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Research Article

Approximation Properties of λ-Gamma Operators Based on q-Integers

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In the present paper, we will introduce λ-Gamma operators based on q-integers. First, the auxiliary results about the moments are presented, and the central moments of these operators are also estimated. Then, we discuss some local approximation properties of these operators by means of modulus of continuity and Peetre $\mathscr{K}$-functional. And the rate of convergence and weighted approximation for these operators are researched. Furthermore, we investigate the Voronovskaja type theorems including the quantitative q-Voronovskaja type theorem and q-Grüss-Voronovskaja theorem.

1. Introduction

Gamma operators are very important positive linear operators and have been widely used in probability theory and computational mathematics. For $f \in C(\mathbb{R}^+)$, $n = 1, 2, 3, \cdots$ where $\mathbb{R}^+ = (0,\infty)$ and $C(\mathbb{R}^+)$ be the space of all continuous functions f on the interval $\mathbb{R}^+$, the Gamma operators were introduced in [1] by

$$G_n(f\,;x) = \frac{1}{n!}\int_0^\infty e^{-t}t^n f\left(\frac{nx}{t}\right)\mathrm{d}t, \quad x \in \mathbb{R}^+. \tag{1}$$

We can learn some properties of Gamma operators and their modified operators in [2–7]. In [8], Qi et al. defined new Gamma operators as follows:

$$G_{n,\lambda}(f\,;x) = \frac{1}{n!}\int_0^\infty e^{-t}t^n \left(\frac{n}{t}\right)^\lambda f\left(\frac{nx}{t}\right)\mathrm{d}t, \quad x \in \mathbb{R}^+. \tag{2}$$

where $f \in C(\mathbb{R}^+)$, $\lambda \in \mathbb{N} = \{0, 1, 2, \cdots\}$. Obviously, if $f^{(\lambda)} \in C(\mathbb{R}^+)$, then $(G_n(f\,;x))^{(\lambda)} = G_{n,\lambda}(f^{(\lambda)}\,;x)$. Meantime, $G_{n,\lambda}(1\,;x) = (n^\lambda(n-\lambda)!/n!) \neq 1$ (while $\lambda \neq 0$). In order to preserve the constant, we defined λ-Gamma operators as follows:

Definition 1. For $f \in C(\mathbb{R}^+)$, $\lambda \in \mathbb{N}$, $n = \lambda, \lambda+1, \cdots$, the λ-Gamma operators are defined by

$$\mathscr{G}_{n,\lambda}(f\,;x) = \frac{1}{(n-\lambda)!}\int_0^\infty e^{-t}t^{n-\lambda} f\left(\frac{nx}{t}\right)\mathrm{d}t, \quad x \in \mathbb{R}^+. \tag{3}$$

Let us recall some useful concepts and notations from q-calculus, which can be founded in [9–11]. For nonnegative integer i, the q-integer $[i]_q$ and q-factorial $[i]_q!$ are defined by

$$[i]_q = 1 + q + \cdots + q^{i-1} = \begin{cases} \dfrac{1-q^i}{1-q}, & q \neq 1, \\ i, & q = 1, \end{cases}$$
$$[i]_q! = \begin{cases} [1]_q[2]_q \cdots [i]_q, & i \geq 1, \\ 1, & i = 0. \end{cases} \tag{4}$$

Further, q-power basis can be defined by

$$(x+y)_q^i = \begin{cases} (x+y)(x+qy)\cdots\left(x+q^{i-1}y\right), & i=1,2,\cdots, \\ 1, & i=0, \end{cases}$$
$$(x-y)_q^i = \begin{cases} (x-y)(x-qy)\cdots\left(x-q^{i-1}y\right), & i=1,2,\cdots, \\ 1, & i=0. \end{cases} \tag{5}$$

The q-derivative $D_q f$ of a function f can be defined by

$$\left(D_q f\right)(x) = \frac{f(x)-f(qx)}{(1-q)x}, \quad \text{if} \quad x \neq 0, \tag{6}$$

and $(D_q f)(0) = f'(0)$ provided $f'(0)$ exists. High-order q-derivatives can be defined by $D_q^0 f = f$, $D_q^i = D_q(D_q^{i-1} f)$, $i = 1, 2, \cdots$. The formula for the q-derivative of a product is $D_q(f(x)g(x)) = D_q(f(x))g(x) + D_q(g(x))f(qx)$. We easily know that if a function f is continuous on an interval which does not include 0, then f is continuous q-differentiable.

The q-improper integral of function f can be defined by

$$\int_0^{\infty/1-q} f(t)\mathrm{d}_q t = \sum_{i=-\infty}^{\infty} f\left(\frac{q^i}{1-q}\right) q^i, \quad q \in (0,1). \tag{7}$$

The q-analogue of the classical exponential function e^x is

$$E_q(x) = \sum_{i=0}^{\infty} q^{((i(i-1))/2)} \frac{x^i}{[i]_q!} = (1+(1-q)x)_q^{\infty}, \quad q \in (0,1). \tag{8}$$

The q-Gamma function is defined by

$$\Gamma_q(s) = \int_0^{(\infty/1-q)} x^{(s-1)} E_q(-qx)\mathrm{d}_q x, \quad s \in \mathbb{R}^+, \tag{9}$$

and satisfies the functional relation: $\Gamma_q(s+1) = [s]_q \Gamma_q(s)$, $\Gamma_q(1) = 1$. Moreover, for any nonnegative integer $i > 0$, the relation holds: $\Gamma_q(i+1) = [i]_q!$.

Now, we construct the q-analogue of λ-Gamma operators using q-Gamma function as follows.

Definition 2. For $f : \mathbb{R}^+ \to \mathbb{R}$, $q \in (0,1)$, $\lambda \in \mathbb{N}$, $n = \lambda, \lambda+1, \cdots$, the q-analogue of λ-Gamma operators (3) are defined as

$$\mathscr{G}_{n,\lambda}^q(f;x) = \frac{1}{[n-\lambda]_q!} \int_0^{\infty/1-q} f\left(\frac{[n]_q x}{t}\right) E_q(-qt) t^{n-\lambda} \mathrm{d}_q t, \quad x \in \mathbb{R}^+. \tag{10}$$

The paper is organized as follows: In Section 1, we introduce the history of Gamma operators, recall some basic notations about the q-calculus, and construct λ-Gamma operators based on q-integers with q-Gamma function. In Section 2, we obtain the auxiliary results about the moment computation formula. The second- and fourth-order central moments computation formula and other quantitative properties are also presented. In Section 3, we discuss local approximation about the operators by means of modulus of continuity and Peetre $\mathscr{K}$-functional. In Section 4 and Section 5, the rate of convergence and weighted approximation for these operators are researched. In the last section, we firstly prove quantitative q-Voronovskaja type theorems in terms of weighted modulus of continuity, and then the q-Grüss-Voronovskaja theorem in the quantitative mean is also presented (for the quantitative q-Voronovskaja type theorem0 and the q-Grüss-Voronovskaja theorem for the other operators, see also [12, 13]).

2. Auxiliary Results

In this section, we will give some lemmas and corollaries, which are necessary to obtain the approximation properties of the operators $\mathscr{G}_{n,\lambda}^q(f;x)$.

Lemma 3. *For $q \in (0,1)$, $\lambda \in \mathbb{N}$, $i \in \mathbb{N}$, $n = \lambda+i, \lambda+i+1, \cdots$, the following formula holds:*

$$\mathscr{G}_{n,\lambda}^q\left(t^i;x\right) = \frac{[n]_q^i [n-\lambda-i]_q!}{[n-\lambda]_q!} x^i. \tag{11}$$

Proof. According to the properties of q-Gamma function, we have

$$\begin{aligned} \mathscr{G}_{n,\lambda}^q\left(t^i;x\right) &= \frac{1}{[n-\lambda]_q!} \int_0^{\infty/1-q} \left(\frac{[n]_q x}{t}\right)^i E_q(-qt) t^{n-\lambda} \mathrm{d}_q t \\ &= \frac{[n]_q^i x^i}{[n-\lambda]_q!} \int_0^{\infty/1-q} t^{n-\lambda-i} E_q(-qt) \mathrm{d}_q t \\ &= \frac{[n]_q^i x^i \Gamma_q(n-\lambda-i+1)}{[n-\lambda]_q!} \\ &= \frac{[n]_q^i [n-\lambda-i]_q!}{[n-\lambda]_q!} x^i. \end{aligned} \tag{12}$$

Lemma 3 is proved.

Corollary 4. *By the lemma given above and some elementary calculations, we can get the results*

$$A(x) = \mathscr{G}_{n,\lambda}^q(t-x;x) = \frac{q^{n-\lambda}[\lambda]_q}{[n-\lambda]_q} x \quad \textit{for} \quad n \geq \lambda+1,$$
$$\begin{aligned} B(x) &= \mathscr{G}_{n,\lambda}^q\left((t-x)^2;x\right) \\ &= \left(\frac{q^{n-\lambda-1}[\lambda+1]_q}{[n-\lambda-1]_q} - \frac{q^{n-\lambda}[\lambda]_q}{[n-\lambda]_q} + \frac{q^{2n-2\lambda-1}[\lambda]_q[\lambda+1]_q}{[n-\lambda-1]_q[n-\lambda]_q}\right) x^2 \quad \textit{for} \quad n \geq \lambda+2. \end{aligned} \tag{13}$$

Lemma 5. *Let $q=(q_n)$ be a sequence satisfying $q_n \in (0,1)$, $q_n \to 1$ and $q_n^n \to a \in [0,1]$. Then, for each $x \in \mathbb{R}^+$, $A_n(x) := \mathscr{G}_{n,\lambda}^{q_n}(t-x;x)$, $B_n(x) := \mathscr{G}_{n,\lambda}^{q_n}((t-x)^2;x)$, we can obtain*

$$\lim_{n\to\infty}[n]_{q_n}A_n(x)=\lambda a x, \quad \lim_{n\to\infty}[n]_{q_n}B_n(x)=a x^2, \tag{14}$$

$$\mathscr{G}_{n,\lambda}^{q_n}\left((t-x)^3;x\right)=O\left(\frac{1}{[n]_{q_n}^2}\right), \tag{15}$$

$$\mathscr{G}_{n,\lambda}^{q_n}\left((t-x)^4;x\right)=O\left(\frac{1}{[n]_{q_n}^2}\right), \tag{16}$$

$$\mathscr{G}_{n,\lambda}^{q_n}\left((t-x)^6;x\right)=O\left(\frac{1}{[n]_{q_n}^3}\right). \tag{17}$$

Proof. By Lemma 3, we can easily get (14). Without loss of generality, we only prove equation (15). Equation (16) and equation (17) can be proved in some way. Set $\mathscr{G}_{n,\lambda}^{q}(t^i;x)=C(i)x^i$, $i=1,2,3$, and $\mathscr{G}_{n,\lambda}^{q_n}((t-x)^3;x)=C(4)x^3$. Using $([n]_{q_n})/([n-\lambda-i]_{q_n})=1+((q_n^{n-\lambda-i}[\lambda+i]_{q_n})/([n-\lambda-i]_{q_n}))$, $i=0,1,2,\cdots,n-\lambda$, we can easily get

$$C(1)=1+\frac{q_n^{n-\lambda}[\lambda]_{q_n}}{[n-\lambda]_{q_n}},$$

$$C(2)=1+\frac{q_n^{n-\lambda}[\lambda]_{q_n}}{[n-\lambda]_{q_n}}+\frac{q_n^{n-\lambda-1}[\lambda+1]_{q_n}}{[n-\lambda-1]_{q_n}}+\frac{q_n^{2n-2\lambda-1}[\lambda]_{q_n}[\lambda+1]_{q_n}}{[n-\lambda-1]_{q_n}[n-\lambda]_{q_n}},$$

$$\begin{aligned}C(3)={}&1+\frac{q_n^{n-\lambda}[\lambda]_{q_n}}{[n-\lambda]_{q_n}}+\frac{q_n^{n-\lambda-1}[\lambda+1]_{q_n}}{[n-\lambda-1]_{q_n}}+\frac{q_n^{n-\lambda-2}[\lambda+2]_{q_n}}{[n-\lambda-2]_{q_n}}\\&+\frac{q_n^{2n-2\lambda-3}[\lambda+2]_{q_n}[\lambda+1]_{q_n}}{[n-\lambda-2]_{q_n}[n-\lambda-1]_{q_n}}+\frac{q_n^{2n-2\lambda-2}[\lambda+2]_{q_n}[\lambda]_{q_n}}{[n-\lambda-2]_{q_n}[n-\lambda]_{q_n}}\\&+\frac{q_n^{2n-2\lambda-1}[\lambda]_{q_n}[\lambda+1]_{q_n}}{[n-\lambda-1]_{q_n}[n-\lambda]_{q_n}}+o\left(\frac{1}{[n]_{q_n}^2}\right).\end{aligned} \tag{18}$$

Combining

$$\begin{aligned}C(4)={}&C(3)-3C(2)+3C(1)-1\\={}&\frac{q_n^{n-\lambda}[\lambda]_{q_n}}{[n-\lambda]_{q_n}}-\frac{2q_n^{n-\lambda-1}[\lambda+1]_{q_n}}{[n-\lambda-1]_{q_n}}+\frac{q_n^{n-\lambda-2}[\lambda+2]_{q_n}}{[n-\lambda-2]_{q_n}}+\frac{q_n^{2n-2\lambda-3}[\lambda+2]_{q_n}[\lambda+1]_{q_n}}{[n-\lambda-2]_{q_n}[n-\lambda-1]_{q_n}}\\&+\frac{q_n^{2n-2\lambda-2}[\lambda+2]_{q_n}[\lambda]_{q_n}}{[n-\lambda-2]_{q_n}[n-\lambda]_{q_n}}-\frac{2q_n^{2n-2\lambda-1}[\lambda]_{q_n}[\lambda+1]_{q_n}}{[n-\lambda-1]_{q_n}[n-\lambda]_{q_n}}\\&+o\left(\frac{1}{[n]_{q_n}^2}\right):=\frac{q_n^{n-\lambda}[\lambda]_{q_n}}{[n-\lambda]_{q_n}}-\frac{2q_n^{n-\lambda-1}[\lambda+1]_{q_n}}{[n-\lambda-1]_{q_n}}\\&+\frac{q_n^{n-\lambda-2}[\lambda+2]_{q_n}}{[n-\lambda-2]_{q_n}}+I=\frac{q_n^{n-\lambda-2}}{\prod_{i=0}^{2}[n-\lambda-i]_{q_n}}\\&\cdot\left([\lambda+2]_{q_n}[n-\lambda]_{q_n}[n-\lambda-1]_{q_n}\right.\\&+q_n^2[\lambda]_{q_n}[n-\lambda-2]_{q_n}[n-\lambda-1]_{q_n}\\&\left.-2q_n[\lambda+1]_{q_n}[n-\lambda-2]_{q_n}[n-\lambda]_{q_n}\right)+I\\={}&\frac{q_n^{n-\lambda-2}}{\prod_{i=0}^{2}[n-\lambda-i]_{q_n}}\left([n-\lambda]_{q_n}\left([\lambda+2]_{q_n}[n-\lambda-1]_{q_n}\right.\right.\\&\left.-q_n[\lambda+1]_{q_n}[n-\lambda-2]_{q_n}\right)-q_n[n-\lambda-2]_{q_n}\\&\cdot\left.\left([\lambda+1]_{q_n}[n-\lambda]_{q_n}-q_n[\lambda]_{q_n}[n-\lambda-1]_{q_n}\right)\right)+I\\={}&\frac{q_n^{n-\lambda-2}}{\prod_{i=0}^{2}[n-\lambda-i]_{q_n}}\left([n-\lambda]_{q_n}[n]_{q_n}-q_n[n-\lambda-2]_{q_n}[n]_{q_n}\right)+I\\={}&\frac{q_n^{n-\lambda-2}[n]_{q_n}\left(1+q_n^{n-\lambda-1}\right)}{\prod_{i=0}^{2}[n-\lambda-i]_{q_n}}+I,\end{aligned}$$

$$\lim_{n\to\infty}[n]_{q_n}^2 I=(3\lambda+2)a^2, \tag{19}$$

we have $\lim_{n\to\infty}[n]_{q_n}^2\mathscr{G}_{n,\lambda}^{q_n}((t-x)^3;x)=3(\lambda+1)a^2x^3+ax^3$. This means that equation (15) is obtained. Thus, the proof of Lemma 5 is accomplished.

Lemma 6. *Let $q=(q_n)$ be a sequence satisfying $q_n\in(0,1)$, $q_n\to 1$ and $q_n^n\to a\in[0,1]$. Then, for each $x\in\mathbb{R}^+$, the following relations*

$$\mathscr{G}_{n,\lambda}^{q_n}\left(\left|(t-x)_{q_n}^2\right|;x\right)\le O\left(\frac{1}{[n]_{q_n}}\right), \tag{20}$$

$$\mathscr{G}_{n,\lambda}^{q_n}\left(\left|(t-x)_{q_n}^2\right|(t-x)^4;x\right)\le O\left(\frac{1}{[n]_{q_n}^3}\right), \tag{21}$$

hold.

Proof. By the definition of q-power basis, we have $(t-x)_{q_n}^2=(t-x)(t-q_nx)=(t-x)^2+(1-q_n^n)x(t-x)=(t-x)^2+x((1-q_n^n)/([n]_{q_n}))(t-x)$. Thus, we can write $|(t-x)_{q_n}^2|\le(t-x)^2+x((1-q_n^n)/([n]_{q_n}))\,|t-x|$. Using the monotonicity of the operators $\mathscr{G}_{n,\lambda}^{q_n}$ and the Cauchy-Schwarz inequality, we can get

$$\mathcal{G}_{n,\lambda}^{q_n}\left(\left|(t-x)_{q_n}^2\right|;x\right) \le \mathcal{G}_{n,\lambda}^{q_n}\left((t-x)^2;x\right) + x\frac{1-q_n^n}{[n]_{q_n}}\mathcal{G}_{n,\lambda}^{q_n}(|t-x|;x) \le \mathcal{G}_{n,\lambda}^{q_n}\left((t-x)^2;x\right) + x\frac{1-q_n^n}{[n]_{q_n}}\sqrt{\mathcal{G}_{n,\lambda}^{q_n}\left((t-x)^2;x\right)} \le O\left(\frac{1}{[n]_{q_n}}\right) + O\left(\frac{1}{[n]_{q_n}^{3/2}}\right) = O\left(\frac{1}{[n]_{q_n}}\right). \tag{22}$$

The inequality (21) can be get in the same way. Using the monotonicity of the operators $\mathcal{G}_{n,\lambda}^{q_n}$, (16)and (17), Cauchy-Schwarz inequality, respectively, we can obtain

$$\mathcal{G}_{n,\lambda}^{q_n}\left(\left|(t-x)_{q_n}^2\right|(t-x)^4;x\right) \le \mathcal{G}_{n,\lambda}^{q_n}\left((t-x)^6;x\right) + \frac{x}{[n]_{q_n}}\mathcal{G}_{n,\lambda}^{q_n}\left(|t-x|^5;x\right) \le \mathcal{G}_{n,\lambda}^{q_n}\left((t-x)^6;x\right) + \frac{x}{[n]_{q_n}}\sqrt{\mathcal{G}_{n,\lambda}^{q_n}\left((t-x)^4;x\right)\mathcal{G}_{n,\lambda}^{q_n}\left((t-x)^6;x\right)} \le O\left(\frac{1}{[n]_{q_n}^3}\right) + O\left(\frac{1}{[n]_{q_n}^{7/2}}\right) = O\left(\frac{1}{[n]_{q_n}^3}\right). \tag{23}$$

Thus, we complete the proof.

3. Local Approximation

Let $C_B(\mathbb{R}^+)$ be the space of all real-valued continuous bounded functions f on $\mathbb{R}^+$, endowed with the norm $\|f\| = \sup_{x\in\mathbb{R}^+} |f(x)|$. Moreover, the Peetre's $\mathcal{K}$-functional is defined by

$$\mathcal{K}_2(f;\delta) = \inf_{h\in C_B^2(\mathbb{R}^+)}\left\{\|f-h\| + \delta\|h''\|\right\}, \tag{24}$$

where $C_B^2(\mathbb{R}^+) := \{h \in C_B(\mathbb{R}^+): h', h'' \in C_B(\mathbb{R}^+)\}$. By ([14], p. 177, Theorem 2.4), there exists an absolute constant $C > 0$ such that

$$\mathcal{K}_2(f;\delta) \le C\omega_2\left(f;\sqrt{\delta}\right), \tag{25}$$

where $\delta > 0$ and the second-order modulus of smoothness is defined by

$$\omega_2\left(f;\sqrt{\delta}\right) = \sup_{0<t\le\delta}\sup_{x\in\mathbb{R}^+}|f(x+2t) - 2f(x+t) + f(x)|, \quad f \in C_B(\mathbb{R}^+). \tag{26}$$

The usual modulus of smoothness is defined by

$$\omega(f;\delta) = \sup_{0<t\le\delta}\sup_{x\in\mathbb{R}^+}|f(x+t) - f(x)|, \quad f \in C_B(\mathbb{R}^+). \tag{27}$$

Theorem 7. *Let $f \in C_B(\mathbb{R}^+)$, $q \in (0,1)$, $\lambda = 1, 2, \cdots$. Then for all $x \in \mathbb{R}^+$ and $n \ge \lambda + 1$, there exists an absolute $C_1 = 4C$ such that*

$$\left|\mathcal{G}_{n,\lambda}^{q}(f;x) - f(x)\right| \le C_1\omega_2\left(f;\sqrt{A^2(x) + B(x)}\right) + \omega(f;|A(x)|). \tag{28}$$

Proof. Using Definition 2, we easily obtain $|\mathcal{G}_{n,\lambda}^{q}(f;x)| \le \|f\|$ for all $f \in C_B(\mathbb{R}^+)$. Next, we define new operators by

$$\mathcal{P}_{n,\lambda}^{q}(f;x) = \mathcal{G}_{n,\lambda}^{q}(f;x) + f(x) - f(A(x)+x), \quad x \in \mathbb{R}^+. \tag{29}$$

We can get $\mathcal{P}_{n,\lambda}^{q}(t-x;x) = \mathcal{G}_{n,\lambda}^{q}(t-x;x) - A(x) = 0$ and $|\mathcal{P}_{n,\lambda}^{q}(f;x)| \le 3\|f\|$ for all $f \in C_B(\mathbb{R}^+)$. For $x, t \in \mathbb{R}^+$ and $h \in C_B^2(\mathbb{R}^+)$, using Taylor's expansion, we can write

$$h(t) = h(x) + h'(x)(t-x) + \int_x^t h''(u)(t-u)\mathrm{d}u. \tag{30}$$

Hence,

$$\begin{aligned}\left|\mathcal{P}_{n,\lambda}^{q}(h;x) - h(x)\right| &= \Big|h'(x)\mathcal{P}_{n,\lambda}^{q}(t-x;x) + \mathcal{P}_{n,\lambda}^{q}\left(\int_x^t h''(u)(t-u)\mathrm{d}u;x\right)\Big| \\ &\le \left|\mathcal{P}_{n,\lambda}^{q}\left(\int_x^t h''(u)(t-u)\mathrm{d}u;x\right)\right| \\ &\le \left|\mathcal{G}_{n,\lambda}^{q}\left(\int_x^t h''(u)(t-u)\mathrm{d}u;x\right) - \int_x^{A(x)+x} h''(u)(A(x)+x-u)\mathrm{d}u\right| \le \mathcal{G}_{n,\lambda}^{q} \\ &\quad\cdot\left(\int_x^t |h''(u)|(t-u)\mathrm{d}u;x\right) + \left|\int_x^{A(x)+x} |h''(u)|(A(x)+x-u)\mathrm{d}u\right| \\ &\le \left(B(x) + A^2(x)\right)\|h''\|.\end{aligned} \tag{31}$$

Further, for all $h \in C_B^2(\mathbb{R}^+)$, we can write

$$\begin{aligned}\left|\mathscr{G}_{n,\lambda}^q(f;x)-f(x)\right| &= \left|\mathscr{P}_{n,\lambda}^q(f;x)+f(A(x)+x)-2f(x)\right| \\ &\le \left|\mathscr{P}_{n,\lambda}^q(f-h;x)-(f-h)(x)\right| \\ &\quad+\left|\mathscr{P}_{n,\lambda}^q(h;x)-h(x)\right|+|f(A(x)+x) \\ &\quad-f(x)|\le 4\|f-h\|+\left(A^2(x)+B(x)\right)\|h''\| \\ &\quad+\omega(f;|A(x)|). \end{aligned} \tag{32}$$

Taking infimum over all h and using (25), we can get the desired conclusion.

Corollary 8. *Let $f \in C_B(\mathbb{R}^+)$, $q \in (0,1)$. Then for all $x \in \mathbb{R}^+$ and $n \ge 1$, there exists an absolute $C_1 = 4C$ such that*

$$\left|\mathscr{G}_{n,0}^q(f;x)-f(x)\right| \le C_1\omega_2\left(f;\sqrt{B(x)}\right). \tag{33}$$

Corollary 9. *Let $f \in C_B(\mathbb{R}^+)$, $q=(q_n)$ be a sequence satisfying $q_n \in (0,1)$, $q_n \longrightarrow 1$ and $q_n^n \longrightarrow a \in [0,1]$ as $n \longrightarrow \infty$, the limit*

$$\lim_{n\to\infty}\mathscr{G}_{n,\lambda}^{q_n}(f;x)=f(x) \tag{34}$$

holds for all $x \in \mathbb{R}^+$.

4. Rate of Convergence

As is known, if f is not uniformly continuous on $\mathbb{R}^+$, we cannot get $\omega(f;\delta)\to 0$ as $\delta\to 0$. To research the rate of convergence of the operators $\mathscr{G}_{n,\lambda}^{q_n}$ on $\mathbb{R}^+$, we recall the weighted modulus of continuity $\Omega(f;\delta)$(see [15] or [16]). First, we shall consider the following three classes of functions:

$$B_2(\mathbb{R}^+) := \left\{f:\mathbb{R}^+\to\mathbb{R}\,;\,|f(x)|\le C_f\left(1+x^2\right)\right\}, \tag{35}$$

where C_f is a positive constant which depends only on f,

$$C_2(\mathbb{R}^+) := \{f\in B_2(\mathbb{R}^+): f \quad \text{is continuous}\},$$

$$C_2^0(\mathbb{R}^+) := \left\{f\in B_2(\mathbb{R}^+): \lim_{x\to\infty}\frac{f(x)}{1+x^2} \quad \text{is finite}\right\}. \tag{36}$$

The space $C_2^0(\mathbb{R}^+)$ is a linear normed space endowed with the norm $\|f\|_2=\sup_{x\in\mathbb{R}^+}((|f(x)|)/(1+x^2))$. For any $f\in C_2(\mathbb{R}^+)$, $\Omega(f;\delta)$ is defined by

$$\Omega(f;\delta)=\sup_{0\le t<\delta,x\in\mathbb{R}^+}\frac{|f(x+t)-f(x)|}{(1+t^2)(1+x^2)}, \tag{37}$$

if $f\in C_2^0(\mathbb{R}^+)$, then $\Omega(f;\delta)$ has the following properties:

(i) $\lim_{\delta\to 0^+}\Omega(f;\delta)=0$

(ii) $\Omega(f;\rho\delta)\le 2(1+\rho)(1+\delta^2)\Omega(f;\delta)$, $\rho\in\mathbb{R}^+$

In [17–19], the following inequality was introduced and used

$$\begin{aligned}|f(t)-f(x)| &\le\left(1+(t-x)^2\right)\left(1+x^2\right)\Omega(f;|t-x|) \\ &\le 2\left(1+\frac{|t-x|}{\delta}\right)\left(1+\delta^2\right)\left(1+(t-x)^2\right)\left(1+x^2\right)\Omega(f;\delta) \\ &\le\begin{cases}4\left(1+\delta^2\right)^2\left(1+x^2\right)\Omega(f;\delta), & |t-x|<\delta,\\ 4\left(1+\delta^2\right)^2\left(1+x^2\right)\dfrac{(t-x)^4}{\delta^4}\Omega(f;\delta), & |t-x|\ge\delta.\end{cases}\end{aligned} \tag{38}$$

Meanwhile, we introduce the modulus of continuity of $f\in C(0,\mathbf{a}](\mathbf{a}>0)$ by $\omega_{\mathbf{a}}(f;\delta)=\sup_{|t-x|\le\delta x,t\in(0,\mathbf{a}]}\ \sup\ |f(t)-f(x)|$.

The following is a theorem of the rate of convergence for the operators $\mathscr{G}_{n,\lambda}^q$:

Theorem 9. *Let $f\in C_2(\mathbb{R}^+)$, $\lambda\in\mathbb{N}$, $n=\lambda+1,\lambda+2,\cdots$, $\boldsymbol{a}\in\mathbb{R}^+$, we have*

$$\|\mathscr{G}_{n,\lambda}^q(f;x)-f\|_{C(0,\mathbf{a}]}\le 4C_f\left(1+\mathbf{a}^2\right)B(\mathbf{a})+2\omega_{\mathbf{a}+1}\left(f,\sqrt{B(\mathbf{a})}\right). \tag{39}$$

Proof. For any $x\in(0,\mathbf{a}]$, $t\in(\mathbf{a}+1,\infty)$, we can easily obtain $1\le(t-\mathbf{a})^2\le(t-x)^2$, therefore

$$\begin{aligned}|f(t)-f(x)| &\le C_f\left(2+x^2+t^2\right) \\ &\le C_f\left(2+3x^2+2(t-x)^2\right) \\ &\le 4C_f\left(1+\mathbf{a}^2\right)(t-x)^2.\end{aligned} \tag{40}$$

If $t\in(0,\mathbf{a}+1)$, for any $\delta\in\mathbb{R}^+$, we can obtain

$$|f(t)-f(x)|\le\omega_{\mathbf{a}+1}(f;|t-x|)\le\left(1+\frac{|t-x|}{\delta}\right)\omega_{\mathbf{a}+1}(f;\delta). \tag{41}$$

Combining (39) with (40), we can get

$$|f(t)-f(x)|\le 4C_f\left(1+\mathbf{a}^2\right)(t-x)^2+\left(1+\frac{|t-x|}{\delta}\right)\omega_{\mathbf{a}+1}(f;\delta). \tag{42}$$

By Cauchy-Schwarz's inequality and Corollary 4, for all $x\in(0,\mathbf{a}]$, we have

$$
\begin{aligned}
|\mathcal{G}_{n,\lambda}^{q}(f;x)-f(x)| &\le \mathcal{G}_{n,\lambda}^{q}(|f(t)-f(x)|;x)\\
&\le 4C_f\left(1+\mathbf{a}^2\right)\mathcal{G}_{n,\lambda}^{q}\left((t-x)^2;x\right)+\mathcal{G}_{n,\lambda}^{q}\left(\left(1+\frac{|t-x|}{\delta}\right);x\right)\omega_{\mathbf{a}+1}(f,\delta)\\
&\le 4C_f\left(1+\mathbf{a}^2\right)\mathcal{G}_{n,\lambda}^{q}\left((t-x)^2;x\right)+\omega_{\mathbf{a}+1}(f,\delta)\left(1+\frac{1}{\delta}\sqrt{\mathcal{G}_{n,\lambda}^{q}\left((t-x)^2;x\right)}\right)\\
&\le 4C_f\left(1+\mathbf{a}^2\right)B(x)+\omega_{\mathbf{a}+1}(f,\delta)\left(1+\frac{1}{\delta}\sqrt{B(x)}\right)\\
&\le 4C_f\left(1+\mathbf{a}^2\right)B(\mathbf{a})+\omega_{\mathbf{a}+1}(f,\delta)\left(1+\frac{1}{\delta}\sqrt{B(\mathbf{a})}\right).
\end{aligned}
\tag{43}
$$

By choosing $\delta=\sqrt{B(\mathbf{a})}$ and taking supremum over all $x\in(0,\mathbf{a}]$, we can get the desired results.

Theorem 10. *$q=(q_n)$ be a sequence satisfying $q_n\in(0,1)$, $q_n\to1$, and $q_n^n\to a$ as $n\to\infty$ and $f\in C_2^0(\mathbb{R}^+)$; then, there exists a positive integer $N\in\mathbb{N}_+$ such that for all $n>N$ and $\upsilon>0$, the inequality*

$$
\sup_{x\in\mathbb{R}^+}\frac{\left|\mathcal{G}_{n,\lambda}^{q_n}(f;x)-f(x)\right|}{\left(1+x^2\right)^{\upsilon}}\le 64\Omega\left(f;\frac{1}{\sqrt{[n]_{q_n}}}\right),\tag{44}
$$

holds.

Proof. Using (14) and (16), there exists a positive integer $N\in\mathbb{N}_+$ such that for all $n>N$,

$$
\begin{aligned}
&\mathcal{G}_{n,\lambda}^{q_n}\left((t-x)^2;x\right)\le\frac{9}{4[n]_{q_n}},\\
&\mathcal{G}_{n,\lambda}^{q_n}\left((t-x)^4;x\right)\le 1.
\end{aligned}
\tag{45}
$$

By Cauchy-Schwarz's inequality, we can get

$$
\mathcal{G}_{n,\lambda}^{q_n}(|t-x|;x)\le\sqrt{\mathcal{G}_{n,\lambda}^{q_n}\left((t-x)^2;x\right)}\le\frac{3}{2\sqrt{[n]_{q_n}}},\tag{46}
$$

$$
\begin{aligned}
\mathcal{G}_{n,\lambda}^{q_n}\left(|t-x|^3;x\right)&\le\sqrt{\mathcal{G}_{n,\lambda}^{q_n}\left((t-x)^2;x\right)}\sqrt{\mathcal{G}_{n,\lambda}^{q_n}\left((t-x)^4;x\right)}\\
&\le\frac{3}{2\sqrt{[n]_{q_n}}}.
\end{aligned}
\tag{47}
$$

Since $\mathcal{G}_{n,\lambda}^{q_n}$ is linear and positive, using (38), (46), and (47), for any $\delta\in(0,1)$, we can obtain

$$
\begin{aligned}
\left|\mathcal{G}_{n,\lambda}^{q_n}(f;x)-f(x)\right|&\le 16\left(1+x^2\right)\Omega(f;\delta)\\
&\quad\cdot\left(1+\frac{\mathcal{G}_{n,\lambda}^{q_n}\left(|t-x|+|t-x|^3;x\right)}{\delta}\right)\\
&\le 16\left(1+x^2\right)\left(1+\frac{3}{\delta\sqrt{[n]_{q_n}}}\right)\Omega(f;\delta).
\end{aligned}
\tag{48}
$$

Taking $\delta=1/\sqrt{[n]_{q_n}}$, we complete the proof.

5. Weighted Approximation

In this section, we will discuss the weighted approximation theorems for the operators $\mathcal{G}_{n,\lambda}^{q_n}$.

Theorem 11. *Let $q=(q_n)$ be a sequence satisfying $q_n\in(0,1)$, $q_n\to1$, and $q_n^n\to a\in[0,1]$ as $n\to\infty$ and $f\in C_2^0(\mathbb{R}^+)$, we have*

$$
\lim_{n\to\infty}\|\mathcal{G}_{n,\lambda}^{q_n}(f;x)-f\|_2=0.\tag{49}
$$

Proof. Using Korovkin's theorem (see [20]), it is sufficient to verify the following three conditions:

$$
\lim_{n\to\infty}\|\mathcal{G}_n^{p_n,q_n}\left(t^k\right)-x^k\|_2=0,k=0,1,2.\tag{50}
$$

Since $\mathcal{G}_n^{p_n,q_n}(1;x)=1$, (51) holds for $k=1$. By Lemma 3 and $\lim_{n\to\infty}\left([n]_{q_n}/[n-\lambda]_{q_n}\right)=\lim_{n\to\infty}\left([n]_{q_n}/[n-\lambda-1]_{q_n}\right)=1$, we can easily obtain

$$\|\mathcal{G}_{n,\lambda}^{q_n}(t;x)-x\|_2=\sup_{x\in\mathbb{R}^+}\frac{1}{1+x^2}\left|\mathcal{G}_{n,\lambda}^{q_n}(t;x)-x\right|$$
$$=\sup_{x\in\mathbb{R}^+}\frac{x}{1+x^2}\left|\frac{[n]_{q_n}}{[n-\lambda]_{q_n}}-1\right|$$
$$\le\left|\frac{[n]_{q_n}}{[n-\lambda]_{q_n}}-1\right|\to 0,\, n\to\infty.$$

$$\|\mathcal{G}_{n,\lambda}^{q_n}(t^2;x)-x^2\|_2=\sup_{x\in\mathbb{R}^+}\frac{1}{1+x^2}\left|\mathcal{G}_{n,\lambda}^{q_n}(t^2;x)-x^2\right|$$
$$=\sup_{x\in\mathbb{R}^+}\frac{x^2}{1+x^2}\left|\frac{[n]_{q_n}^2}{[n-\lambda]_{q_n}[n-\lambda-1]_{q_n}}-1\right|$$
$$\le\left|\frac{[n]_{q_n}^2}{[n-\lambda]_{q_n}[n-\lambda-1]_{q_n}}-1\right|\to 0,\, n\to\infty. \tag{51}$$

We can draw the final conclusion through all the estimates above.

Theorem 12. *Let $q=(q_n)$ be a sequence satisfying $q_n\in(0,1)$, $q_n\to 1$ and $q_n^n\to 1$ as $n\to\infty$ and $f\in C_2^0(\mathbb{R}^+)$. For any $f\in C_2^0(\mathbb{R}^+)$ and $v>0$, we have*

$$\lim_{n\to\infty}\sup_{x\in\mathbb{R}^+}\frac{|\mathcal{G}_n^{p_n,q_n}(f;x)-f(x)|}{(1+x^2)^{1+v}}=0. \tag{52}$$

Proof. Let $x_0\in\mathbb{R}^+$ be arbitrary but fixed. Then,

$$\sup_{x\in\mathbb{R}^+}\frac{|\mathcal{G}_{n,\lambda}^{q_n}(f;x)-f(x)|}{(1+x^2)^{1+v}}\le\sup_{x\in(0,x_0)}\frac{|\mathcal{G}_{n,\lambda}^{q_n}(f;x)-f(x)|}{(1+x^2)^{1+v}}+\sup_{x\in[x_0,\infty)}\frac{|\mathcal{G}_{n,\lambda}^{q_n}(f;x)-f(x)|}{(1+x^2)^{1+v}}$$
$$\le\|\mathcal{G}_{n,\lambda}^{q_n}(f;x)-f\|_{C(0,x_0]}+\|f\|_2\sup_{x\in[x_0,\infty)}\frac{|\mathcal{G}_{n,\lambda}^{q_n}((1+t^2);x)|}{(1+x^2)^{1+v}}+\sup_{x\in[x_0,\infty)}\frac{|f(x)|}{(1+x^2)^{1+v}}. \tag{53}$$

Since $|f(x)|\le\|f\|_2(1+x^2)$, we have $\sup_{x\in[x_0,\infty)}((|f(x)|)/(1+x^2)^{1+v})\le((\|f\|_2)/(1+x_0^2)^v)$. Let $\varepsilon>0$ be arbitrary, we can choose x_0 to be so large that

$$\frac{\|f\|_2}{(1+x_0^2)^v}<\varepsilon. \tag{54}$$

In view of Corollary 9, while $x\in[x_0,\infty)$, we obtain

$$\|f\|_2\lim_{n\to\infty}\frac{|\mathcal{G}_{n,\lambda}^{q_n}((1+t^2);x)|}{(1+x^2)^{1+v}}=\|f\|_2\frac{(1+x^2)}{(1+x^2)^{1+v}}=\frac{\|f\|_2}{(1+x^2)^v}\le\frac{\|f\|_2}{(1+x_0^2)^v}<\varepsilon. \tag{55}$$

Using Theorem 9, we can see that the first term of the inequality (53) implies that

$$\|\mathcal{G}_{n,\lambda}^{q_n}(f;x)-f\|_{C(0,x_0]}<\varepsilon,\quad \text{as}\quad n\to\infty. \tag{56}$$

Combining (53)–(56), we get the desired result.

6. Voronovskaja Type Theorems

As is known, Voronovskaja type theorems of many positive operators are widely researched and discussed (see [21–28]). In this section, we will discuss the quantitative q-Voronovskaja theorem and q-Grüss-Voronovskaja theorem.

6.1. Quantitative q-Voronovskaja Theorem. In this subsection, we will obtain the Quantitative q-Voronovskaja theorem and Voronovskaja type asymptotic formula for the operators $\mathcal{G}_{n,\lambda}^{q_n}$.

Theorem 13. *Let $q=(q_n)$ be a sequence satisfying $q_n\in(0,1)$, $q_n\to 1$ and $q_n^n\to a\in[0,1]$ as $n\to\infty$ and $f\in C_2^0(\mathbb{R}^+)$ satisfy $D_{q_n}^2 f\in C_2^0(\mathbb{R}^+)$. Then, the inequality*

$$\left|[n]_{q_n}\left(\mathcal{G}_{n,\lambda}^{q_n}(f;x)-f(x)-D_{q_n}f(x)A_n(x)\right)-\frac{[n]_{q_n}B_n(x)+(1-q_n^n)A_n(x)x}{[2]_{q_n}!}D_{q_n}^2 f(x)\right|\le O(1)\Omega\left(D_{q_n}^2 f;\frac{1}{\sqrt{[n]_{q_n}}}\right), \tag{57}$$

holds for any $x\in\mathbb{R}^+$.

Proof. Using the q-Taylor expansion formula (58), we have

$$f(t)=f(x)+D_{q_n}f(x)(t-x)+\frac{D_{q_n}^2 f(\xi)}{[2]_{q_n}!}(t-x)_{q_n}^2$$
$$=f(x)+D_{q_n}f(x)(t-x)+\frac{D_{q_n}^2 f(x)}{[2]_{q_n}!}(t-x)_{q_n}^2+R_2(t,x;q_n), \tag{58}$$

where ξ is a number between t and x and

$$R_2(t,x;q_n)=\frac{D_{q_n}^2 f(\xi)-D_{q_n}^2 f(x)}{[2]_{q_n}!}(t-x)_{q_n}^2. \tag{59}$$

Applying the operators $\mathcal{G}_{n,\lambda}^{q_n}$ to both sides of (58) and using $(t-x)_{q_n}^2=(t-x)^2+((1-q_n^n)/([n]_{q_n}))(t-x)x$, we have

$$\begin{aligned}&\left|\mathcal{G}_{n,\lambda}^{q_n}(f;x)-f(x)-D_{q_n}f(x)A_n(x)-\frac{D_{q_n}^2f(x)}{[2]_{q_n}!}\mathcal{G}_{n,\lambda}^{q_n}\left((t-x)_{q_n}^2;x\right)\right|\\&=\left|\mathcal{G}_{n,\lambda}^{q_n}(f;x)-f(x)-D_{q_n}f(x)A_n(x)\right.\\&\quad\left.-\frac{B_n(x)+\left(((1-q_n^n)A_n(x))/\left([n]_{q_n}\right)\right)x}{[2]_{q_n}!}D_{q_n}^2f(x)\right|\\&\le\mathcal{G}_{n,\lambda}^{q_n}(|R_2(t,x;q_n)|;x).\end{aligned}\tag{60}$$

Multiplying the above inequality by $[n]_{q_n}$, we have

$$\begin{aligned}&\left|[n]_{q_n}\left(\mathcal{G}_{n,\lambda}^{q_n}(f;x)-f(x)-D_{q_n}f(x)A_n(x)\right)\right.\\&\quad\left.-\frac{[n]_{q_n}B_n(x)+(1-q_n^n)A_n(x)x}{[2]_{q_n}!}D_{q_n}^2f(x)\right|\\&\le[n]_{q_n}\mathcal{G}_{n,\lambda}^{q_n}(|R_2(t,x;q_n)|;x).\end{aligned}\tag{61}$$

Furthermore,

$$\begin{aligned}\left|\frac{D_{q_n}^2f(\xi)-D_{q_n}^2f(x)}{[2]_{q_n}!}\right|&\le\frac{1}{[2]_{q_n}!}\Omega\left(D_{q_n}^2f;|\xi-x|\left(1+(\xi-x)^2\right)\left(1+x^2\right)\right)\\&\le\frac{1}{[2]_{q_n}!}\Omega\left(D_{q_n}^2f;|t-x|\left(1+(t-x)^2\right)\left(1+x^2\right)\right)\\&\le\frac{2}{[2]_{q_n}!}\left(1+\frac{|t-x|}{\delta}\right)\left(1+(t-x)^2\right)\\&\quad\cdot\left(1+\delta^2\right)\left(1+x^2\right)\Omega\left(D_{q_n}^2f;\delta\right)\\&\le16\left(1+x^2\right)\left(1+\frac{(t-x)^4}{\delta^4}\right)\Omega\left(D_{q_n}^2f;\delta\right),\end{aligned}\tag{62}$$

for all $\delta\in(0,1)$. Hence,

$$|R_2(t,x;q_n)|\le16\left(1+x^2\right)\left(|(t-x)_{q_n}^2|+\frac{|(t-x)_{q_n}^2|(t-x)^4}{\delta^4}\right)\Omega\left(D_{q_n}^2f;\delta\right).\tag{63}$$

Using (20), (21), for any $x\in\mathbb{R}^+$, we can write

$$\begin{aligned}\mathcal{G}_{n,\lambda}^{q_n}(|R_2(t,x;q_n)|;x)&\le16\left(1+x^2\right)\left(\mathcal{G}_{n,\lambda}^{q_n}\left(|(t-x)_{q_n}^2|;x\right)\right.\\&\quad\left.+\frac{\mathcal{G}_{n,\lambda}^{q_n}\left(|(t-x)_{q_n}^2|(t-x)^4;x\right)}{\delta^4}\right)\Omega\left(D_{q_n}^2f;\delta\right)\\&\le\left(O\left(\frac{1}{[n]_{q_n}}\right)+\frac{1}{\delta^4}O\left(\frac{1}{[n]_{q_n}^3}\right)\right)\Omega\left(D_{q_n}^2f;\delta\right).\end{aligned}\tag{64}$$

If we choose $\delta=1/\sqrt{[n]_{q_n}}$, we can easily get

$$[n]_{q_n}\mathcal{G}_{n,\lambda}^{q_n}(|R_2(t,x;q_n)|;x)\le O(1)\Omega\left(D_{q_n}^2f;\frac{1}{\sqrt{[n]_{q_n}}}\right),\tag{65}$$

which completes the proof of Theorem 13.

Corollary 14. *Let (q_n) be a sequence satisfying $q_n\in(0,1)$, $q_n\to1$ and $q_n^n\to a\in[0,1]$ as $n\to\infty$ and $f\in C_2^0(\mathbb{R}^+)$ satisfy $f''\in C_2^0(\mathbb{R}^+)$. Then, we can obtain*

$$\lim_{n\to\infty}[n]_{q_n}\left(\mathcal{G}_{n,\lambda}^{q_n}(f;x)-f(x)\right)=\lambda axf'(x)+\frac{1}{2}ax^2f''(x).\tag{66}$$

6.2. q-Grüss-Voronovskaja Theorem. In this subsection, we will obtain the q-Grüss-Voronovskaja theorem and its quantitative version for the operators $\mathcal{G}_{n,\lambda}^{q_n}$.

Theorem 15. *Let $q=(q_n)$ be a sequence satisfying $q_n\in(0,1)$, $q_n\to1$ and $q_n^n\to a\in[0,1]$ as $n\to\infty$ and $f,g\in C_2^0(\mathbb{R}^+)$ satisfy $D_{q_n}^2f,D_{q_n}^2g,D_{q_n}^2(fg)\in C_2^0(\mathbb{R}^+)$. Then, the following inequality*

$$\begin{aligned}&[n]_{q_n}\left|\mathcal{G}_{n,\lambda}^{q_n}(fg;x)-\mathcal{G}_{n,\lambda}^{q_n}(f;x)\mathcal{G}_{n,\lambda}^{q_n}(g;x)\right.\\&\quad-\frac{D_{q_n}(f(q_nx))\left(D_{q_n}(g(x))+D_{q_n}(g(q_nx))\right)}{[2]_{q_n}!}\mathcal{G}_{n,\lambda}^{q_n}\\&\quad\left.\cdot\left((t-x)_{q_n}^2;x\right)\right|\le O(1)\Omega\left(D_{q_n}^2(fg);\frac{1}{\sqrt{[n]_{q_n}}}\right)\\&\quad+O(1)\left(\|f\|_2+O\left(\frac{1}{[n]_{q_n}}\right)\left(\|D_{q_n}f\|_2+\|D_{q_n}^2f\|_2\right)\right)\Omega\\&\quad\cdot\left(D_{q_n}^2g;\frac{1}{\sqrt{[n]_{q_n}}}\right)+O(1)\left(\|g\|_2+O\left(\frac{1}{[n]_{q_n}}\right)\right.\\&\quad\left.\cdot\left(\|D_{q_n}g\|_2+\|D_{q_n}^2g\|_2\right)\right)\Omega\left(D_{q_n}^2f;\frac{1}{\sqrt{[n]_{q_n}}}\right)\\&\quad+O\left(\frac{1}{[n]_{q_n}}\right)\left(\|D_{q_n}f\|_2+\|D_{q_n}^2f\|_2\right)\left(\|D_{q_n}g\|_2+\|D_{q_n}^2g\|_2\right)\\&\quad+O\left(\frac{1}{[n]_{q_n}}\right)\Omega\left(D_{q_n}^2f;\frac{1}{\sqrt{[n]_{q_n}}}\right)\Omega\left(D_{q_n}^2g;\frac{1}{\sqrt{[n]_{q_n}}}\right),\end{aligned}\tag{67}$$

holds for any $x\in\mathbb{R}^+$.

Proof. Using the equalities

$$\begin{aligned}D_{q_n}(f(x)g(x))&=D_{q_n}(f(x))g(x)+f(q_nx)D_{q_n}(g(x)),\\D_{q_n}^2(f(x)g(x))&=D_{q_n}^2(f(x))g(x)+D_{q_n}(f(q_nx))D_{q_n}(g(x))\\&\quad+f(q_nx)D_{q_n}^2(g(x))+D_{q_n}(f(q_nx))D_{q_n}(g(q_nx)),\end{aligned}\tag{68}$$

by simple computations, for $x \in \mathbb{R}^+$ and $n = \lambda + 1, \cdots$, we can obtain

$$\begin{aligned}
&\mathscr{G}_{n,\lambda}^{q_n}(fg;x) - \mathscr{G}_{n,\lambda}^{q_n}(f;x)\mathscr{G}_{n,\lambda}^{q_n}(g;x) = \mathscr{G}_{n,\lambda}^{q_n}(fg;x) \\
&\quad - f(x)g(x) - \mathscr{G}_{n,\lambda}^{q_n}(t-x;x)D_{q_n}(f(x)g(x)) \\
&\quad - \frac{\mathscr{G}_{n,\lambda}^{q_n}\left((t-x)_{q_n}^2;x\right)}{[2]_{q_n}!}D_{q_n}^2(f(x)g(x)) \\
&\quad - g(x)\left(\mathscr{G}_{n,\lambda}^{q_n}(f;x) - f(x) - \mathscr{G}_{n,\lambda}^{q_n}(t-x;x)D_{q_n}(f(x))\right. \\
&\quad \left. - \frac{\mathscr{G}_{n,\lambda}^{q_n}\left((t-x)_{q_n}^2;x\right)}{[2]_{q_n}!}D_{q_n}^2(f(x))\right) \\
&\quad - \mathscr{G}_{n,\lambda}^{q_n}(f;x)\left(\mathscr{G}_{n,\lambda}^{q_n}(g;x) - g(x) - \mathscr{G}_{n,\lambda}^{q_n}(t-x;x)D_{q_n}(g(x))\right. \\
&\quad \left. - \frac{\mathscr{G}_{n,\lambda}^{q_n}\left((t-x)_{q_n}^2;x\right)}{[2]_{q_n}!}D_{q_n}^2(g(x))\right) \\
&\quad + \frac{D_{q_n}(f(q_nx))\left(D_{q_n}(g(x)) + D_{q_n}(g(q_nx))\right)}{[2]_{q_n}!}\mathscr{G}_{n,\lambda}^{q_n}\left((t-x)_{q_n}^2;x\right) \\
&\quad - D_{q_n}^2(g(x))\left(\mathscr{G}_{n,\lambda}^{q_n}(f;x) - f(x)\right)\frac{\mathscr{G}_{n,\lambda}^{q_n}\left((t-x)_{q_n}^2;x\right)}{[2]_{q_n}!} \\
&\quad + D_{q_n}^2(g(x))D_{q_n}(f(x))\frac{\mathscr{G}_{n,\lambda}^{q_n}\left((t-x)_{q_n}^2;x\right)}{[2]_{q_n}!}(q_n-1)x \\
&\quad - \mathscr{G}_{n,\lambda}^{q_n}(t-x;x)\left(\mathscr{G}_{n,\lambda}^{q_n}(f;x) - f(q_nx)\right)D_{q_n}(g(x)).
\end{aligned} \tag{69}$$

Hence, we can write

$$\begin{aligned}
&\mathscr{G}_{n,\lambda}^{q_n}(fg;x) - \mathscr{G}_{n,\lambda}^{q_n}(f;x)\mathscr{G}_{n,\lambda}^{q_n}(g;x) \\
&\quad - \frac{D_{q_n}(f(q_nx))\left(D_{q_n}(g(x)) + D_{q_n}(g(q_nx))\right)}{[2]_{q_n}!}\mathscr{G}_{n,\lambda}^{q_n}\left((t-x)_{q_n}^2;x\right) \\
&\quad = \mathscr{G}_{n,\lambda}^{q_n}(fg;x) - f(x)g(x) \\
&\quad - \mathscr{G}_{n,\lambda}^{q_n}(t-x;x)D_{q_n}(f(x)g(x)) \\
&\quad - \frac{\mathscr{G}_{n,\lambda}^{q_n}\left((t-x)_{q_n}^2;x\right)}{[2]_{q_n}!}D_{q_n}^2(f(x)g(x)) \\
&\quad - g(x)\left(\mathscr{G}_{n,\lambda}^{q_n}(f;x) - f(x) - \mathscr{G}_{n,\lambda}^{q_n}(t-x;x)D_{q_n}(f(x))\right. \\
&\quad \left. - \frac{\mathscr{G}_{n,\lambda}^{q_n}\left((t-x)_{q_n}^2;x\right)}{[2]_{q_n}!}D_{q_n}^2(f(x))\right) \\
&\quad - f(x)\left(\mathscr{G}_{n,\lambda}^{q_n}(g;x) - g(x) - \mathscr{G}_{n,\lambda}^{q_n}(t-x;x)D_{q_n}(g(x))\right. \\
&\quad \left. - \frac{\mathscr{G}_{n,\lambda}^{q_n}\left((t-x)_{q_n}^2;x\right)}{[2]_{q_n}!}D_{q_n}^2(g(x))\right) + (f(x) \\
&\quad - \mathscr{G}_{n,\lambda}^{q_n}(f;x))\left(\mathscr{G}_{n,\lambda}^{q_n}(g;x) - g(x)\right) \\
&\quad + \mathscr{G}_{n,\lambda}^{q_n}(t-x;x)(q_n-1)xD_{q_n}(f(x))D_{q_n}(g(x)) \\
&\quad + \frac{\mathscr{G}_{n,\lambda}^{q_n}\left((t-x)_{q_n}^2;x\right)}{[2]_{q_n}!}(q_n-1)xD_{q_n}f(x)D_{q_n}^2g(x) \coloneqq I_1 \\
&\quad + I_2 + I_3 + I_4 + I_5 + I_6.
\end{aligned} \tag{70}$$

By Theorem 13, for any fixed $x \in \mathbb{R}^+$, we can easily have the following estimates

$$[n]_{q_n}|I_1| \le O(1)\Omega\left(D_{q_n}^2(fg);\frac{1}{\sqrt{[n]_{q_n}}}\right), \tag{71}$$

$$\begin{aligned}
[n]_{q_n}|I_2| &\le |g(x)|O(1)\Omega\left(D_{q_n}^2(f);\frac{1}{\sqrt{[n]_{q_n}}}\right) \\
&\le \|g\|_2 O(1)\Omega\left(D_{q_n}^2(f);\frac{1}{\sqrt{[n]_{q_n}}}\right),
\end{aligned} \tag{72}$$

$$\begin{aligned}
[n]_{q_n}|I_3| &\le |f(x)|O(1)\Omega\left(D_{q_n}^2(g);\frac{1}{\sqrt{[n]_{q_n}}}\right) \\
&\le \|f\|_2 O(1)\Omega\left(D_{q_n}^2(g);\frac{1}{\sqrt{[n]_{q_n}}}\right).
\end{aligned} \tag{73}$$

Using (14), (20), and $|q_n - 1| = (|q_n^n - 1|/[n]_{q_n}) \le O(1/[n]_{q_n})$, we have

$$\begin{aligned}
[n]_{q_n}|I_5| &\le O\left(\frac{1}{[n]_{q_n}}\right)\|D_{q_n}f\|_2\|D_{q_n}g\|_2, \\
[n]_{q_n}|I_6| &\le O\left(\frac{1}{[n]_{q_n}}\right)\|D_{q_n}f\|_2\|D_{q_n}^2g\|_2.
\end{aligned} \tag{74}$$

Using (14), (20), and Theorem 13, we can get

$$\begin{aligned}
\left|\mathscr{G}_{n,\lambda}^{q_n}(f;x) - f(x)\right| &\le O\left(\frac{1}{[n]_{q_n}}\right)\left(\|D_{q_n}f\|_2 + \|D_{q_n}^2f\|_2\right) \\
&\quad + O\left(\frac{1}{[n]_{q_n}}\right)\Omega\left(D_{q_n}^2(f);\frac{1}{\sqrt{[n]_{q_n}}}\right),
\end{aligned} \tag{75}$$

hence, we can know

$$\begin{aligned}
[n]_{q_n}|I_4| &\le O\left(\frac{1}{[n]_{q_n}}\right)\left(\left(\|D_{q_n}f\|_2 + \|D_{q_n}^2f\|_2\right)\left(\|D_{q_n}g\|_2 + \|D_{q_n}^2g\|_2\right)\right) \\
&\quad + O\left(\frac{1}{[n]_{q_n}}\right)\Omega\left(D_{q_n}^2(f);\frac{1}{\sqrt{[n]_{q_n}}}\right)\left(\|D_{q_n}g\|_2 + \|D_{q_n}^2g\|_2\right) \\
&\quad + O\left(\frac{1}{[n]_{q_n}}\right)\Omega\left(D_{q_n}^2(g);\frac{1}{\sqrt{[n]_{q_n}}}\right)\left(\|D_{q_n}f\|_2 + \|D_{q_n}^2f\|_2\right) \\
&\quad + O\left(\frac{1}{[n]_{q_n}}\right)\Omega\left(D_{q_n}^2(f);\frac{1}{\sqrt{[n]_{q_n}}}\right)\Omega\left(D_{q_n}^2(g);\frac{1}{\sqrt{[n]_{q_n}}}\right).
\end{aligned} \tag{76}$$

Combining (71)–(76), we complete the proof of Theorem 15.

Corollary 16. *Let $q=(q_n)$ be a sequence satisfying $q_n \in (0,1)$, $q_n \rightarrow 1$ and $q_n^n \rightarrow a \in [0,1]$ as $n \rightarrow \infty$ and $f, g \in C_2^0(\mathbb{R}^+)$ satisfy $f'', g'', (fg)'' \in C_2^0(\mathbb{R}^+)$. Then, the following limit equality*

$$\lim_{n\to\infty} [n]_{q_n} \left(\mathscr{G}_{n,\lambda}^{q_n}(fg;x) - \mathscr{G}_{n,\lambda}^{q_n}(f;x)\mathscr{G}_{n,\lambda}^{q_n}(g;x) \right) = af'(x)g'(x)x^2, \quad (77)$$

holds for any $x \in \mathbb{R}^+$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All authors read and approved the final manuscript.

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