(D, A_0, x^*)$. Since $x^* \in A_0$ and $T(A_0) \subseteq B_0$, $Tx^* \in B_0$. It follows that there exists $a \in A$ such that

$$D(a, Tx^*) = D(A, B). \tag{24}$$

By (11) and (iii), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x^*) \in E(G)$ for all $k \in \mathbb{N}$. Then, from (11),

$$D\left(x_{n_k+1}, Tx_{n_k}\right) = D(A,B) \text{ for all } k \in \mathbb{N}. \tag{25}$$

By the weak P-property of (A, B), (24) and (25), we have that $D(x_{n_k+1}, a) \le D(Tx_{n_k}, Tx^*)$.

Since $(x_{n_k}, x^*) \in E(G)$ and T is a (G, D)-proximal Geraghty mapping, we obtain that

$$\begin{aligned} D\left(x_{n_k+1}, a\right) &\le D\left(Tx_{n_k}, Tx^*\right) \le \beta\left(D\left(x_{n_k}, x^*\right)\right) D\left(x_{n_k}, x^*\right) \\ &\le D\left(x_{n_k}, x^*\right) < \infty \text{for all } n \ge 1. \end{aligned} \tag{26}$$

Taking $k \longrightarrow \infty$ in (26), $\lim_{k\to\infty} D(x_{n_k}, a) = 0$.

Therefore, $\{x_{n_k}\} \in C(D, A_0, x^*) \cap C(D, A_0, a)$. It follows from Proposition 3 that $x^* = a$. From (24), there exists $x^* \in A$ such that $D(x^*, Tx^*) = D(A, B)$. Finally, if $x^*, y^* \in B(T)$ such that $(x^*, y^*) \in E(G)$. By Lemma 11, we have that $x^* = y^*$. The proof is now completed.

Example 1. Let $X = \mathbb{R}$ be equipped with a JS-metric D given by

$$D(x,y) = \begin{cases} |x+y|, & x \ne 0 \text{ and } y \ne 0, \\ \left|\dfrac{x}{2}\right|, & y = 0, \\ \left|\dfrac{y}{2}\right|, & x = 0. \end{cases} \tag{27}$$

Let $A = [-5, 0]$ and $B = [0, 10]$. We can see that the pair (A, B) have the weak P-property. Also, we have that

$$A_0 = [-5, 0], B_0 = [0, 5] \text{ and } A_0 \text{ is } D\text{-complete}. \tag{28}$$

Let $T : A \longrightarrow B$ be a mapping defined by $T(x) = -x/10$, for all $x \in A$. Then,

$$T(A_0) = \left[0, \frac{1}{2}\right] \subseteq B_0 = [0, 5]. \tag{29}$$

Define a graph $G = (V(G), E(G))$ by $V(G) = X$ and $E(G) = \{(x, y): x \neq 0 \text{ or } y = 0\}$. Clearly, G is transitive and there is $0 \in A_0$ such that

$$D(0, T(0)) = D(0, 0) = 0 = D(A, B) \text{ and } (0, 0) \in E(G). \quad (30)$$

We first check that T is (G, D)-proximal. Let $x_1, x_2, u_1, u_2 \in A$ such that $(x_1, x_2) \in E(G)$ and

$$D(u_1, T(x_1)) = D(u_2, T(x_2)) = D(A, \mathrm{B}). \quad (31)$$

Thus,

$$D\left(u_1, \frac{-x_1}{10}\right) = D\left(u_2, \frac{-x_2}{10}\right) = 0. \quad (32)$$

Now, suppose that $(u_1, u_2) \notin E(G)$. Then, $u_1 = 0$ and $u_2 \neq 0$. Since $(x_1, x_2) \in E(G)$, we consider the following two cases.

If $x_2 \neq 0$, then, $x_1 \neq 0$ and $D(0, -x_1/10) = D(u_2, -x_2/10) = 0$. It follows that $|-x_1/20| = |u_2 + (-x_2/10)| = 0$. Thus, $x_1 = 0$ which is a contradiction.

If $x_2 = 0$, then $D(0, -x_1/10) = D(u_2, 0) = 0$ and so $|-x_1/20| = |u_2/2| = 0$. This implies that $u_2 = 0$ which is a contradiction.

Therefore, $(u_1, u_2) \in E(G)$ and so T is (G, D)-proximal.

We consider a constant map $\beta(t) = 1/10 \in \mathscr{B}$.

Let $(x, y) \in E(G)$. Then, $x \neq 0$ or $y = 0$.

If $y = 0$, then

$$\begin{aligned} D(T(x), T(y)) &= D(T(x), T(0)) = D\left(\frac{-x}{10}, 0\right) \\ &= \left|\frac{-x}{20}\right| = \frac{1}{10}\left|\frac{x}{2}\right| \leq \frac{1}{10} D(x, y). \end{aligned} \quad (33)$$

If $x \neq 0$, then

$$\begin{aligned} D(T(x), T(y)) &= D\left(\frac{-x}{10}, \frac{-y}{10}\right) = \left|\frac{-x}{10} + \frac{-y}{10}\right| \\ &= \frac{1}{10}|x + y| \leq \frac{1}{10} D(x, y). \end{aligned} \quad (34)$$

Thus, T is a (G, D)-proximal Geraghty map.

Finally, we will show that the condition (iii) in Theorem 13 holds. Let $\{z_n\} \in C(D, A_0, a)$ for some $a \in A_0$ such that $(z_n, z_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$. Then,

$$z_n \neq 0 \text{ or } z_{n+1} = 0 \text{ for each } n \in \mathbb{N}. \quad (35)$$

If $z_n \neq 0$ for all $n \in \mathbb{N}$, then $(z_n, a) \in E(G)$ for all $n \in \mathbb{N}$. Assume that there exists an $n_0 \in \mathbb{N}$ such that $z_{n_0} = 0$. By (35), $z_k = 0$ for all $k \geq n_0$. Suppose that $a \neq 0$. Then,

$$D(z_k, a) = D(0, a) = \left|\frac{a}{2}\right| \neq 0 \text{ for all } k \geq n_0. \quad (36)$$

This contradicts to the fact that $\{z_n\} \in C(D, A_0, a)$. Thus, $a = 0$ and so $(z_n, a) \in E(G)$. Therefore, $0 \in B(T)$ by Theorem 13.

We next present some consequences from our main results.

Definition 14. A mapping $T : A \longrightarrow B$ is said to be a (G, D)-proximal contraction if the following hold:

(i) T is (G, D)-proximal

(ii) For all $x, y \in A$ such that $(x, y) \in E(G)$, there exists $k \in [0, 1)$ such that

$$D(Tx, Ty) \leq kD(x, y) \quad (37)$$

We immediately have the following corollaries.

Corollary 15. *Let X be a JS^*-metric space and A_0 be D-complete. Let $T : A \longrightarrow B$ be a (G, D)-proximal contraction. Suppose that the following conditions hold:*

(i) $T(A_0) \subseteq B_0$ and (A, B) has the weak P-property

(ii) There exist $x, y \in A_0$ such that $d(x, Ty) = d(A, B)$, $(y, x) \in E(G)$, and $D(y, x) < \infty$

(iii) T is G-continuous and for $C > 0$ such that $D(x, Tx) \leq C \limsup_{n\to\infty} D(x_n, Tx_{n-1})$, $C\lambda \leq 1$ for some $\lambda \geq 1$

Then, there exists $x^* \in B(T)$. Moreover, if $(x^*, y^*) \in E(G)$ for all $x^*, y^* \in B(T)$, T has a unique best proximity point.

Corollary 16. *Let A_0 be D-complete and $T : A \longrightarrow B$ be a (G, D)-proximal contraction. Suppose that the following conditions hold:*

(i) $T(A_0) \subseteq B_0$ and (A, B) has the weak P-property

(ii) There exist $x, y \in A_0$ such that $d(x, Ty) = d(A, B)$, $(y, x) \in E(G)$, and $D(y, x) < \infty$

(iii) For any sequence $\{x_n\}$ in $C(D, A_0, x^*)$, if $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, then there exists a subsequence $\{x_{n_k}\}$ with $(x_{n_k}, x^*) \in E(G)$ for all $k \in \mathbb{N}$

Then, there exists $x^* \in B(T)$. Moreover, if $(x^*, y^*) \in E(G)$ for all $x^*, y^* \in B(T)$, T has a unique best proximity point.

Definition 17. A mapping $T : A \longrightarrow B$ is said to be a (G, D)-proximal, R-type mapping if the following hold:

(i) T is (G, D)-proximal

(ii) For all $x, y \in A$ such that $(x, y) \in E(G)$,

$$D(Tx, Ty) \leq \frac{D(x, y)}{D(x, y) + 1} \quad (38)$$

Applying $\beta(t) = 1/(t + 1)$ in Theorems 12 and 13, we obtain two corollaries as follows.

Corollary 18. *Let X be a JS^*-metric space and A_0 be D-complete. Let $T : A \longrightarrow B$ be a (G, D)-proximal, R-type mapping. Suppose that the following conditions hold:*

(i) *$T(A_0) \subseteq B_0$ and (A, B) has the weak P-property*

(ii) *There exist $x, y \in A_0$ such that $d(x, Ty) = d(A, B)$, $(y, x) \in E(G)$, and $D(y, x) < \infty$*

(iii) *T is G-continuous and for $C > 0$ such that $D(x, Tx) \leq C\limsup_{n\to\infty} D(x_n, Tx_{n-1})$, $C\lambda \leq 1$ for some $\lambda \geq 1$*

Then, there exists $x^* \in B(T)$. Moreover, if $(x^*, y^*) \in E(G)$ for all $x^*, y^* \in B(T)$, T has a unique best proximity point.

Corollary 19. *Let A_0 be D-complete and $T : A \longrightarrow B$ be a (G, D)-proximal, R-type mapping. Suppose that the following conditions hold:*

(i) *$T(A_0) \subseteq B_0$ and (A, B) has the weak P-property*

(ii) *There exist $x, y \in A_0$ such that $d(x, Ty) = d(A, B)$, $(y, x) \in E(G)$, and $D(y, x) < \infty$*

(iii) *If for any sequence $\{x_n\}$ in $C(D, A_0, x^*)$, if $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, then there is a subsequence $\{x_{n_k}\}$ with $(x_{n_k}, x^*) \in E(G)$ for all $k \in \mathbb{N}$*

Then, there exists $x^* \in B(T)$. Moreover, if $(x^*, y^*) \in E(G)$ for all $x^*, y^* \in B(T)$, T has a unique best proximity point.

4. Application

In this section, we apply our result on best proximity points on a metric space endowed with binary relation. Let A and B be nonempty subset of a JS-metric space X with a binary relation $\mathscr{R}$, and let $T : A \longrightarrow B$ be a nonself mapping. The mapping T is said to be a D-proximally comparative if xRg and $D(u_1, Tx) = D(u_2, Ty) = D(A, B) \Rightarrow u_1 R u_2$ and $D(u_1, u_2) < \infty$ for all $x, y, u_1, u_2 \in A$.

Definition 20. The mapping T is said to be D-proximally comparative, Geraghty mapping if the following hold:

(1) T is a D-proximally comparative

(2) There exists $\beta \in \mathscr{B}$ such that for all $x, y \in A$, if xRy, then

$$D(Tx, Ty) \leq \beta(D(x, y))D(x, y) \quad (39)$$

Corollary 21. *Let R be symmetric and transitive, and let A_0 be D-complete. Let $T : A \rightarrow B$ be a D-proximally comparative, Geraghty map. Suppose that the following conditions hold:*

(i) *$T(A_0) \subseteq B_0$ and (A, B) has the weak P-property*

(ii) *There exist $x, y \in A_0$ such that $d(x, Ty) = d(A, B)$, yRx, and $D(y, x) < \infty$*

(iii) *For any sequence $\{x_n\}$ in $C(D, A_0, x^*)$, if $x_n R x_{n+1}$ for all $n \in \mathbb{N}$, then there exists a subsequence $\{x_{n_k}\}$ with $x_{n_k} R x^*$ for all $k \in \mathbb{N}$*

Then, there exists $x^* \in B(T)$. Moreover, if $(x^*, y^*) \in E(G)$ for all $x^*, y^* \in B(T)$, T has a unique best proximity point.

Proof. Define a graph $G = (V(G), E(G))$ by $V(G) = X$ and $E(G) = \{(x, y) \in X \times X : xRy\}$. Let $x_1, x_2, u_1, u_2 \in A$ such that $(x_1, x_2) \in E_G$ and $D(u_1, Tx_1) = D(u_2, Tx_2) = D(A, B)$.

By the definition of $E(G)$, we have that $x\mathscr{R}y$. Since T is a D-proximally comparative, $u_1 R u_2$. It follows that $(u_1, u_2) \in E(G)$. Therefore, T is a (G, D)-proximal Geraghty mapping. The condition (ii) implies that there exist $x_0, x_1 \in A_0$ such that $D(x_1, Tx_0) = D(A, B)$ and $(x_0, x_1) \in E(G)$. Also, the condition (iii) of Theorem 13 follows from the property of $E(G)$ and the condition (iii). By applying Theorem 13, we have that $B(T) \neq \varnothing$. Moreover, if $x^*, y^* \in B(T)$, then $x^* R y^*$ which implies that $(x^*, y^*) \in E(G)$. Again, by Theorem 13, $x^* = y^*$.

Note that when X is a JS^*-metric space, and the condition (iii) in the above corollary is replaced by the condition (iii) in Theorem 12, the result also follows.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors have no conflict of interests regarding the publication of this paper.

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