Research Article

Uniqueness of Iterative Positive Solution to Nonlinear Fractional Differential Equations with Negatively Perturbed Term

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We prove that there are unique positive solutions for a new kind of fractional differential equation with a negatively perturbed term boundary value problem. Our methods rely on an iterative algorithm which requires constructing an iterative scheme to approximate the solution. This allows us to calculate the estimation of the convergence rate and the approximation error.

1. Introduction

In this work, we present the uniqueness of a positive solution for the following boundary value problem (BVP for short):

$$\begin{cases} D_{0+}^p x(t) + p(t)f(t, x(t), y(t)) + q(t) = 0, & 0 < t < 1, \\ x(0) = x'(0) = 0, & x(1) = 0, \end{cases} \quad (1)$$

where D_{0+}^p is the standard fractional derivative of order p satisfying $2 < p \leq 3$, and $f(t, x, y)$ may be singular at $y = 0$, $t = 0, 1$.

Fractional calculus differential equations are an important branch of differential equations. In recent years, it has attracted the interest of many researchers and has become a hot-button issue [1–14]. Compared with the integer order, it has a wider range of applications as it can be used to describespecific problems more precisely, such as the problem in complex analysis, polymer rheology, physical chemistry, electrical networks, and many other branches of science. For specific applications, see [15, 16, 20–28]. In [3], the authors study the following BVP:

$$\begin{cases} -D_{0+}^p x(t) = p(t)f(t, x(t)) - q(t), & 0 < t < 1, \\ x(0) = x'(0) = 0, & x(1) = 0. \end{cases} \quad (2)$$

They obtained the existence of multiple positive solutions by means of the Gou-Krasnoselskii fixed point theorem. In [4], the authors also study the same BVP. By constructing a special u_0 -positive operator and using its properties, they obtained a unique solution for BVP (2).

BVP (1) is more general than the problem in papers [3] and [4] in four aspects. First, the nonlinear term has two space variables and can be singular with respect to the second space variable. Second, the method we used is different from papers [3] and [4]. Through constructing an iterative process which can be from any initial value, only by the iterative algorithm, we can prove that it converges uniformly to the unique positive solution. Third, we calculate the estimation of the convergence rate and the approximation error. Finally, we compare with it [3] and [4]. We lower the conditional constraint on the nonlinear term since we do not need the monotone of the nonlinear term. So the result of this paper is most general, not only did it weaken the restrictions but it also strengthened the conclusions in [3] and [4]. Also, we can show that the main result in [4] is a corollary of our work.

The rest of our presentation is as follows. In Section 2, we recall some definitions and lemmas. In Section 3, we establish the result of the uniqueness of the positive solutions to BVP (1).

Finally, in Section 4, an illustrative example is also presented.

2. Preliminaries

We first list some definitions and lemmas which will be used later.

Definition 1 (see [5]). Let $p > 0$, and the Riemann-Liouville standard fractional integral and the Riemann-Liouville standard fractional derivative of order $p > 0$ of a function $f : (0, \infty) \rightarrow \mathbb{R}$ are given by

$$\begin{aligned} I_{0+}^p f(x) &= \frac{1}{\Gamma(p)} \int_0^x \frac{f(t)}{(x-t)^{1-p}} dt, \\ D_{0+}^p f(t) &= \frac{1}{\Gamma(n-p)} \left(\frac{d}{dt} \right)^n \int_0^t \frac{f(s)}{(t-s)^{p-n+1}} ds, \end{aligned} \quad (3)$$

where $n = [p] + 1$ and $[p]$ denotes the integer part of the real number p and provides that the right side integral is pointwise defined in $[0, \infty)$.

Lemma 2 (see [3]). Suppose that $2 < p \leq 3$ and $h \in C(0, 1)^T L^1[0, 1]$, then the boundary value problem

$$\begin{cases} D_{0+}^p x(t) + h(t) = 0, & 0 < t < 1, \\ x(0) = x'(0) = 0, & x(1) = 0, \end{cases} \quad (4)$$

is given by

$$x(t) = \int_0^1 G(t, s) h(s) ds, \quad (5)$$

where

$$G(t, s) = \frac{1}{\Gamma(p)} \begin{cases} t^{p-1}(1-s)^{p-1} - (t-s)p-1, & 0 \leq s \leq t \leq 1, \\ t^{p-1}(1-s)^{p-1}, & 0 \leq t \leq s \leq 1. \end{cases} \quad (6)$$

Lemma 3 (see [3]). The function $G(t, s)$ obtained in Lemma 2 is continuous on $[0, 1] \times [0, 1]$ and satisfies the following properties:

- (1) for all $t, s \in [0, 1]$, $G(t, s) \geq 0$
- (2) for all $t, s \in [0, 1]$

$$\frac{t^{p-1}(1-t)(1-s)^{p-1}s}{\Gamma(p)} \leq G(t, s) \leq \frac{(p-1)t^{p-1}(1-t)}{\Gamma(p)}. \quad (7)$$

In the rest of this paper, we assume that the following conditions hold:

H_1 . $f(t, x, y) \in C[(0, 1) \times [0, +\infty) \times (0, +\infty); (0, +\infty)]$ is increasing with respect to x , decreasing with respect to y , and for any $(t, x, y) \in (0, 1) \times [0, +\infty) \times (0, +\infty)$, $\lambda \in (0, 1)$, there exists a constant $k \in (0, 1)$ such that

$$f(t, \lambda x, \lambda^{-1} y) \geq \lambda^k f(t, x, y). \quad (8)$$

H_2 . $p, q \in C[(0, 1); (0, +\infty)]$ are Lebesgue integrable, and p does not vanish identically on any subinterval of $(0, 1)$.

H_3 . The third condition is as follows:

$$0 < \int_0^1 [p(s)f(s, (1-s)s^{p-1}, (1-s)s^{p-1}) + q(s)] ds < \infty. \quad (9)$$

The basic space used in this paper is $E = C[0, 1]$. Define a set P in E as follows:

$$P = \{x \in E \mid \text{there exists a positive constant } l_x \in (0, 1), \text{ such that } l_x(1-t)t^{p-1} \leq x(t) \leq (l_x)^{-1}(1-t)t^{p-1}, t \in (0, 1)\}. \quad (10)$$

Denote $h(t) = (1-t)t^{p-1}$. Evidently, $(h(t), h(t)) \in P \times P$, i.e., $P \times P$ is not empty.

Let the operator $T : E \times E \rightarrow E$ be defined by

$$(T(x, y))(t) = \int_0^1 G(t, s) [p(s)f(s, x(s), y(s)) + q(s)] ds. \quad (11)$$

3. Main Results

In this section, we will prove the existence of the positive solution to BVP (1) and calculate the approximation error and the convergence rate.

Theorem 4. Assume that H_1 , H_2 , and H_3 hold. Then, BVP (1) has a unique positive solution $x^*(t)$ satisfying

$$l(1-t)t^{p-1} \leq x^*(t) \leq l^{-1}(1-t)t^{p-1}, \quad t \in [0, 1], \quad (12)$$

where l is a constant that belongs to $(0, 1)$.

Proof. The solution of BVP (1) coincides with the fixed point of operator T . So our goal is to show that operator T has a unique fixed point in P . We divide our proof in four steps.

Step 1. We verify that operator T is well defined. From H_1 , H_2 , and H_3 , for any $(x, y) \in P \times P$ and $t \in (0, 1)$, we know that

$$\begin{aligned} (T(x, y))(t) &= \int_0^1 G(t, s) [p(s)f(s, x(s), y(s)) + q(s)] \\ &\cdot ds \leq \frac{(p-1)t^{p-1}(1-t)}{\Gamma(p)} \int_0^1 \\ &\cdot [p(s)f(s, (l_x)^{-1}(1-s)s^{p-1}, l_x(1-s)s^{p-1}) \\ &+ q(s)] ds \leq \frac{(l_x)^{-k}(p-1)t^{p-1}(1-t)}{\Gamma(p)} \int_0^1 \\ &\cdot [p(s)f(s, (1-s)s^{p-1}, (1-s)s^{p-1}) \\ &+ q(s)] ds < +\infty. \end{aligned} \quad (13)$$

So, operator T is well defined.

Step 2. We verify the properties of T .

$$\begin{aligned}
 (T(x, y))(t) &= \int_0^1 G(t, s) [p(s)f(s, x(s), y(s)) + q(s)] \\
 &\quad \cdot ds \geq \frac{(t^{(p-1)}(1-t))}{\Gamma(p)} \int_0^1 (1-s)^{(p-1)} \\
 &\quad \cdot s \left[p(s)f\left(s, l_x(1-s)s^{(p-1)}, (l_x)^{-1}(1-s)s^{(p-1)}\right) \right. \\
 &\quad \left. + q(s) \right] ds \geq \frac{(l_x)^k t^{(p-1)}(1-t)}{\Gamma(p)} \int_0^1 (1-s)^{(p-1)} \\
 &\quad \cdot s \left[p(s)f\left(s, (1-s)s^{(p-1)}, (1-s)s^{(p-1)}\right) + q(s) \right] ds.
 \end{aligned} \tag{14}$$

Let

$$\begin{aligned}
 0 < l_{Tx} < \min \left\{ 1, \frac{(l_x)^k}{\Gamma(p)} \int_0^1 (1-s)^{p-1} s [p(s) \right. \\
 &\quad \left. \cdot f(s, (1-s)s^{p-1}, (1-s)s^{p-1}) + q(s)] ds \right\},
 \end{aligned} \tag{15}$$

and

$$\begin{aligned}
 (l_{Tx})^{-1} > \max \left\{ 1, \frac{(l_x)^{-k}(p-1)}{\Gamma(p)} \int_0^1 [p(s) \right. \\
 &\quad \left. \cdot f(s, (1-s)s^{p-1}, (1-s)s^{p-1}) + q(s)] ds \right\}.
 \end{aligned} \tag{16}$$

Consequently, we can prove that there exists constants $0 < l_{Tx} < 1$ such that

$$l_{Tx}(1-t)t^{p-1} \leq (T(x, y))(t) \leq (l_{Tx})^{-1}(1-t)t^{p-1}, \quad t \in (0, 1). \tag{17}$$

Therefore, the operator $T : P \times P \longrightarrow P$ is well defined. It follows from H1 that T is nondecreasing with respect to x and nonincreasing with respect to y .

Step 3. We will establish the existence of a positive solution to BVP (1). Since $(h, h) \in P \times P$, from the above steps, we have $T(h, h) \in P$. According to the definition of P , there exists a constant $0 < l_{Th} < 1$ such that

$$l_{Th}(1-t)t^{p-1} \leq T(h, h)(t) \leq (l_{Th})^{-1}(1-t)t^{p-1}. \tag{18}$$

Taking

$$\begin{aligned}
 x_0 &= \rho h(t), \\
 y_0 &= \rho^{-1} h(t),
 \end{aligned} \tag{19}$$

$$\begin{aligned}
 x_n &= T(x_{n-1}, y_{n-1}), \\
 y_n &= T(y_{n-1}, x_{n-1}), \\
 n &= 1, 2, \dots,
 \end{aligned} \tag{20}$$

where ρ is a fixed number satisfying

$$0 < \rho \leq l_{Th}^{1/1-k}. \tag{21}$$

In fact, $0 < \rho < 1$. From (19), we get $x_0, y_0 \in P$ and $x_0 \leq y_0$. Moreover,

$$\begin{aligned}
 x_1 &= (T(x_0, y_0))(t) = \int_0^1 G(t, s) [p(s)f(s, \rho h(s), \rho^{-1} h(s)) \\
 &\quad + q(s)] ds \geq \rho^k \int_0^1 G(t, s) [p(s)f(s, h(s), h(s)) + q(s)] \\
 &\quad \cdot ds = \rho^k T(h, h)(t) \geq \rho^k l_{Th} h(t) \geq \rho^k \rho^{1-k} h(t) = x_0,
 \end{aligned} \tag{22}$$

and

$$\begin{aligned}
 y_1 &= (T(y_0, x_0))(t) = \int_0^1 G(t, s) [p(s)f(s, \rho^{-1} h(s), \rho h(s)) \\
 &\quad + q(s)] ds \leq \rho^{-k} \int_0^1 G(t, s) [p(s)f(s, h(s), h(s)) + q(s)] \\
 &\quad \cdot ds = \rho^{-k} T(h, h)(t) \leq \rho^{-k} (l_{Th})^{-1} h(t) \leq \rho^{-k} \rho^{k-1} h(t) = y_0.
 \end{aligned} \tag{23}$$

Combining $x_0 \leq y_0$ with T is nondecreasing with respect to x and nonincreasing with respect to y , so we have

$$x_0 \leq x_1 \leq \dots \leq x_n \leq \dots \leq y_n \leq \dots \leq y_1 \leq y_0. \tag{24}$$

On the other hand, for any nature number n , denote ρ^2 by c and we have

$$\begin{aligned}
 x_n &= T(x_{n-1}, y_{n-1}) = T_n(y_0, x_0) \\
 &= T_n(\rho h(t), \rho^{-1} h(t)) = T_n(\rho^2 \rho^{-1} h(t), \rho^{-2} \rho h(t)) \\
 &\geq (\rho^2)^{k_n} T_n(\rho^{-1} h(t), \rho h(t)) = c_{k_n} y_n.
 \end{aligned} \tag{25}$$

Therefore, for any nature number n and n^* , we obtain

$$0 \leq x_{n+n^*} - x_n \leq y_n - x_n \leq (1 - c^{k_n}) y_n \leq (1 - c^{k_n}) \rho^{-1} h(t). \tag{26}$$

So, there exists $x^* \in P$ such that

$$x_n(t) \longrightarrow x^*(t), \tag{27}$$

uniformly on $(0, 1)$. By the same argument, we can also prove that

$$y_n(t) \longrightarrow x^*(t), \tag{28}$$

uniformly on $(0, 1)$. Since T is continuous, we can take the limits in $x_n = T(x_n, y_n)$ and we get $x^* = T(x^*, x^*)$. Therefore, x^* is a positive solution of BVP (1). Owing to $x^* \in P$, for any $t \in (0, 1)$, there exists a constant $l \in (0, 1)$ such that

$$l(1-t)t^{p-1} \leq x^*(t) \leq l^{-1}(1-t)t^{p-1}, \quad (29)$$

holds.

Step 4. We further show its uniqueness. Let $y^*(t)$ be another positive solution of BVP (1), then for any $t \in (0, 1)$, there exists a constant $r \in (0, 1)$ such that

$$r(1-t)t^{p-1} \leq y^*(t) \leq r^{-1}(1-t)t^{p-1}. \quad (30)$$

Taking ρ defined in (21) as being small enough such that $\rho < r$. So

$$x_0(t) \leq y^*(t) \leq y_0(t), \quad t \in (0, 1). \quad (31)$$

In view of $T(y^*, y^*) = y^*$, by means of the nondecreasing T , we obtain

$$x_n(t) \leq y^*(t) \leq y_n(t), \quad t \in (0, 1). \quad (32)$$

Taking limits to the above inequality, we get $x^* = y^*$. Therefore, the solutions of BVP (1) are unique which completes the proof of Theorem 4.

Now, we are in a position to construct the successive sequence which converges to the unique solution.

Theorem 5. Suppose conditions H1, H2, and H3 are satisfied. Then, for any initial value $u_0 \in P$, the successive sequence

$$u_n(t) = \int_0^1 G(t, s)[p(s)f(s, u_{n-1}(s), u_{n-1}(s)) + q(s)]ds, \quad (33)$$

uniformly converges to the unique positive solution $x^*(t)$ where the error estimation is the same order infinitesimal of $(1 - c^{k_n})$, where $c \in (0, 1)$ and determined by u_0 .

Proof. According to Theorem 4, notice that the positive solution x^* is unique; for any $u_0 \in P$, there exists a constant $l \in (0, 1)$, such that

$$l(1-t)t^{p-1} \leq u_0(t) \leq l^{-1}(1-t)t^{p-1}. \quad (34)$$

We shall adopt the similar argument as in the proof of Theorem 4; take $\rho < l$ to be a fixed number $0 < \rho \leq l^{1/(1-k)}$, thus

$$x_0(t) \leq u_0(t) \leq y_0(t), \quad t \in (0, 1). \quad (35)$$

Let

$$u_n = \int_0^1 G(t, s)[p(s)f(s, u_{n-1}(s), u_{n-1}(s)) + q(s)]ds, \quad n = 1, 2, \dots, \quad (36)$$

then

$$x_n(t) \leq u_n(t) \leq y_n(t), \quad t \in (0, 1). \quad (37)$$

Taking limits to inequality (37), by the same method of the proof to (26), we can show that $\{u_n(t)\}$ uniformly converges to the unique positive solution x^* of BVP (1), and

$$\max \{|u_n(t) - x^*(t)|\} = o(1 - c^{k_n}), \quad (38)$$

which means the error estimation is the same order infinitesimal of $(1 - c^{k_n})$, where $c = \rho^2$ and determined by u_0 .

The proof is completed.

4. Example

Let us illustrate the main results with an example.

Example. Take $p = (5/2)$, $p(t) = (1-t)^2$, $q(t) = t^{2/3}$, and $f(t, x, y) = a(t)x^{1/4} + b(t)y^{-1/3}$.

We consider the following BVP:

$$\begin{cases} D_{0+}^{5/2}x(t) + (1-t)^2[a(t)x^{1/4} + b(t)y^{-1/3}] + t^{2/3} = 0, & 0 < t < 1, \\ x(0) = x'(0) = 0, & x(1) = 0. \end{cases} \quad (39)$$

For any $\lambda \in (0, 1)$, take $k = 1/2$, we can prove that

$$f(t, \lambda x, \lambda^{-1}y) \geq \lambda^k f(t, x, y). \quad (40)$$

Combined with the expression of f , p , and q , it is easy to see that H1 and H2 are holding. In addition,

$$0 < \int_0^1 [(1-t)^2 f(t, (1-t)t^{3/2}, (1-t)t^{3/2}) + t^{3/2}]dt < \infty. \quad (41)$$

So all of the assumptions in Theorem 4 are satisfied. Then, BVP (39) has a unique positive solution x^* . For any initial value $x_0 \in P$, we can construct the successive iterative sequence $\{x_n(t)\}$ as follows

$$x_n(t) = \int_0^1 G(t, s)[p(s)f(s, x_{n-1}(s), x_{n-1}(s)) + q(s)]ds, \quad n = 1, 2, \dots, \quad (42)$$

which uniformly converges to the unique positive solution x^* on $(0, 1)$. The error estimation is the same order infinitesimal of $(1 - c^{k_n})$, i.e.,

$$\max \{|x_n(t) - x^*(t)|\} = o\left(1 - c^{(1/4)^n}\right), \quad (43)$$

where $c \in (0, 1)$ is a constant and determined by the initial value x_0 . In addition, for any $t \in (0, 1)$, there exists a constant $l \in (0, 1)$ that satisfies

$$l(1-t)t^{p-1} \leq x^*(t) \leq l^{-1}(1-t)t^{p-1}. \quad (44)$$

5. Conclusions

We obtain the uniqueness of the positive solutions for a new kind of fractional differential equation with a negatively perturbed term. Through constructing an iterative process u_n , which can be from any initial value $u_0 \in P$, only by the iterative algorithm can we prove that it converges uniformly to the unique positive solution where the error estimation is the same order infinitesimal of $(1 - c^{k_n})$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' Contributions

All authors typed, read, and approved the final manuscript.

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