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Research Article

On the Exact Solutions of Two (3+1)-Dimensional Nonlinear Differential Equations

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In this article, exact solutions of two (3+1)-dimensional nonlinear differential equations are derived by using the complex method. We change the (3+1)-dimensional B-type Kadomtsev-Petviashvili (BKP) equation and generalized shallow water (gSW) equation into the complex differential equations by applying traveling wave transform and show that meromorphic solutions of these complex differential equations belong to class W, and then, we get exact solutions of these two (3+1)-dimensional equations.

1. Introduction and Main Results

Nonlinear differential equations (NLDEs) play an important role in the research of nonlinear science, which has attracted a lot of attentions of the researchers [1–8]. The investigation of NLDEs is helpful for well understanding of nonlinear physical phenomena [9–16]. Numerous methods have been developed for seeking traveling wave exact solutions to NLDEs, such as sine-Gordon expansion method [17], Kudryashov method [18], modified simple equation method [19], Jacobi elliptic function expansion [20], $\exp(-\psi(z))$-expansion method [21, 22], modified extended tanh method [23, 24], generalized (G'/G) expansion method [25], and improved F-expansion method [26].

In recent years, Yuan et al. [27] introduced an efficient method named complex method to get exact solutions for NLDEs. The complex method is developed by complex analysis and complex differential equations. More details about the complex method can be found in [28–34]. In this work, we will utilize the complex method to achieve exact solutions of the following two (3+1)-dimensional NLDEs.

The (3+1)-dimensional BKP equation [35] is given by

$$u_{xxxy} + \theta\left(u_x u_y\right)_x + \left(u_x + u_y + u_s\right)_t - \left(u_{xx} + u_{ss}\right) = 0, \quad (1)$$

where θ is a constant.

The (3+1)-dimensional gSW equation [36] is given by

$$u_{xxxy} + 3u_{xx}u_y + 3u_x u_{xy} - u_{yt} - u_{xs} = 0. \quad (2)$$

Class W consists of elliptic function or their degeneration. Substituting traveling wave transform

$$u(x, y, s, t) = U(z), \quad z = n_1 x + n_2 y + n_3 s + n_4 t, \quad (3)$$

into Eq. (1), and then integrating it we get

$$n_1^3 n_2 U''' + \theta n_1^2 n_2 \left(U'\right)^2 + \left(n_4\left(n_1 + n_2 + n_3\right) - n_1^2 - n_3^2\right)U' + r = 0, \quad (4)$$

where r is the integration constant.

Theorem 1. *If $\theta n_1 n_2 \neq 0$, then meromorphic solutions w of Eq. (4) belong to class W and Eq. (4) has the following solutions where $c_i (i = 1, 2, 3, 4)$ are the integral constants.*

(i) *The rational function solutions*

$$U_r(z) = \frac{6n_1}{\theta}\frac{1}{z - z_0} + \frac{n_4(n_1 + n_2 + n_3) - n_1^2 - n_3^2}{2\theta n_1^2 n_2}(z - z_0) + c_1, \tag{5}$$

where $r = -(n_4(n_1 + n_2 + n_3) - n_1^2 - n_3^2)^2 / 4\theta n_1^2 n_2$, $z_0 \in \mathbb{C}$.

(ii) *The simply periodic solutions*

$$\begin{aligned} U_{1s}(z) = &-\frac{3n_1\mu}{\theta}\coth\frac{\mu(z - z_0)}{2} - \frac{3n_1\mu}{2\theta}\ln\left(\frac{\coth(\mu/2)(z - z_0) - 1}{\coth(\mu/2)(z - z_0) + 1}\right) \\ &+ \frac{\left(n_4(n_1 + n_2 + n_3) - n_1^2 - n_3^2\right)^2 - 2\mu^2 n_1^3 n_2}{2\theta n_1^2 n_2}(z - z_0) + c_2, \\ U_{2s}(z) = &-\frac{3n_1\mu}{\theta}\tanh\frac{\mu(z - z_0)}{2} - \frac{3n_1\mu}{2\theta}\ln\left(\frac{\tanh(\mu/2)(z - z_0) - 1}{\tanh(\mu/2)(z - z_0) + 1}\right) \\ &+ \frac{\left(n_4(n_1 + n_2 + n_3) - n_1^2 - n_3^2\right)^2 - 2\mu^2 n_1^3 n_2}{2\theta n_1^2 n_2}(z - z_0) + c_3, \end{aligned} \tag{6}$$

where $r = \mu^4 n_1^6 n_2^2 - (n_4(n_1 + n_2 + n_3) - n_1^2 - n_3^2)^2 / 4\theta n_1^2 n_2$, $z_0 \in \mathbb{C}$.

(iii) *The elliptic function solutions*

$$\begin{aligned} U_d(z) = &-\frac{6n_1}{\theta}[\zeta(z) - \zeta(z_0)] - \frac{3n_1}{\theta}\frac{\wp'(z) + G}{\wp(z) - H} \\ &+ \frac{n_4(n_1 + n_2 + n_3) - n_1^2 - n_3^2}{2\theta n_1^2 n_2}(z - z_0) + c_4, \end{aligned} \tag{7}$$

where $G^2 = 4H^3 - g_2 H - g_3$, $g_2 = 4n_1^2 n_2 r\theta + (n_4(n_1 + n_2 + n_3) - n_1^2 - n_3^2)^2 / 12 n_1^6 n_2^2$, *$H$ and g_3 are arbitrary.*

Substituting traveling wave transform

$$u(x, y, s, t) = V(z), z = m_1 x + m_2 y + m_3 s + m_4 t, \tag{8}$$

into Eq. (2), and then integrating it we get

$$m_1^3 m_2 V''' + 3m_1^2 m_2 \left(V'\right)^2 - (m_1 m_3 + m_2 m_4)V' + \lambda = 0, \tag{9}$$

where λ is the integration constant.

Theorem 2. *If $m_1 m_2 \neq 0$, then meromorphic solutions w of Eq. (9) belong to the class W and Eq. (9) has the following solutions where $c_i (i = 1, 2, 3, 4)$ are the integral constants.*

(i) *The rational function solutions*

$$V_r(z) = \frac{2m_1}{z - z_0} + \frac{m_1 m_3 + m_2 m_4}{6m_1^2 m_2}(z - z_0) + c_1, \tag{10}$$

where $\lambda = (m_1 m_3 + m_2 m_4)^2 / 12 m_1^2 m_2$, $z_0 \in \mathbb{C}$.

(ii) *The simply periodic solutions*

$$\begin{aligned} V_{1s}(z) = &-4m_1\mu\coth\frac{\mu(z - z_0)}{2} - 4m_1\mu\ln\left(\frac{\coth(\mu/2)(z - z_0) - 1}{\coth(\mu/2)(z - z_0) + 1}\right) \\ &+ \frac{2m_1^3 m_2\mu^2 + m_1 m_3 + m_2 m_4}{6m_1^2 m_2}(z - z_0) + c_2, \\ V_{2s}(z) = &-4m_1\mu\tanh\frac{\mu(z - z_0)}{2} - 4m_1\mu\ln\left(\frac{\tanh(\mu/2)(z - z_0) - 1}{\tanh(\mu/2)(z - z_0) + 1}\right) \\ &+ \frac{2m_1^3 m_2\mu^2 + m_1 m_3 + m_2 m_4}{6m_1^2 m_2}(z - z_0) + c_3, \end{aligned} \tag{11}$$

where $\lambda = (m_1 m_3 + m_2 m_4)^2 - m_1^6 m_2^2 \mu^4 / 12 m_1^2 m_2$, $z_0 \in \mathbb{C}$..

(iii) *The elliptic function solutions*

$$\begin{aligned} V_d(z) = &-2m_1[\zeta(z) - \zeta(z_0)] - m_1\frac{\wp'(z) + E}{\wp(z) - F} \\ &+ \frac{m_1 m_3 + m_2 m_4}{6m_1^2 m_2}(z - z_0) + c_4, \end{aligned} \tag{12}$$

where $E^2 = 4F^3 - g_2 F - g_3$, $g_2 = (m_1 m_3 + m_2 m_4)^2 - 12\lambda m_1^2 m_2 / 12 m_1^6 m_2^2$, *$F$ and g_3 is arbitrary.*

2. Preliminaries

Set $m \in \mathbb{N} := \{1, 2, 3, \cdots\}$, $r_i \in \{0, 1, 2, \cdots\}$, $i = 0, 1, \cdots, m$, $r = (r_0, r_1, \cdots, r_m)$, and

$$K_r[U](z) := \prod_{i=0}^{m}\left[U^{(i)}(z)\right]^{r_i}. \tag{13}$$

The degree of $Kr[U]$ is defined as $d(r) := \sum_{i=0}^{m} r_i$. The differential polynomial is given by

$$P\left(U, U', \cdots, U^{(m)}\right) := \sum_{r \in J} a_r K_r[U], \tag{14}$$

where J is a finite index set, then $\deg P(U, U', \cdots, U^{(m)}) := \max_{r\in J}\{d(r)\}$ is the degree of $P(U, U', \cdots, U^{(m)})$.

Considering the following equation:

$$P\left(U, U', \cdots, U^{(m)}\right) = aU^n + d, \tag{15}$$

where $n \in \mathbb{N}$, and $a \neq 0, d$ are constants.

Assume that meromorphic solutions w of Eq. (13) have at least one pole and let $p, q \in \mathrm{N}$. Substitute the Laurent series

$$U(z) = \sum_{k=-q}^{\infty} \beta_k z^k, \beta_{-q} \neq 0, q > 0, \tag{16}$$

into Eq. (15) to determine p distinct Laurent principal parts

$$\sum_{k=-q}^{-1} \beta_k z^k, \tag{17}$$

then, Eq. (15) is said to satisfy weak $<p, q>$ condition.

It is know that Weierstrass elliptic function $\wp(z) := \wp(z, g_2, g_3)$ has double periods and satisfies:

$$\left(\wp'(z)\right)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3. \tag{18}$$

Weierstrass zeta function $\zeta(z)$ is a meromorphic function which satisfies

$$\wp(z) = -\zeta'(z). \tag{19}$$

These two Weierstrass functions have the following addition formulas:

$$\begin{aligned} \wp(z - z_0) &= -\wp(z) + \frac{1}{4}\left[\frac{\wp'(z)+\wp'(z_0)}{\wp(z)-\wp(z_0)}\right]^2 - \wp(z_0), \\ \zeta(z - z_0) &= \zeta(z) - \zeta(z_0) + \frac{1}{2}\left[\frac{\wp'(z)+\wp'(z_0)}{\wp(z)-\wp(z_0)}\right]. \end{aligned} \tag{20}$$

Eremenko et al. [37] had investigated the following m -order Briot-Bouquet equation (BBEq)

$$P\left(U, U^{(m)}\right) = \sum_{j=0}^{n} P_j(U)\left(U^{(m)}\right)^j = 0, \tag{21}$$

in which $m \in \mathrm{N}$, and $P_j(U)$ are constant coefficient polynomials.

Lemma 1 [38–40]. *Let $m, n, p, s \in N$, $degP(U,U^{(m)}) < n$. If the m-order BBEq*

$$P\left(U, U^{(m)}\right) = aU^n + c \tag{22}$$

satisfies weak $<p, q>$ condition; then, meromorphic solutions w belong to class W. Assume that some values of parameters such solutions w exist; then, other meromorphic solutions should form 1 parameter family $U(z - z_0), z0 \in C$. In addition, each elliptic solution U with a pole at $z = 0$ is.

$$\begin{aligned} U(z) = &\sum_{i=1}^{s-1}\sum_{j=2}^{q} \frac{(-1)^j\beta_{-ij}}{(j-1)!}\frac{d^{j-2}}{dz^{j-2}}\left(\frac{1}{4}\left[\frac{\wp'(z)+D_i}{\wp(z)-B_i}\right]^2 - \wp(z)\right) \\ &+ \sum_{i=1}^{s-1}\frac{\beta_{-i1}}{2}\frac{\wp'(z)+D_i}{\wp(z)-B_i} + \sum_{j=2}^{q}\frac{(-1)^j\beta_{-sj}}{(j-1)!}\frac{d^{j-2}}{dz^{j-2}}\wp(z) + \beta_0, \end{aligned} \tag{23}$$

where β_{-ij} are determined by (16), $D_i^2 = 4B_i^3 - g_2B_i - g_3$ and $\sum_{i=1}^{s}\beta_{-i1} = 0$.

Each rational function solution is

$$R(z) = \sum_{i=1}^{s}\sum_{j=1}^{q}\frac{\beta_{ij}}{(z - z_i)^j} + \beta_0, \tag{24}$$

which contains $s(\leq p)$ distinct poles of multiplicity q.

Each simply periodic solution is a rational function $R(\eta)$ of $\eta = e^{\alpha z}(\alpha \in \mathbb{C})$, that is

$$R(\eta) = \sum_{i=1}^{s}\sum_{j=1}^{q}\frac{\beta_{ij}}{(\eta - \eta_i)^j} + \beta_0, \tag{25}$$

which contains $s(\leq p)$ distinct poles of multiplicity q.

3. Proofs of Main Results

Proof of Theorem 1. Let $u = U'$, then Eq. (4) becomes

$$n_1^3 n_2 u'' + \theta n_1^2 n_2 u^2 + \left(n_4(n_1 + n_2 + n_3) - n_1^2 - n_3^2\right)u + r = 0. \tag{26}$$

Substituting (16) into Eq. (4), we have $p = 1, q = 2, \beta_{-2} = 6n_1/\theta, \beta_{-1} = 0, \beta_0 = n_4(n_1 + n_2 + n_3) - n_1^2 - n_3^2/2\theta n_1^2 n_2, \beta_1 = 0, \beta_2 = 4n_1^2 n_2 r\theta + (n_4(n_1 + n_2 + n_3) - n_1^2 - n_3^2)^2/40\theta n_1^5 n_2^2$, $\beta_3 = 0$, and β_4 is an arbitrary constant. Thus, Eq. (26) is a second-order BBEq as well as satisfies weak $h1, 2i$ condition. Therefore, by Lemma 1, we know that the meromorphic solutions of Eq. (26) belong to class W.

From (23) of Lemma 1, we have the form of elliptic solutions of Eq. (26)

$$u_{d0}(z) = \beta_{-2}\wp(z) + \beta_{10} \tag{27}$$

with pole at $z = 0$.

Put $u_{d0}(z)$ into Eq. (26) to yield

$$u_{d0}(z) = \frac{6n_1}{\theta}\wp(z) + \frac{n_4(n_1 + n_2 + n_3) - n_1^2 - n_3^2}{2\theta n_1^2 n_2}, \tag{28}$$

where $g_2 = 4n_1^2 n_2 r\theta + (n_4(n_1+n_2+n_3) - n_1^2 - n_3^2)^2/12n_1^6 n_2^2$ and g_3 is arbitrary.

Therefore, the elliptic solutions of Eq. (26) with arbitrary pole are

$$u_d(z) = \frac{6n_1}{\theta}\wp(z-z_0) + \frac{n_4(n_1+n_2+n_3) - n_1^2 - n_3^2}{2\theta n_1^2 n_2}, \quad (29)$$

where $z_0 \in C$.

Therefore, the solutions of Eq. (4) are

$$\begin{aligned} U_d(z) &= \int u_d(z)dz = \int \left(\frac{6n_1}{\theta}\wp(z-z_0) + \frac{n_4(n_1+n_2+n_3) - n_1^2 - n_3^2}{2\theta n_1^2 n_2}\right)dz \\ &= \frac{6n_1}{\theta}\zeta(z-z_0) + \frac{n_4(n_1+n_2+n_3) - n_1^2 - n_3^2}{2\theta n_1^2 n_2}(z-z_0) + c_4 \\ &= -\frac{6n_1}{\theta}[\zeta(z) - \zeta(z_0)] - \frac{3n_1}{\theta}\frac{\wp'(z) + G}{\wp(z) - H} \\ &\quad + \frac{n_4(n_1+n_2+n_3) - n_1^2 - n_3^2}{2\theta n_1^2 n_2}(z-z_0) + c_4, \end{aligned} \quad (30)$$

where $G^2 = 4H^3 - g_2 H - g_3$, $g_2 = 4n_1^2 n_2 r\theta + (n_4(n_1+n_2+n_3) - n_1^2 - n_3^2)^2/12n_1^6 n_2^2$, c_4 is the integral constant, and H and g_3 are arbitrary.

By (24), we infer that the indeterminant rational solutions of Eq.(26) are

$$R_1(z) = \frac{\beta_{11}}{z^2} + \frac{\beta_{12}}{z} + \beta_{20} \quad (31)$$

with pole at $z = 0$.

Substitute $R_1(z)$ into Eq. (26) to yield

$$R_1(z) = \frac{6n_1}{\theta}\frac{1}{z^2} + \frac{n_4(n_1+n_2+n_3) - n_1^2 - n_3^2}{2\theta n_1^2 n_2}, \quad (32)$$

where $r = -(n_4(n_1+n_2+n_3) - n_1^2 - n_3^2)^2/4\theta n_1^2 n_2$.

So the rational solutions of Eq. (26) with arbitrary pole are

$$u_r(z) = \frac{6n_1}{\theta}\frac{1}{(z-z_0)^2} + \frac{n_4(n_1+n_2+n_3) - n_1^2 - n_3^2}{2\theta n_1^2 n_2}. \quad (33)$$

Therefore, the solutions of Eq. (4) are

$$\begin{aligned} U_r(z) &= \int u_r(z)dz = \int \left(\frac{6n_1}{\theta}\frac{1}{(z-z_0)^2} + \frac{n_4(n_1+n_2+n_3) - n_1^2 - n_3^2}{2\theta n_1^2 n_2}\right)dz \\ &= -\frac{6n_1}{\theta}\frac{1}{z-z_0} + \frac{n_4(n_1+n_2+n_3) - n_1^2 - n_3^2}{2\theta n_1^2 n_2}(z-z_0) + c_1, \end{aligned} \quad (34)$$

where c_1 is the integral constant, $r = -(n_4(n_1+n_2+n_3) - n_1^2 - n_3^2)^2/4\theta n_1^2 n_2$, $z_0 \in \mathbb{C}$.

Let $\eta = e^{\mu z}$. To obtain simply periodic solutions, we insert $u = R(\eta)$ into Eq. (26) and get

$$\begin{aligned} n_1^3 n_2 \mu^2 \left(\eta R' + \eta^2 R''\right) + \theta n_1^2 n_2 R^2 + (n_4(n_1+n_2+n_3) \\ - n_1^2 - n_3^2)R + r = 0. \end{aligned} \quad (35)$$

Substituting $R_2(z)$ into the Eq.(35), we obtain that

$$\begin{aligned} R_{21}(z) &= \frac{6n_1}{\theta}\frac{\mu^2}{(\eta-1)^2} + \frac{6n_1}{\theta}\frac{\mu^2}{(\eta-1)} \\ &\quad + \frac{\mu^2 n_1^3 n_2 + (n_4(n_1+n_2+n_3) - n_1^2 - n_3^2)^2}{2\theta n_1^2 n_2}, \end{aligned} \quad (36)$$

$$\begin{aligned} R_{22}(z) &= \frac{6n_1}{\theta}\frac{\mu^2}{(\eta+1)^2} - \frac{6n_1}{\theta}\frac{\mu^2}{(\eta+1)} \\ &\quad + \frac{\mu^2 n_1^3 n_2 + (n_4(n_1+n_2+n_3) - n_1^2 - n_3^2)^2}{2\theta n_1^2 n_2}, \end{aligned} \quad (37)$$

where $r = \mu^4 n_1^6 n_2^2 - (n_4(n_1+n_2+n_3) - n_1^2 - n_3^2)^2/4\theta n_1^2 n_2$.

Substituting $\eta = e^{\mu z}$ into Eq. (36) and Eq. (37) yields simply periodic solutions to Eq. (26) with pole at $z = 0$

$$\begin{aligned} u_{1s0}(z) &= \frac{6n_1}{\theta}\frac{\mu^2}{(e^{\mu z}-1)^2} + \frac{6n_1}{\theta}\frac{\mu^2}{(e^{\mu z}-1)} \\ &\quad + \frac{\mu^2 n_1^3 n_2 + (n_4(n_1+n_2+n_3) - n_1^2 - n_3^2)^2}{2\theta n_1^2 n_2} \\ &= \frac{6n_1}{\theta}\mu^2\frac{e^{\mu z}}{(e^{\mu z}-1)^2} \\ &\quad + \frac{\mu^2 n_1^3 n_2 + (n_4(n_1+n_2+n_3) - n_1^2 - n_3^2)^2}{2\theta n_1^2 n_2} \\ &= \frac{3n_1}{2\theta}\mu^2 \coth^2\frac{\mu z}{2} \\ &\quad + \frac{(n_4(n_1+n_2+n_3) - n_1^2 - n_3^2)^2 - 2\mu^2 n_1^3 n_2}{2\theta n_1^2 n_2}, \\ u_{2s0}(z) &= \frac{6n_1}{\theta}\frac{\mu^2}{(e^{\mu z}+1)^2} - \frac{6n_1}{\theta}\frac{\mu^2}{(e^{\mu z}+1)} \\ &\quad + \frac{\mu^2 n_1^3 n_2 + (n_4(n_1+n_2+n_3) - n_1^2 - n_3^2)^2}{2\theta n_1^2 n_2} \\ &= -\frac{6n_1}{\theta}\mu^2\frac{e^{\mu z}}{(e^{\mu z}+1)^2} \\ &\quad + \frac{\mu^2 n_1^3 n_2 + (n_4(n_1+n_2+n_3) - n_1^2 - n_3^2)^2}{2\theta n_1^2 n_2} \\ &= \frac{3n_1}{2\theta}\mu^2 \tanh^2\frac{\mu z}{2} \\ &\quad + \frac{(n_4(n_1+n_2+n_3) - n_1^2 - n_3^2)^2 - 2\mu^2 n_1^3 n_2}{2\theta n_1^2 n_2}, \end{aligned} \quad (38)$$

where $r=\mu^4 n_1^6 n_2^2-(n_4(n_1+n_2+n_3)-n_1^2-n_3^2)^2/4\theta n_1^2 n_2$.

So simply periodic solutions of Eq. (4) with arbitrary pole are

$$u_{1s}(z)=\frac{3n_1}{2\theta}\mu^2\coth^2\frac{\mu(z-z_0)}{2}+\frac{(n_4(n_1+n_2+n_3)-n_1^2-n_3^2)^2-2\mu^2 n_1^3 n_2}{2\theta n_1^2 n_2}, \tag{39}$$

and

$$u_{2s}(z)=\frac{3n_1}{2\theta}\mu^2\tanh^2\frac{\mu(z-z_0)}{2}+\frac{(n_4(n_1+n_2+n_3)-n_1^2-n_3^2)^2-2\mu^2 n_1^3 n_2}{2\theta n_1^2 n_2}, \tag{40}$$

where $r=\mu^4 n_1^6 n_2^2-(n_4(n_1+n_2+n_3)-n_1^2-n_3^2)^2/4\theta n_1^2 n_2$.

Therefore, the solutions of Eq. (4) are

$$\begin{aligned}U_{1s}(z)&=\int u_{1s}(z)dz=\int\left(\frac{3n_1}{2\theta}\mu^2\coth^2\frac{\mu(z-z_0)}{2}+\frac{(n_4(n_1+n_2+n_3)-n_1^2-n_3^2)^2-2\mu^2 n_1^3 n_2}{2\theta n_1^2 n_2}\right)dz\\&=-\frac{3n_1\mu}{\theta}\coth\frac{\mu(z-z_0)}{2}-\frac{3n_1\mu}{2\theta}\ln\left(\frac{\coth(\mu/2)(z-z_0)-1}{\coth(\mu/2)(z-z_0)+1}\right)\\&\quad+\frac{(n_4(n_1+n_2+n_3)-n_1^2-n_3^2)^2-2\mu^2 n_1^3 n_2}{2\theta n_1^2 n_2}(z-z_0)+c_2,\end{aligned} \tag{41}$$

and

$$\begin{aligned}U_{2s}(z)&=\int u_{2s}(z)dz=\int\left(\frac{3n_1}{2\theta}\mu^2\tanh^2\frac{\mu(z-z_0)}{2}+\frac{(n_4(n_1+n_2+n_3)-n_1^2-n_3^2)^2-2\mu^2 n_1^3 n_2}{2\theta n_1^2 n_2}\right)dz\\&=-\frac{3n_1\mu}{\theta}\tanh\frac{\mu(z-z_0)}{2}-\frac{3n_1\mu}{2\theta}\ln\left(\frac{\tanh(\mu/2)(z-z_0)-1}{\tanh(\mu/2)(z-z_0)+1}\right)\\&\quad+\frac{(n_4(n_1+n_2+n_3)-n_1^2-n_3^2)^2-2\mu^2 n_1^3 n_2}{2\theta n_1^2 n_2}(z-z_0)+c_3,\end{aligned} \tag{42}$$

where c_2 and c_3 are the integral constants, $r=\mu^4 n_1^6 n_2^2-(n_4(n_1+n_2+n_3)-n_1^2-n_3^2)^2/4\theta n_1^2 n_2$, $z_0\in\mathbb{C}$.

Proof of Theorem 2. Let $v=V'$, then Eq. (9) becomes

$$m_1^3 m_2 v'' + 3m_1^2 m_2 v^2-(m_1m_3+m_2m_4)v+\lambda=0. \tag{43}$$

Substituting (16) into Eq.(9), we have $p=1, q=2, \beta_{-2}=-2m_1, \beta_{-1}=0, \beta_0=m_1m_3+m_2m_4/6m_1^2m_2, \beta_1=0, \beta_2=12\lambda m_1^2m_2-(m_1m_3+m_2m_4)^2/120m_1^5m_2, \beta_3=0$, and β_4 is an arbitrary constant. Thus, Eq. (43) is a second-order BBEq as well as satisfies weak $h1,2i$ condition. Therefore, by Lemma 1, we know that the meromorphic solutions of Eq. (43) belong to class W.

From (23) of Lemma 1, we have the form of elliptic solutions of Eq. (43)

$$v_{d0}(z)=\beta_{-2}\wp(z)+\beta_{10} \tag{44}$$

with pole at $z=0$.

Put $v_{d0}(z)$ into Eq. (43) to yield

$$v_{d0}(z)=-2m_1\wp(z)+\frac{m_1m_3+m_2m_4}{6m_1^2m_2}, \tag{45}$$

where $g_2=(m_1m_3+m_2m_4)^2-12\lambda m_1^2m_2/12m_1^6m_2^2$ and g_3 is arbitrary.

Therefore, the elliptic solutions of Eq. (43) with arbitrary pole are

$$v_d(z)=-2m_1\wp(z-z_0)+\frac{m_1m_3+m_2m_4}{6m_1^2m_2}, \tag{46}$$

where $z_0\in C$.

Therefore, the solutions of Eq. (9) are

$$\begin{aligned}V_d(z)&=\int v_d(z)dz=\int\left(-2m_1\wp(z-z_0)+\frac{m_1m_3+m_2m_4}{6m_1^2m_2}\right)dz\\&=-2m_1\zeta(z-z_0)+\frac{m_1m_3+m_2m_4}{6m_1^2m_2}(z-z_0)+c_4\\&=-2m_1[\zeta(z)-\zeta(z_0)]-m_1\frac{\wp'(z)+E}{\wp(z)-F}+\frac{m_1m_3+m_2m_4}{6m_1^2m_2}(z-z_0)+c_4,\end{aligned} \tag{47}$$

where $E^2=4F^3-g_2F-g_3$, $g_2=(m_1m_3+m_2m_4)^2-12\lambda m_1^2 m_2/12m_1^6m_2^2$, c_4 is the integral constant, and F and g_3 are arbitrary.

By (24), we infer that the indeterminant rational solutions of Eq. (43) are

$$R_1(z)=\frac{\beta_{11}}{z^2}+\frac{\beta_{12}}{z}+\beta_{20} \tag{48}$$

with pole at $z=0$.

Substitute $R_1(z)$ into Eq. (43) to yield

$$R_1(z)=-\frac{2m_1}{z^2}+\frac{m_1m_3+m_2m_4}{6m_1^2m_2}, \tag{49}$$

where $\lambda=(m_1m_3+m_2m_4)^2/12m_1^2m_2$.

So the rational solutions of Eq. (43) with arbitrary pole are

$$v_r(z) = -\frac{2m_1}{(z-z_0)^2} + \frac{m_1m_3 + m_2m_4}{6m_1^2m_2}. \quad (50)$$

Therefore, the solutions of Eq. (9) are

$$\begin{aligned} V_r(z) &= \int v_r(z)dz = \int \left(-\frac{2m_1}{(z-z_0)^2} + \frac{m_1m_3+m_2m_4}{6m_1^2m_2}\right)dz \\ &= \frac{2m_1}{z-z_0} + \frac{m_1m_3+m_2m_4}{6m_1^2m_2}(z-z_0) + c_1, \end{aligned} \quad (51)$$

where c_1 is the integral constant, $\lambda = (m_1m_3 + m_2m_4)^2/12m_1^2m_2$, $z_0 \in \mathbb{C}$.

Let $\eta = e^{\mu z}$. To obtain simply periodic solutions, we insert $v = R(\eta)$ into Eq. (43) and get

$$m_1^3m_2\mu^2\left(\eta R' + \eta^2 R''\right) + 3m_1^2m_2R^2 - (m_1m_3+m_2m_4)R + \lambda = 0. \quad (52)$$

Substituting $R_2(z)$ into the Eq. (35), we obtain that

$$R_{21}(z) = -\frac{2m_1\mu^2}{(\eta-1)^2} - \frac{2m_1\mu^2}{(\eta-1)} + \frac{m_1m_3+m_2m_4-m_1^3m_2\mu^2}{6m_1^2m_2}, \quad (53)$$

$$R_{22}(z) = -\frac{2m_1\mu^2}{(\eta+1)^2} + \frac{2m_1\mu^2}{(\eta+1)} + \frac{m_1m_3+m_2m_4-m_1^3m_2\mu^2}{6m_1^2m_2}, \quad (54)$$

where $\lambda = (m_1m_3+m_2m_4)^2 - m_1^6m_2^2\mu^4/12m_1^2m_2$.

Substituting $\eta = e^{\mu z}$ into Eq. (53) and Eq. (54) yields simply periodic solutions to Eq. (43) with pole at $z=0$

$$\begin{aligned} v_{1s0}(z) &= -\frac{2m_1\mu^2}{(e^{\mu z}-1)^2} - \frac{2m_1\mu^2}{(e^{\mu z}-1)} + \frac{m_1m_3+m_2m_4-m_1^3m_2\mu^2}{6m_1^2m_2} \\ &= -\frac{2m_1\mu^2e^{\mu z}}{(e^{\mu z}-1)^2} + \frac{m_1m_3+m_2m_4-m_1^3m_2\mu^2}{6m_1^2m_2} \\ &= -2m_1\mu^2\coth^2\frac{\mu z}{2} + \frac{2m_1^3m_2\mu^2+m_1m_3+m_2m_4}{6m_1^2m_2}, \\ v_{2s0}(z) &= -\frac{2m_1\mu^2}{(e^{\mu z}+1)^2} + \frac{2m_1\mu^2}{(e^{\mu z}+1)} + \frac{m_1m_3+m_2m_4-m_1^3m_2\mu^2}{6m_1^2m_2} \\ &= \frac{2m_1\mu^2e^{\mu z}}{(e^{\mu z}+1)^2} + \frac{m_1m_3+m_2m_4-m_1^3m_2\mu^2}{6m_1^2m_2} \\ &= -2m_1\mu^2\tanh^2\frac{\mu z}{2} + \frac{2m_1^3m_2\mu^2+m_1m_3+m_2m_4}{6m_1^2m_2}, \end{aligned} \quad (55)$$

where $\lambda = (m_1m_3+m_2m_4)^2 - m_1^6m_2^2\mu^4/12m_1^2m_2$.

So simply periodic solutions of Eq. (9) with arbitrary pole are

$$v_{1s}(z) = -2m_1\mu^2\coth^2\frac{\mu(z-z_0)}{2} + \frac{2m_1^3m_2\mu^2+m_1m_3+m_2m_4}{6m_1^2m_2}, \quad (56)$$

and

$$v_{2s}(z) = -2m_1\mu^2\tanh^2\frac{\mu(z-z_0)}{2} + \frac{2m_1^3m_2\mu^2+m_1m_3+m_2m_4}{6m_1^2m_2}, \quad (57)$$

where $\lambda = (m_1m_3+m_2m_4)^2 - m_1^6m_2^2\mu^4/12m_1^2m_2$.

Therefore, the solutions of Eq. (9) are

$$\begin{aligned} V_{1s}(z) &= \int v_{1s}(z)dz = \int \Big(-2m_1\mu^2\coth^2\frac{\mu(z-z_0)}{2} \\ &\quad + \frac{2m_1^3m_2\mu^2+m_1m_3+m_2m_4}{6m_1^2m_2}\Big)dz \\ &= -4m_1\mu\coth\frac{\mu(z-z_0)}{2} \\ &\quad - 4m_1\mu\ln\left(\frac{\coth(\mu/2)(z-z_0)-1}{\coth(\mu/2)(z-z_0)+1}\right) \\ &\quad + \frac{2m_1^3m_2\mu^2+m_1m_3+m_2m_4}{6m_1^2m_2}(z-z_0) + c_2, \end{aligned} \quad (58)$$

and

$$\begin{aligned} V_{2s}(z) &= \int v_{2s}(z)dz = \int \Big(-2m_1\mu^2\tanh^2\frac{\mu(z-z_0)}{2} \\ &\quad + \frac{2m_1^3m_2\mu^2+m_1m_3+m_2m_4}{6m_1^2m_2}\Big)dz \\ &= -4m_1\mu\tanh\frac{\mu(z-z_0)}{2} \\ &\quad - 4m_1\mu\ln\left(\frac{\tanh(\mu/2)(z-z_0)-1}{\tanh(\mu/2)(z-z_0)+1}\right) \\ &\quad + \frac{2m_1^3m_2\mu^2+m_1m_3+m_2m_4}{6m_1^2m_2}(z-z_0) + c_3, \end{aligned} \quad (59)$$

where c_2 and c_3 are the integral constants, $\lambda = (m_1m_3+m_2m_4)^2 - m_1^6m_2^2\mu^4/12m_1^2m_2$, $z_0 \in \mathbb{C}$.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflict of interest.

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