Research Article

Existence and Compactness Results for a System of Fractional Differential Equations

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The existence and uniqueness, boundedness, and continuous dependence of solutions for fractional differential equations with Caputo fractional derivative is proven by Perov's fixed point theorem in vector Banach spaces. We study the existence and compactness of solution sets and the u.s.c. of operator solutions.

1. Introduction

In the past twenty years, the fractional differential equation has aroused great consideration not only in its application in mathematics but also in other applications in physics, engineering, finance, fluid mechanics, viscoelastic mechanics, electroanalytical chemistry, and biological and other sciences [1–7].

In recent decades, the Riemann-Liouville, Caputo, and Hadamard fractional calculus are paid more attention; see the monographs [5, 8–13].

Applied problems requiring definitions of fractional derivatives are those that are physically interpretable for initial conditions containing $y(0)$, $y'(0)$, etc. The same requirements are true for boundary conditions. Caputo's fractional derivative satisfies these demands. For more details on the geometric and physical interpretation for fractional derivatives of both the Riemann-Liouville type and the Caputo type, see Podlubny [12] and Diethelm [14].

The theory of fractional differential equations and inclusions has been extensively studied and developed by many authors; see [15–21] and the references therein.

Perov in 1964 [22] and Perov and Kibenko [23] extended the classical Banach contraction principle for contractive maps on space endowed with a vector-valued metric. Later, they attempted to generalize the Perov fixed point theorem in several directions which has a number of applications in various fields of nonlinear analysis, semilinear differential equations, and system of ordinary differential equations.

In [24], Dezideriu and Precup studied the following system of semilinear equations

$$\begin{cases} A_1 x = F^1(x, y) \\ A_2 y = F^2(x, y) \end{cases} \quad (1)$$

where $A_1, A_2 : D(A) \subset X \longrightarrow X$ are linear operators and $F^1, F^2 : J \times X \times X \longrightarrow X$ are nonlinear operators.

Precup, in [25], explained the advantage of vector-valued norms and the role of matrices that are convergent to zero in the study of semilinear operator systems.

Many authors studied the existence of solutions for a system of differential equations and impulsive differential

equations by using the vector version fixed point theorem; their results are given in [26–30].

Our goal of this paper is to treat the systems of fractional differential equations. More precisely, we will consider the following problem:

$$\begin{cases} {}^c D^\alpha x(t) = f(t, x(t), y(t)), \\ {}^c D^\beta y(t) = g(t, x(t), y(t)), \quad t \in J, \\ x(0) = x_0, \\ y(0) = y_0, \end{cases} \quad (2)$$

where ${}^c D^\alpha$ and ${}^c D^\beta$ are the Caputo fractional derivatives, $\alpha, \beta \in (0, 1]$, $J = [0, \infty)$, $f, g : J \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are given functions, and $x_0, y_0 \in \mathbb{R}$.

In the case where $\alpha = \beta = 1$, the above system was used to analyze initial value problems and boundary value problems for nonlinear competitive or cooperative differential systems from mathematical biology [31] and mathematical economics [32] which can be set in the operator from ((2)).

The plan of this paper is as follows: in Section 2, we introduce all the background material used in this paper such as some properties of generalized Banach spaces, fixed point theory, and fractional calculus theory. In Section 3, we state and prove our main results by using Perov's fixed point type theorem in generalized Banach spaces. By the Leray-Schauder fixed point in vector Banach space, we prove the existence and compactness of solution sets of the above problems.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

Definition 1 [22]. Let X be a nonempty set. The mapping $d : X \times X \rightarrow \mathbb{R}_+^m$ which satisfies all the usual axioms of the metric is called a generalized metric in Perov's sense and (X, d) is called a generalized metric space.

In a generalized metric space in Perov's sense, the concepts of Cauchy sequence, convergent sequence, completeness, and open and closed subsets are similarly defined as those for usual metric space.

If $v, r \in \mathbb{R}^m$, $v := (v_1, v_2, \dots, v_m)$ and $r := (r_1, r_2, \dots, r_m)$, then by $v \leq r$, we mean $v_i \leq r_i$ for each $i \in \{1, \dots, m\}$, and by $v < r$, we mean $v_i < r_i$ for each $i \in \{1, \dots, m\}$. Also $|v| := (|v_1|, \dots, |v_m|)$ and $\max(u, v) := (\max(u_1, v_1), \dots, \max(u_m, v_m))$. If $c \in \mathbb{R}$, then $v \leq c$ means $v_i \leq c$ for each $i \in \{1, \dots, m\}$. Denote by

$$B(x_0, r) = \{x \in X : d(x_0, x) < r\}, \quad (3)$$

the open ball centered in x_0 with radius r , and

$$\overline{B(x_0, r)} = \{x \in X : d(x_0, x) \leq r\}, \quad (4)$$

the closed ball centered in x_0 with radius r .

Definition 2. A square matrix M of real numbers is said to be convergent to zero if and only if $M^n \rightarrow 0$ as $n \rightarrow \infty$.

Lemma 3 [33]. Let $M \in \mathcal{M}_{m,m}(\mathbb{R}_+)$. The following statements are equivalent:

- (i) M is a matrix convergent to zero
- (ii) The eigenvalues of M are in open disc, i.e., $|\mu| < 1$, for every $\mu \in \mathbb{C}$ with $\det(M - \mu I) = 0$
- (iii) The matrix $I - M$ is nonsingular and $(I - M)^{-1} = I + M + \dots + M^n + \dots$
- (iv) The matrix $I - M$ is nonsingular and $(I - M)^{-1}$ has nonnegative elements
- (v) $M^n q \rightarrow 0$ and ${}^t q M^n \rightarrow 0$ as $n \rightarrow \infty$, for any $q \in \mathbb{R}^m$

Example 4. Some examples of matrix convergent to zero are

$$1) \quad M = \begin{pmatrix} a & b \\ a & b \end{pmatrix}, \quad (5)$$

where $a, b \in \mathbb{R}_+$ and $a + b < 1$,

$$2) \quad M = \begin{pmatrix} a & a \\ b & b \end{pmatrix}, \quad (6)$$

where $a, b \in \mathbb{R}_+$ and $a + b < 1$, and

$$3) \quad M = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, \quad (7)$$

where $a, b, c \in \mathbb{R}_+$ and $\max\{a, c\} < 1$.

Definition 5 [34]. Let (X, d) be a generalized metric space. An operator $N : X \rightarrow X$ is said to be contractive if there exists a convergent to zero matrix M such that

$$d(N(x), N(y)) \leq M d(x, y) \quad \text{for all } x, y \in X. \quad (8)$$

Notice now that the Banach fixed point theorem can be extended to generalized metric spaces in the sense of Perov.

Theorem 6 [22, 28]. Let (X, d) be a complete generalized metric space and $N : X \rightarrow X$ be a contractive operator with Lipschitz matrix M . Then, N has a unique fixed point x^* , and for each $x_0 \in X$, we have

$$d(N^k(x_0), x^*) \leq M^k (I - M)^{-1} d(x_0, N(x_0)) \quad \text{for all } k \in \mathbb{N}. \quad (9)$$

We recall now the following Leray-Schauder type theorem.

Theorem 7 [28, 35]. Let X be a generalized Banach space and let $N : X \longrightarrow X$ be a completely continuous operator. Then, either

- (i) the equation $N(x) = x$ has at least one solution, or
- (ii) the set $\mathcal{M} = \{x \in X \mid \mu N(x) = x, \mu \in (0, 1)\}$ is unbounded

We will use the following notations. Let (X, d) and (Y, ρ) be two metric spaces and $G : X \longrightarrow \mathcal{P}(Y)$.

$$\begin{aligned} \mathcal{P}(X) &= \{Y \subset X : Y = \emptyset\}, \mathcal{P}_d(X) \\ &= \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}, \mathcal{P}_b(X) \\ &= \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}. \end{aligned} \quad (10)$$

Definition 8 [28, 36]. A multivalued map $\varphi : X \longrightarrow \mathcal{P}(Y)$ is called upper semicontinuous (u.s.c.) at a point $x_0 \in X$ provided that for every open subset $V \subset Y$ with $V \supseteq \varphi(x_0)$, there exists $U \in \mathcal{V}(x_0)$ such that

$$\forall x \in U, \varphi(x) \subset V. \quad (11)$$

φ is called upper semicontinuous if it is u.s.c. at every point $x \in X$.

The mapping G is said to be completely continuous if it is u.s.c., and for every bounded subset $C \subseteq X$, $G(C)$ is relatively compact, i.e., there exists a relatively compact set $K = K \subset X$ such that

$$G(C) = \cup \{G(x) : x \in C\} \subset K. \quad (12)$$

Also, G is compact if $G(X)$ is relatively compact, and it is called *locally compact* if for each $x \in X$, there exists an open set W containing x , such that $G(W)$ is relatively compact.

Theorem 9 [36]. Let $G : X \longrightarrow \mathcal{P}_{cp}(Y)$ be a closed locally compact multifunction. Then, G is u.s.c.

Now, we recall some notations and definitions of fractional calculus theory.

Definition 10 [5]. The Riemann-Liouville fractional integral of the function $h \in L^1([0, T], \mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by

$$I^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds, \quad (13)$$

where Γ is the Euler gamma function defined by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$, $\alpha > 0$.

Definition 11 [5]. For a function $h \in AC^n(J, \mathbb{R})$, the Caputo fractional-order derivative of order α of h is defined by

$$({}^c D_0^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} h^{(n)}(s) ds, \quad (14)$$

where $n = [\alpha] + 1$.

We recall Gronwall's lemma for singular kernels, whose proof can be found in Lemma 7.1.1 of [37].

Lemma 12. Let $v : [0, b) \longrightarrow [0, \infty)$ be a real function; $w(\cdot)$ is a nonnegative, locally integrable function on $[0, b)$ (some $b \leq +\infty$); $a(t)$ is a nonnegative, nondecreasing continuous function defined on $0 \leq t < b$; $a(t) \leq M$ (constant); and suppose $v(t)$ is nonnegative and locally integrable on $0 \leq t < b$. Assume $\gamma > 0$ such that

$$v(t) \leq w(t) + a(t) \int_0^t \frac{v(s)}{(t-s)^{1-\gamma}} ds, \quad (15)$$

then

$$v(t) \leq w(t) + \int_0^t \sum_{n=1}^{\infty} \frac{(a(s)\Gamma(\gamma))^n}{\Gamma(n\gamma)} (t-s)^{n\gamma-1} w(s) ds, \quad (16)$$

for every $t \in [0, b)$.

3. Existence, Uniqueness, and Bounded Solutions

In order to define a solution for problem (2), consider the following functional spaces. Let $J = [0, \infty)$ and $C(J, \mathbb{R})$ be the space of all continuous functions from J into $\mathbb{R}$.

$$C_b = \{y \in C(J, \mathbb{R}) : y \text{ is bounded}\}. \quad (17)$$

C_b is a Banach with norm

$$\|y\|_{C_b} = \sup \{|y(t)| : t \in J\}. \quad (18)$$

We need the following auxiliary result.

Lemma 13 [14]. Concerning the problem,

$$\begin{cases} {}^c D^\alpha x(t) = f(t, x), & 0 < \alpha < 1, \\ x(0) = x_0 \in \mathbb{R}, \end{cases} \quad (19)$$

where the function $f : \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous. The function $x : \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}$ is the unique solution of the problem (19) if and only if

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s)) ds, \quad t \in \mathbb{R}_+. \quad (20)$$

Definition 14. A function $x, y \in C_b$ is said to be a solution of (2) if and only if

$$\begin{cases} x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), y(s)) ds, \\ y(t) = y_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s, x(s), y(s)) ds, \\ t \in J. \end{cases} \quad (21)$$

In this section, we assume the following conditions.

(H1). There exists functions $h_{i,\alpha}, h_{j,\beta} \in L^1(J, \mathbb{R}^+)$, $i = 1, 2$, $j = 3, 4$ such that

$$\begin{aligned} |f(s, x, y) - f(s, \bar{x}, \bar{y})| &\leq h_{i,\alpha}(s)|x - \bar{x}| + h_{2,\alpha}(s)|y - \bar{y}| \quad \text{for all } x, \bar{x}, y, \bar{y} \in \mathbb{R}, \\ |g(s, x, y) - g(s, \bar{x}, \bar{y})| &\leq h_{3,\beta}(s)|x - \bar{x}| + h_{4,\beta}(s)|y - \bar{y}| \quad \text{for all } x, \bar{x}, y, \bar{y} \in \mathbb{R}, \end{aligned} \quad (22)$$

where

$$\begin{aligned} \|h_{i,\alpha}\|_\infty &= \sup_{t \in J} \int_0^t \theta^\alpha(t-s) h_{i,\alpha}(s) ds < \infty, \quad i = 1, 2, \\ \|h_{j,\beta}\|_\infty &= \sup_{t \in J} \int_0^t \theta^\beta(t-s) h_{j,\beta}(s) ds < \infty, \quad j = 3, 4, \end{aligned} \quad (23)$$

where for $\gamma = \alpha, \beta$,

$$\theta_\gamma(t) = \begin{cases} t^{\gamma-1}, & t > 0, \\ 0, & t \leq 0. \end{cases} \quad (24)$$

(H2). The functions $f_\alpha, g_\beta : J \rightarrow \mathbb{R}$ are defined by

$$\begin{aligned} f_\alpha(t) &= \theta_\alpha(t-s) f(s, 0, 0), \quad g_\beta(t) \\ &= \theta_\beta(t-s) g(s, 0, 0), \quad t \in J, \end{aligned} \quad (25)$$

satisfies

$$\begin{aligned} \|f_\alpha\|_\infty &= \sup_{t \in J} \int_0^t \theta_\alpha(t-s) |f(s, 0, 0)| ds < \infty, \\ \|g_\beta\|_\infty &= \sup_{t \in J} \int_0^t \theta_\beta(t-s) |g(s, 0, 0)| ds < \infty. \end{aligned} \quad (26)$$

Now, we are in a position to prove our existence and uniqueness solution for the problem (2) using the Perov fixed point theorem and show that for each initial condition (x_0, y_0) , the solution is bounded.

Theorem 15. Assume that (H1)-(H2) are satisfied. If the matrix

$$M = \begin{pmatrix} \frac{\|h_{1,\alpha}\|_\infty}{\Gamma(\alpha)} & \frac{\|h_{2,\alpha}\|_\infty}{\Gamma(\alpha)} \\ \frac{\|h_{3,\beta}\|_\infty}{\Gamma(\beta)} & \frac{\|h_{4,\beta}\|_\infty}{\Gamma(\beta)} \end{pmatrix} \in \mathcal{M}_{2 \times 2}(\mathbb{R}^+), \quad (27)$$

converges to zero. Then, the problem (2) has a unique bounded solution.

Proof. Transform the problem (2) into a fixed point theorem of the operator $N : C_b \times C_b \rightarrow C_b \times C_b$ defined by $N(x, y) = (N_1(x, y), N_2(x, y))$, where

$$\begin{aligned} N_1(x, y)(t) &= x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), y(s)) ds, \\ N_2(x, y)(t) &= y_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s, x(s), y(s)) ds, \\ t &\in [0, \infty). \end{aligned} \quad (28)$$

First, we show that the operator N is well-defined. Let $(x, y) \in C_b \times C_b$ and $t \in [0, \infty)$, then we have

$$\begin{aligned} |N_1(x, y)(t)| &\leq |x_0| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, x(s), y(s))| ds \\ &\leq |x_0| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\ &\quad \cdot |f(s, x(s), y(s)) - f(s, 0, 0)| ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{|f(s, 0, 0)|}{(t-s)^{1-\alpha}} ds. \end{aligned} \quad (29)$$

Then,

$$\begin{aligned} \|N_1(x, y)\|_{C_b} &\leq |x_0| + \frac{\|h_{1,\alpha}\|_\infty}{\Gamma(\alpha)} \|x\|_{C_b} \\ &\quad + \frac{\|h_{2,\alpha}\|_\infty}{\Gamma(\alpha)} \|y\|_{C_b} + \frac{\|f_\alpha\|_\infty}{\Gamma(\alpha)}, \\ \|N_2(x, y)\|_{C_b} &\leq |y_0| + \frac{\|h_{3,\beta}\|_\infty}{\Gamma(\beta)} \|x\|_{C_b} \\ &\quad + \frac{\|h_{4,\beta}\|_\infty}{\Gamma(\beta)} \|y\|_{C_b} + \frac{\|g_\beta\|_\infty}{\Gamma(\beta)}. \end{aligned} \quad (30)$$

Hence, the operator N is well-defined.

Clearly, the fixed points of operator N are solutions of problem (2). Now, we show that N is a contraction. For all $(x, y), (\bar{x}, \bar{y}) \in C_b \times C_b$, we have

$$\begin{aligned}
& |N_1(x, y)(t) - N_1(\bar{x}, \bar{y})(t)| \\
& \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, x(s), y(s)) - f(s, \bar{x}(s), \bar{y}(s))| ds \\
& \leq \frac{1}{\Gamma(\alpha)} \int_0^t [h_{1,\alpha}(s)|x(s) - \bar{x}(s)| + h_{2,\alpha}(s)|y(s) - \bar{y}(s)|] ds.
\end{aligned} \tag{31}$$

Then,

$$\begin{aligned}
& \|N_1(x, y) - N_1(\bar{x}, \bar{y})\|_{C_b} \\
& \leq \frac{\|h_{1,\alpha}\|_\infty}{\Gamma(\alpha)} \|x - \bar{x}\|_{C_b} + \frac{\|h_{2,\alpha}\|_\infty}{\Gamma(\alpha)} \|y - \bar{y}\|_{C_b}.
\end{aligned} \tag{32}$$

Similarly, we have

$$\begin{aligned}
& \|N_2(x, y) - N_2(\bar{x}, \bar{y})\|_{C_b} \\
& \leq \frac{\|h_{3,\beta}\|_\infty}{\Gamma(\beta)} \|x - \bar{x}\|_{C_b} + \frac{\|h_{4,\beta}\|_\infty}{\Gamma(\beta)} \|y - \bar{y}\|_{C_b}.
\end{aligned} \tag{33}$$

Therefore,

$$\|N(x, y) - N(\bar{x}, \bar{y})\|_{C_b \times C_b} \leq M \begin{pmatrix} \|x - \bar{x}\|_{C_b} \\ \|y - \bar{y}\|_{C_b} \end{pmatrix}. \tag{34}$$

According to Theorem 6, we deduce that the operator N has unique fixed point which is a solution of problem (2). Now, we will prove that the solution (x, y) of problem (2) is bounded. For all $t \in [0, \infty)$, we have

$$\begin{aligned}
|x(t)| & \leq |x_0| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, x(s), y(s))| ds \\
& \leq |x_0| + \frac{1}{\Gamma(\alpha)} \int_0^t [(h_{1,\alpha}(s)|x(s)| + h_{2,\alpha}(s)|y(s)|)] ds \\
& \quad + \frac{1}{\Gamma(\alpha)} \int_0^t |f_\alpha(s, 0, 0)| ds \\
& \leq |x_0| + \frac{1}{\Gamma(\alpha)} \int_0^t [(h_{1,\alpha}(s)|x(s)| + h_{2,\alpha}(s)|y(s)|)] ds + \frac{\|f_\alpha\|_\infty}{\Gamma(\alpha)}, \\
|y(t)| & \leq |y_0| + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} |g(s, x(s), y(s))| ds \\
& \leq |y_0| + \frac{1}{\Gamma(\beta)} \int_0^t [(h_{3,\beta}(s)|x(s)| + h_{4,\beta}(s)|y(s)|)] ds \\
& \quad + \frac{1}{\Gamma(\beta)} \int_0^t |g_\beta(s, 0, 0)| ds \\
& \leq |y_0| + \frac{1}{\Gamma(\beta)} \int_0^t [(h_{3,\beta}(s)|x(s)| + h_{4,\beta}(s)|y(s)|)] ds + \frac{\|g_\beta\|_\infty}{\Gamma(\beta)}.
\end{aligned} \tag{35}$$

Therefore,

$$\begin{aligned}
\|x\|_{C_b} & \leq |x_0| + \frac{\|h_{1,\alpha}\|_\infty}{\Gamma(\alpha)} \|x\|_{C_b} + \frac{\|h_{2,\alpha}\|_\infty}{\Gamma(\alpha)} \|y\|_{C_b} + \frac{\|f_\alpha\|_\infty}{\Gamma(\alpha)}, \\
\|y\|_{C_b} & \leq |y_0| + \frac{\|h_{3,\beta}\|_\infty}{\Gamma(\beta)} \|x\|_{C_b} + \frac{\|h_{4,\beta}\|_\infty}{\Gamma(\beta)} \|y\|_{C_b} + \frac{\|g_\beta\|_\infty}{\Gamma(\beta)}.
\end{aligned} \tag{36}$$

Hence,

$$\begin{pmatrix} \|x\|_{C_b} \\ \|y\|_{C_b} \end{pmatrix} \leq M \begin{pmatrix} \|x\|_{C_b} \\ \|y\|_{C_b} \end{pmatrix} + \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}, \tag{37}$$

where

$$\sigma_1 = |x_0| + \frac{\|f_\alpha\|_\infty}{\Gamma(\alpha)}, \quad \sigma_2 = |y_0| + \frac{\|g_\beta\|_\infty}{\Gamma(\beta)}. \tag{38}$$

Then,

$$\begin{pmatrix} \|x\|_{C_b} \\ \|y\|_{C_b} \end{pmatrix} \leq (I - M)^{-1} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}. \tag{39}$$

From (H1) and (H2), we deduce that the solution (x, y) is bounded.

For the next result, we prove the continuous dependence of solutions on initial conditions.

Theorem 16. Assume that (H1) and (H2) hold. If $f(t, 0, 0) = g(t, 0, 0) = 0$, $t \in J$ and the matrix M defined in (27) converges to zero.

For every $(x_0, y_0) \in \mathbb{R} \times \mathbb{R}$, we denoted by $(x(t, x_0), y(t, y_0))$ the solution of problem (2). Then, the map $(x_0, y_0) \longrightarrow (x(\cdot, x_0), y(\cdot, y_0))$ is continuous.

Proof. From Theorem 15, for each initial condition (x_0, y_0) , $(\bar{x}_0, \bar{y}_0) \in \mathbb{R} \times \mathbb{R}$, there exists unique solution $(x(\cdot, x_0), y(\cdot, y_0))$, $(\bar{x}(\cdot, \bar{x}_0), \bar{y}(\cdot, \bar{y}_0))$, then we get

$$\begin{cases} x(t, x_0) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), y(s)) ds, \\ y(t, y_0) = y_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s, x(s), y(s)) ds, \\ \bar{x}(t, \bar{x}_0) = \bar{x}_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \bar{x}(s), \bar{y}(s)) ds, \\ \bar{y}(t, \bar{y}_0) = \bar{y}_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s, \bar{x}(s), \bar{y}(s)) ds. \end{cases} \tag{40}$$

Therefore,

$$\begin{aligned} |x(t, x_0) - \bar{x}(t, \bar{x}_0)| &\leq |x_0 - \bar{x}_0| + \frac{1}{\Gamma(\alpha)} \int_0^t [(h_{1,\alpha}(s)|x(s) - \bar{x}(s)| \\ &\quad + h_{2,\alpha}(s)|y(s) - \bar{y}(s)|)] ds, \\ |y(t, y_0) - \bar{y}(t, \bar{y}_0)| &\leq |y_0 - \bar{y}_0| + \frac{1}{\Gamma(\beta)} \int_0^t [(h_{3,\beta}(s)|x(s) - \bar{x}(s)| \\ &\quad + h_{4,\beta}(s)|y(s) - \bar{y}(s)|)] ds. \end{aligned} \quad (41)$$

Hence,

$$\begin{aligned} \left(\begin{array}{l} \|x(\cdot, x_0) - \bar{x}(\cdot, \bar{x}_0)\|_{C_b} \\ \|y(\cdot, y_0) - \bar{y}(\cdot, \bar{y}_0)\|_{C_b} \end{array} \right) &\leq (I - M)^{-1} \left(\begin{array}{l} |x_0 - \bar{x}_0| \\ |y_0 - \bar{y}_0| \end{array} \right) \\ &\longrightarrow 0, \quad \text{as } (x_0, y_0) \longrightarrow (\bar{x}_0, \bar{y}_0). \end{aligned} \quad (42)$$

4. Existence and Compactness of Solution Sets

For the existence and compactness result of problem (2), we consider the following Banach space:

$$BC_b(J, \mathbb{R}) = \left\{ u \in C(J, \mathbb{R}): \sup_{t \in J} \frac{|u(t)|}{1+t} < \infty \right\}, \quad (43)$$

with norm

$$\|u\|_B = \sup_{t \in J} \frac{|u(t)|}{1+t}. \quad (44)$$

It is evident that $BC_b(J, \mathbb{R})$ is a Banach space. The following compactness criterion on unbounded domains is called Corduneanu compactness criterion in which the proof is easy and similar to the classical one in $C_b(\mathbb{R}^+, \mathbb{R})$ (see [38]).

Lemma 17. *Let $H \subset BC_b(\mathbb{R}^+, \mathbb{R})$. Then, H is relatively compact if the following conditions hold:*

- (a) H is uniformly bounded in $BC_b(\mathbb{R}^+, \mathbb{R}^n)$
- (b) The functions belonging to H are almost equicontinuous on $\mathbb{R}^+$, i.e., for all $I \subset \mathbb{R}_+$ compact interval, for any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that for every $t_1, t_2 \in I$ with $|t_1 - t_2| < \delta(\epsilon)$, we have for all $u \in H$

$$\left| \frac{u(t_2)}{1+t_2} - \frac{u(t_1)}{1+t_1} \right| < \epsilon \quad (45)$$

- (c) The functions from H are equiconvergent, that is, given $\epsilon > 0$, there corresponds $T(\epsilon) > 0$ such that

$$\left| \frac{u(t_1)}{1+t_1} - \frac{u(t_2)}{1+t_2} \right| < \epsilon \quad \text{for any } t_1, t_2 \geq T(\epsilon) \text{ and } u \in H \quad (46)$$

In the sequel of this section, we will consider the following assumption.

(H3). For every $l > 0$, the functions $(x, y) \longrightarrow f(t, x, y)$, $g(t, x, y)$ are uniformly continuous on the sets $[-l, l] \times [-l, l]$ uniformly with respect to $t \in \mathbb{R}_+$, i.e., f and g satisfied the following condition: for all $l > 0$, $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that for all $(x, y), (\bar{x}, \bar{y}) \in [-l, l] \times [-l, l]$ and for all $t \in \mathbb{R}_+$ with $|x/(1+t) - \bar{x}/(1+t)| < \delta(\epsilon)$, $|y/(1+t) - \bar{y}/(1+t)| < \delta(\epsilon)$, we have

$$|f(t, x, y) - f(t, \bar{x}, \bar{y})| < \epsilon, |g(t, x, y) - g(t, \bar{x}, \bar{y})| < \epsilon. \quad (47)$$

(H4). There exist $a_i, b_i, c_i > 0$, $i = 1, 2$ such that

$$\begin{aligned} |f(t, x, y)| &\leq a_1 \frac{|x|}{1+t} + b_1 \frac{|y|}{1+t} + c_1, \\ |g(t, x, y)| &\leq a_2 \frac{|x|}{1+t} + b_2 \frac{|y|}{1+t} + c_2, \\ &\forall x, y \in \mathbb{R}, t \in \mathbb{R}_+. \end{aligned} \quad (48)$$

Theorem 18. *Assume that (H3) and (H4) hold. Then, the problem (2) has at least one bounded solution. Moreover, the solution set*

$$\begin{aligned} S(x_0, y_0) &= \{(x, y) \in BC_b(J, \mathbb{R}) \\ &\quad \times BC_b(J, \mathbb{R}): (x, y) \text{ is solution of (2)}\}, \end{aligned} \quad (49)$$

is compact and the multivalued map $S : (x_0, y_0) \longrightarrow S(x_0, y_0)$ is u.s.c.

Proof. Let $N = (N_1, N_2)$ is defined in the proof of Theorem 15.

Step 1. N is well defined. Let $(x, y) \in BC_b(J, \mathbb{R}) \times BC_b(J, \mathbb{R})$, then

$$\begin{aligned} \frac{|N_1(x(t), y(t))|}{1+l} &\leq \frac{|x_0|}{1+l} + \frac{1}{(1+l)\Gamma(\alpha)} \int_0^t \\ &\quad \cdot (t-s)^{\alpha-1} |f(s, x(s), y(s))| ds \\ &\leq \frac{|x_0|}{1+l} \frac{1}{(1+l)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\ &\quad \cdot \left(a_1 \frac{|x(s)|}{1+s} + b_1 \frac{|y(s)|}{1+s} + c_1 \right) ds \\ &\leq \frac{|x_0|}{1+l} + \frac{c_1 t^\alpha}{\Gamma(\alpha+1)(1+t)} \\ &\quad + \frac{1}{(1+t)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\ &\quad \cdot \left(a_1 \frac{|x(s)|}{1+s} + b_1 \frac{|y(s)|}{1+s} \right) ds \\ &\leq |x_0| + \frac{c_1}{\Gamma(\alpha+1)} + \frac{1}{(1+t)\Gamma(\alpha)} \int_0^t \\ &\quad \cdot (t-s)^{\alpha-1} ds (a_1 \|x\|_B + b_1 \|y\|_B). \end{aligned} \quad (50)$$

Hence,

$$\begin{aligned} \sup_{t \in J} \frac{|N_1(x(t), y(t))|}{1+t} &\leq |x_0| + \frac{c_1}{\Gamma(\alpha+1)} + \frac{a_1 \|x\|_B + b_1 \|y\|_B}{\Gamma(\alpha+1)}, \\ \sup_{t \in J} \frac{|N_2(x(t), y(t))|}{1+t} &\leq |y_0| + \frac{c_1}{\Gamma(\beta+1)} + \frac{a_2 \|x\|_B + b_2 \|y\|_B}{\Gamma(\beta+1)}. \end{aligned} \quad (51)$$

Step 2. N is continuous.

Let $(x_n, y_n) \longrightarrow (x, y)$ in $BC_b(J, \mathbb{R}) \times BC_b(J, \mathbb{R})$. Then, there exists $l > 0$ such that for any $t \in \mathbb{R}_+$ and $n \in \mathbb{N}$, we have

$$\frac{x_n(t)}{1+t}, \frac{y_n(t)}{1+t}, \frac{x(t)}{1+t}, \frac{y(t)}{1+t} \in [-l, l]. \quad (52)$$

By (H3), for every $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that for all $x, y, \bar{x}, \bar{y} \in [-l, l]$ and for all $t \in \mathbb{R}_+$ with $|x/(1+t) - \bar{x}/(1+t)| < \delta(\epsilon)$, $|y/(1+t) - \bar{y}/(1+t)| < \delta(\epsilon)$ we have

$$\begin{aligned} |f(t, x, y) - f(t, \bar{x}, \bar{y})| \\ < \epsilon \Gamma(\alpha+1), |g(t, x, y) - g(t, \bar{x}, \bar{y})| < \epsilon \Gamma(\beta+1). \end{aligned} \quad (53)$$

Since (x_n, y_n) converge to (x, y) , then there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$,

$$\left| \frac{x_n(t)}{1+t} - \frac{x(t)}{1+t} \right| < \delta(\epsilon), \left| \frac{y_n(t)}{1+t} - \frac{y(t)}{1+t} \right| < \delta(\epsilon), \quad \forall t \in \mathbb{R}_+. \quad (54)$$

Hence,

$$\begin{aligned} \left| \frac{N_1(x_n, y_n)(t)}{1+t} - \frac{N_1(x, y)(t)}{1+t} \right| \\ \leq \frac{1}{\Gamma(\alpha)} \int_0^t \frac{(t-s)^{\alpha-1}}{1+t} |f(s, x_n(s), y_n(s)) - f(s, x(s), y(s))| ds \\ \leq \frac{1}{\Gamma(\alpha)} \int_0^t \frac{(t-s)^{\alpha-1}}{1+t} ds \Gamma(\alpha+1) \epsilon \leq \frac{t^\alpha}{1+t} \epsilon \leq \epsilon. \end{aligned} \quad (55)$$

Thus

$$\begin{aligned} \|N_1(x_n, y_n) - N_1(x, y)\|_B &\longrightarrow 0, \\ \|N_2(x_n, y_n) - N_2(x, y)\|_B &\longrightarrow 0, \end{aligned} \quad (56)$$

as $n \longrightarrow \infty$.

Step 3. We will show that N maps bounded sets into bounded sets in $BC_b(J, \mathbb{R}) \times BC_b(J, \mathbb{R})$.

Let $B_r := \{(x, y) \in BC_b(J, \mathbb{R}) \times BC_b(J, \mathbb{R}) : \|(x, y)\|_\infty \leq r\}$, where $r = (r_1, r_2)$ and if $(x, y) \in B_r$, then we obtain

$$\begin{aligned} \left| \frac{N_1(x(t), y(t))}{1+t} \right| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t \frac{(t-s)^{\alpha-1}}{1+t} |f(s, x(s), y(s))| ds \\ &\leq \frac{1}{\Gamma(\alpha)} [a_1 r_1 + b_1 r_2 + c_1]. \end{aligned} \quad (57)$$

Similarly, we have

$$\left| \frac{N_2(x(t), y(t))}{1+t} \right| \leq \frac{1}{\Gamma(\beta)} [a_2 r_1 + r_2 b_2 + c_2]. \quad (58)$$

Hence,

$$(\|N_1(x, y)\|, \|N_2(x, y)\|) \leq (l_1, l_2), \quad (59)$$

where

$$l_1 = \frac{1}{\Gamma(\alpha)} [a_1 r_1 + b_1 r_2 + c_1], l_2 = \frac{1}{\Gamma(\beta)} [a_2 r_1 + r_2 b_2 + c_2]. \quad (60)$$

Step 4. Now, we prove that N maps bounded sets in $BC_b(J, \mathbb{R}) \times BC_b(J, \mathbb{R})$ into almost equicontinuous sets of $BC_b(J, \mathbb{R}) \times BC_b(J, \mathbb{R})$.

Let $(x, y) \in B_r$. Then, for all $t_1 < t_2$,

$$\begin{aligned} \left| \frac{N_1(x(t_2), y(t_2))}{1+t_2} - \frac{N_1(x(t_1), y(t_1))}{1+t_1} \right| \\ \leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left| \frac{(t_2-s)^{\alpha-1}}{1+t_2} - \frac{(t_1-s)^{\alpha-1}}{1+t_1} \right| |f(s, x(s), y(s))| ds \\ + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{1+t_2} |f(s, x(s), y(s))| ds \\ \leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left| \frac{(t_2-s)^{\alpha-1}}{1+t_2} - \frac{(t_1-s)^{\alpha-1}}{1+t_1} \right| (a_1 r_1 + b_1 r_2 + c_1) ds \\ + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{1+t_2} (a_1 r_1 + b_1 r_2 + c_1) ds \\ \leq \frac{a_1 r_1 + b_1 r_2 + c_1}{\Gamma(\alpha+1)} \left(\left| \frac{t_2^\alpha}{1+t_2} - \frac{t_1^\alpha}{1+t_1} \right| + \frac{2(t_2 - t_1)^\alpha}{1+t_2} \right). \end{aligned} \quad (61)$$

Thus,

$$\left| \frac{N_1(x(t_2), y(t_2))}{1+t_2} - \frac{N_1(x(t_1), y(t_1))}{1+t_1} \right| \longrightarrow 0 \quad \text{as } t_1 \longrightarrow t_2. \quad (62)$$

Similarly, we have

$$\left| \frac{N_2(x(t_2), y(t_2))}{1+t_2} - \frac{N_2(x(t_1), y(t_1))}{1+t_1} \right| \longrightarrow 0 \quad \text{as } t_1 \longrightarrow t_2. \quad (63)$$

Step 5. The set $N(\bar{B}(0, r))$ is equiconvergent, i.e., for every $\epsilon > 0$, there exists $T(\epsilon) > 0$ such that $|N(x(t_2), y(t_2)) - N(x(t_1), y(t_1))| < \epsilon$, for every $t_1, t_2 > T(\epsilon)$ and each $(x, y) \in \bar{B}(0, r)$.

$$\begin{aligned} & \left| \frac{N_1(x(t_2), y(t_2))}{1+t_2} - \frac{N_1(x(t_1), y(t_1))}{1+t_1} \right| \\ & \leq \frac{a_1 r_1 + b_1 r_2 + c_1}{\Gamma(\alpha+1)} \left(\frac{3t_2^\alpha}{1+t_2} + \frac{t_1^\alpha}{1+t_1} \right), \\ & \left| \frac{N_2(x(t_2), y(t_2))}{1+t_2} - \frac{N_2(x(t_1), y(t_1))}{1+t_1} \right| \\ & \leq \frac{a_2 r_1 + b_1 r_2 + c_1}{\Gamma(\alpha+1)} \left(\frac{3t_2^\beta}{1+t_2} + \frac{t_1^\beta}{1+t_1} \right). \end{aligned} \quad (64)$$

It is clear that

$$\frac{3t_2^\beta}{1+t_2} \longrightarrow 0, \quad \frac{t_1^\beta}{1+t_1} \longrightarrow 0, \quad \text{as } t_2, t_1 \longrightarrow \infty. \quad (65)$$

Then, for any $\epsilon > 0$, there exists $T(\epsilon) > 0$ such that for all $t_1, t_2 > T(\epsilon)$, we have

$$\begin{aligned} & \left| \frac{N_1(x(t_2), y(t_2))}{1+t_2} - \frac{N_1(x(t_1), y(t_1))}{1+t_1} \right| \\ & \leq \epsilon, \quad \text{for all } (x, y) \in \bar{B}(0, r), \\ & \left| \frac{N_2(x(t_2), y(t_2))}{1+t_2} - \frac{N_2(x(t_1), y(t_1))}{1+t_1} \right| \\ & \leq \epsilon, \quad \text{for any } (x, y) \in \bar{B}(0, r). \end{aligned} \quad (66)$$

Step 6. The set is defined as follows:

$\mathcal{M} = \{(x, y) \in \text{BC}_b(J, \mathbb{R}) \times \text{BC}_b(J, \mathbb{R}) : (x, y) = \mu N(x, y), \mu \in (0, 1)\}$ is bounded. Let $(x, y) \in \mathcal{M}$, then

$$\begin{aligned} \frac{|x(t)|}{1+t} & \leq \frac{|x_0|}{1+t} + \frac{\mu}{(1+t)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, x(s), y(s))| ds \\ & \leq \frac{|x_0|}{1+t} + \frac{\mu}{(1+t)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\ & \quad \cdot \left(a_1 \frac{|x(s)|}{1+s} + b_1 \frac{|y(s)|}{1+s} + c_1 \right) ds \end{aligned}$$

$$\begin{aligned} & \leq \frac{|x_0|}{1+t} + \frac{c_1 t^\alpha}{\Gamma(\alpha+1)(1+t)} \\ & \quad + \frac{1}{(1+t)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\ & \quad \cdot \left(a_1 \frac{|x(s)|}{1+s} + b_1 \frac{|y(s)|}{1+s} \right) ds. \end{aligned} \quad (67)$$

Similarly, we get

$$\begin{aligned} & \frac{|y(t)|}{1+t} + \frac{|y_0|}{1+t} + \frac{c_2 t^\beta}{\Gamma(\beta+1)(1+t)} \\ & \quad + \frac{1}{(1+t)\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \left(a_2 \frac{|x(s)|}{1+s} + b_2 \frac{|y(s)|}{1+s} \right) ds. \end{aligned} \quad (68)$$

Thus,

$$\begin{aligned} \frac{|x(t)|}{1+t} + \frac{|y(t)|}{1+t} & \leq \frac{X_0}{1+t} + \frac{Ct^\gamma}{1+t} + \frac{C_*}{1+t} \int_0^t (t-s)^{\gamma-1} \\ & \quad \cdot \left(\frac{|x(s)|}{1+s} + b_2 \frac{|y(s)|}{1+s} \right) ds, \end{aligned} \quad (69)$$

where

$$\begin{aligned} X_0 & = |x_0| + |y_0|, \\ C & = \frac{c_1}{\Gamma(\alpha+1)} + \frac{c_2}{\Gamma(\beta+1)}, \\ C_* & = \frac{a_1 + b_1}{\Gamma(\alpha)} + \frac{a_2 + b_2}{\Gamma(\beta)}, \\ \gamma & = \max(\alpha, \beta). \end{aligned} \quad (70)$$

Hence,

$$\begin{aligned} \frac{|x(t)|}{1+t} + \frac{|y(t)|}{1+t} & \leq X_0 + C + \frac{C_*}{1+t} \int_0^t (t-s)^{\gamma-1} \\ & \quad \cdot \left(\frac{|x(s)|}{1+s} + \frac{|y(s)|}{1+s} \right) ds. \end{aligned} \quad (71)$$

By the Gronwall-Bellman Lemma 12, we have

$$\begin{aligned} \frac{|x(t)|}{1+t} + \frac{|y(t)|}{1+t} & \leq X_0 + C + \int_0^t \sum_{n=1}^{\infty} \frac{(C_* \Gamma(\gamma))^n}{\Gamma(n\gamma)} (t-s)^{n\gamma-1} ds \\ & \leq X_0 + C + \sum_{n=1}^{\infty} \frac{(t^\gamma C_* \Gamma(\gamma))^n}{(1+t)^n \Gamma(n\gamma+1)} (X_0 + C) \\ & \leq X_0 + C + \sum_{n=0}^{\infty} \frac{(C_* \Gamma(\gamma))^n}{\Gamma(n\gamma+1)} (X_0 + C). \end{aligned} \quad (72)$$

Then, for every $t \in [0, \infty)$, we have

$$\frac{|x(t)|}{1+t} + \frac{|y(t)|}{1+t} \leq (X_0 + C)(1 + E_\gamma(C_* \Gamma(\gamma)) := M, \quad (73)$$

where E_γ is the Mittag-Leffler function defined by

$$E_\gamma(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\gamma + 1)}. \quad (74)$$

Then,

$$\|x\|_{C_b} \leq M, \|y\|_{C_b} \leq M. \quad (75)$$

According to Theorem 7, problem (2) has at least one solution.

Now, we show that the set

$$S(x_0, y_0) = \{(x, y) \in BC_b(J, \mathbb{R}) \times BC_b(J, \mathbb{R}) : (x, y) \text{ is solution of (2)}\}, \quad (76)$$

is compact. It is clear that $S(x_0, y_0) \subset N(S(x_0, y_0))$. From (75), we deduced that $S(x_0, y_0)$ is bounded sets in $BC_b(J, \mathbb{R})$. Since N is compact, then $S(x_0, y_0)$ is compact if and only if $S(x_0, y_0)$ is closed. Let $\{(x_n, y_n)\}_{n \geq 1} \subset S(x_0, y_0)$ be a sequence converge to (x, y) . Thus,

$$\begin{aligned} x_n(t) &= x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x_n(s), y_n(s)) ds, \\ y_n(t) &= y_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s, x_n(s), y_n(s)) ds, \end{aligned} \quad (77)$$

$t \in J$.

Similarly to Step 2, we can prove that

$$\begin{aligned} x(t) &= x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), y(s)) ds, \\ y(t) &= y_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s, x(s), y(s)) ds, \end{aligned} \quad (78)$$

$t \in J$.

This implies that $S(x_0, y_0)$ is compact.

The solution operator S is u.s.c.

(a) S has a closed graph:

To see this, first note that the graph of S is the set

$$G_S = \{((a, b), (x, y)) \in (\mathbb{R} \times \mathbb{R}) \times (BC_b(J, \mathbb{R}) \times BC_b(J, \mathbb{R})) \mid (x, y) \in S(a, b)\}. \quad (79)$$

Let $((a_q, b_q), (x_q, y_q))_q$ be a sequence in G_S , and let $((a_q, b_q), (x_q, y_q))_q \longrightarrow ((a, b), (x, y))$ as $q \longrightarrow \infty$.

Since $(x_q, y_q) \in S(a_q, b_q)$, then we have

$$\begin{aligned} x_q(t) &= a_q + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x_q(s), y_q(s)) ds, \\ y_q(t) &= b_q + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s, x_q(s), y_q(s)) ds, \end{aligned} \quad (80)$$

$t \in J$.

Let

$$\begin{aligned} Z(t) = (Z_1(t), Z_2(t)) &= \left(a + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), y(s)) ds, b \right. \\ &\quad \left. + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s, x(s), y(s)) ds \right) \quad t \in J. \end{aligned} \quad (81)$$

As in Step 2, we can prove that

$$(x_q, y_q) \longrightarrow (x, y) \quad \text{as } q \longrightarrow \infty. \quad (82)$$

(b) Using the same method as Steps 3 to 4, we find that for each bounded set $C \subset BC_b(J, \mathbb{R}) \times BC_b(J, \mathbb{R})$, we can show that $S(C)$ is compact.

From (a) and (b), we concluded that, S is u.s.c.

5. Applications

In this section, we show the applicability of our main result. We begin by illustrating by Theorem 15.

Example 19. Consider the problem:

$$\begin{cases} {}^c D^{1/2} x = h_1(t)(1 + k_1 \sin |x| + k_2 \cos |x|), \\ {}^c D^{1/2} y = h_2(t)(1 + k_3 |x| + k_4 |y|)^{1/2}, \quad t \in \mathbb{R}, \\ x(0) = x_0, \\ y(0) = y_0, \end{cases} \quad (83)$$

where

$$\begin{aligned} h_1(t) &= \frac{\sqrt{y} \int_0^t s e^{-s^2} ds}{100(1+t^2)} \text{ and } h_2(s) \\ &= \frac{t\sqrt{t}}{200(1+t^2)}, \quad t \in \mathbb{R}_+, k_i \geq 0, i = 1, 2, 3, 4, \end{aligned}$$

$$\begin{aligned} f(t, x, y) &= h_1(t)(1 + k_1 \sin |x| + k_2 \cos |y|), \quad g(t, x, y) \\ &= h_2(t)(1 + k_3 |x| + k_4 |y|). \end{aligned}$$

(84)

It is clear that

$$\begin{aligned}
 \int_0^t \frac{h_1(s)}{(t-s)^{1/2}} ds &= \int_0^t \frac{\sqrt{s} \int_0^s r e^{-r^2} dr}{100(1+s^2)(t-s)^{1/2}} \\
 &\leq \frac{\int_0^\infty e^{-r^2} dr}{100(1+s^2)(t-s)^{1/2}} \int_0^t \frac{s\sqrt{s}}{(1+s^2)(t-s)^{1/2}} ds \\
 &= \frac{\int_0^\infty e^{-r^2} dr}{100} \int_0^t \frac{t^2 l \sqrt{l}}{(1+(tl)^2)(1-l)^{1/2}} dl \\
 &\leq \frac{\int_0^\infty e^{-r^2} dr}{100} \int_0^t (1-l)^{-(1/2)} dl \\
 &\leq \frac{\int_0^\infty e^{-r^2} dr}{50} = \frac{\sqrt{\pi}}{100}, \\
 \int_0^t \frac{h_2(s)}{(t-s)^{1/2}} ds &= \int_0^t \frac{s\sqrt{s}}{200(1+s^2)(t-s)^{1/2}} ds \leq \frac{1}{100}.
 \end{aligned} \tag{85}$$

Then,

$$\sup_{t \in \mathbb{R}_+} \int_0^t \frac{h_i(s)}{(t-s)^{1/2}} ds < \infty, \quad i = 1, 2. \tag{86}$$

Observe that every $x, y, \bar{x}, \bar{y} \in \mathbb{R}$ and $t \in \mathbb{R}_+$, we have

$$\begin{aligned}
 |f(t, x, y) - f(t, \bar{x}, \bar{y})| &\leq k_1 h_1(t) |x - \bar{x}| + k_2 h_1(t) |y - \bar{y}|, \\
 |g(t, x, y) - g(t, \bar{x}, \bar{y})| &\leq k_3 h_2(t) |x - \bar{x}| + k_4 h_2(t) |y - \bar{y}|.
 \end{aligned} \tag{87}$$

Hence, the condition (H1) holds.

We see that

$$|f(t, 0, 0)| = h_1(t), \quad |g(t, 0, 0)| = h_2(t), \quad t \in \mathbb{R}_+. \tag{88}$$

Hence, the condition (H2) holds. Assume that $k_1 k_4 - k_2 k_3 \neq 0$. For this, we have

$$\tilde{M} = \begin{pmatrix} \frac{k_1 \sqrt{\pi}}{\Gamma(1/2)100} & \frac{k_2 \sqrt{\pi}}{\Gamma(1/2)100} \\ \frac{k_3 \sqrt{\pi}}{\Gamma(1/2)100} & \frac{k_4 \sqrt{\pi}}{\Gamma(1/2)100} \end{pmatrix}. \tag{89}$$

$\det \tilde{M} \neq 0$. If we add that $k_2 = 0$ or k_3 and $0 < k_1 < (\Gamma(1/2)100)/\sqrt{\pi}$, $0 < k_4 < \Gamma(1/2)100 \leq 1$. The $\rho(\tilde{M}) < 1$. By Theorem 15, it follows that problem (83) has a unique solution.

In this last example, we illustrate the applicability of Theorem 18.

Lemma 20. Let us consider the function $f : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f(t, x, y) = a(t)\phi(x, y), \tag{90}$$

where $a : \mathbb{R}_+ \rightarrow \mathbb{R}$, $\phi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ satisfy the following assumptions:

(S1) $a \in C_b(\mathbb{R})$.

(S2) The function ϕ is continuous and bounded. Then, the function

$$f_*(t, x, y) = a(t)\phi\left(\frac{x}{1+t}, \frac{y}{1+t}\right), \quad t \in \mathbb{R}_+, (x, y) \in \mathbb{R} \times \mathbb{R}, \tag{91}$$

satisfied the condition (H3).

Proof. Since a is bounded, then there exists $A > 0$ such that

$$|a(t)| < A, \quad \forall t \in \mathbb{R}_+. \tag{92}$$

Let $l > 0$, then for any $(x, y), (\bar{x}, \bar{y}) \in [-l, l] \times [-l, l]$, we obtain

$$|f(t, x, y) - f(t, \bar{x}, \bar{y})| = |a(t)| |\phi(x, y) - \phi(\bar{x}, \bar{y})|. \tag{93}$$

Hence,

$$|f(t, x, y) - f(t, \bar{x}, \bar{y})| \leq A |\phi(x, y) - \phi(\bar{x}, \bar{y})|. \tag{94}$$

It is clear that from (S2) ϕ is uniformly continuous on the $[-l, l] \times [-l, l]$. Thus, for any $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that for all $(x, y), (\bar{x}, \bar{y}) \in [-l, l] \times [-l, l]$ with $|x - \bar{x}| < \delta(\epsilon)$, $|y - \bar{y}| < \delta(\epsilon)$ then,

$$|f(t, x, y) - f(t, \bar{x}, \bar{y})| \leq A\epsilon. \tag{95}$$

This implies that for every $t \in \mathbb{R}_+$ such that for all $(x, y), (\bar{x}, \bar{y}) \in [-l, l] \times [-l, l]$ with $|x/(1+t) - \bar{x}/(1+t)| < \delta(\epsilon)$, $|y/(1+t) - \bar{y}/(1+t)| < \delta(\epsilon)$, we get

$$|f(t, x, y) - f(t, \bar{x}, \bar{y})| \leq A\epsilon. \tag{96}$$

Theorem 21. Let $a_1, a_2 : \mathbb{R}_+ \rightarrow \mathbb{R}$, $\phi_1, \phi_2 : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be functions that satisfy the conditions (S1) and (S2).

Assume

(S3) There exists $d_i > 0, i = 1, 4$ such that

$$\begin{aligned}
 |\phi_1(x, y)| &\leq d_1|x| + d_2|y| + d_3, \quad \forall x, y \in \mathbb{R}, t \in \mathbb{R}_+, \\
 |\phi_2(x, y)| &\leq d_4|x| + d_5|y| + d_6, \quad \forall x, y \in \mathbb{R}, t \in \mathbb{R}_+.
 \end{aligned} \tag{97}$$

Then, the following problem

$$\begin{cases} {}^c D^\alpha x(t) = f(t, x, (t)), \\ {}^c D^\beta x(t) = g(t, x, (t)), \quad t \in J, \\ x(0) = x_0, \\ y(0) = y_0, \end{cases} \tag{98}$$

where

$$\begin{aligned} f(t, x, y) &= a_1(t)\phi_1\left(\frac{x}{1+t}, \frac{y}{1+t}\right), \quad t \in \mathbb{R}_+, x, y \in \mathbb{R}, \\ g(t, x, y) &= a_2(t)\phi_2\left(\frac{x}{1+t}, \frac{y}{1+t}\right), \quad t \in \mathbb{R}_+, x, y \in \mathbb{R}. \end{aligned} \quad (99)$$

Proof. From Lemma 20, we deduce that f and g satisfies (H3). By (S1) and (S3), there exist $A_1, A_2 > 0$ such that

$$\begin{aligned} f(t, x, y) &\leq A_1 \left(\frac{|x|}{1+t}, \frac{|y|}{1+t} \right), \\ g(t, x, y) &\leq A_2 \left(\frac{x}{1+t}, \frac{y}{1+t} \right), \quad t \in \mathbb{R}_+, x, y \in \mathbb{R}. \end{aligned} \quad (100)$$

Therefore, all the conditions of Theorem 18 hold. Then, problem (98) has at least one solution in BC_b .

Remark 22. For another application, we can replace the condition (S2) with (S4). The functions φ_1 and φ_2 are uniformly continuous and bounded.

6. Conclusion

In this paper, we investigated a system of fractional differential equations under various assumptions on the right-hand-side nonlinearity and we obtain a number of results regarding the existence and uniqueness of solutions in an appropriate space of continuous functions. In this paper, we have focused on the dependence continuity of a solution, compactness of solution sets, and upper semicontinuity of operator solutions. We hope this paper can provide some contribution to the questions of existence and topological structure for the system of fractional differential equations on unbounded domains.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest.

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