Research Article

Nonexistence of Global Weak Solutions of a System of Nonlinear Wave Equations with Nonlinear Fractional Damping

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We consider the system of nonlinear wave equations with nonlinear time fractional damping

$$\begin{cases} u_{tt} + (-\Delta)^m u + {}^C D_{0,t}^\alpha (t^\sigma |u|^q) = |v|^p, & t > 0, x \in \mathbb{R}^N, \\ v_{tt} + (-\Delta)^m v + {}^C D_{0,t}^\beta (t^\delta |v|^r) = |u|^s, & t > 0, x \in \mathbb{R}^N, \\ (u(0, x), u_t(0, x)) = (u_0(x), u_1(x)), & x \in \mathbb{R}^N, \\ (v(0, x), v_t(0, x)) = (v_0(x), v_1(x)), & x \in \mathbb{R}^N, \end{cases}$$

where $(u, v) = (u(t, x), v(t, x))$, m and N are positive natural numbers, $p, q, r, s > 1$, $\sigma, \delta \geq 0$, $0 < \alpha, \beta < 1$, and ${}^C D_{0,t}^\kappa$, $0 < \kappa < 1$, is the Caputo fractional derivative of order κ . Namely, sufficient criteria are derived so that the system admits no global weak solution. To the best of our knowledge, the considered system was not previously studied in the literature.

1. Introduction

In this paper, we investigate the system of nonlinear wave equations with nonlinear time fractional damping:

$$\begin{cases} u_{tt} + (-\Delta)^m u + {}^C D_{0,t}^\alpha (t^\sigma |u|^q) = |v|^p, & t > 0, x \in \mathbb{R}^N, \\ v_{tt} + (-\Delta)^m v + {}^C D_{0,t}^\beta (t^\delta |v|^r) = |u|^s, & t > 0, x \in \mathbb{R}^N, \\ (u(0, x), u_t(0, x)) = (u_0(x), u_1(x)), & x \in \mathbb{R}^N, \\ (v(0, x), v_t(0, x)) = (v_0(x), v_1(x)), & x \in \mathbb{R}^N, \end{cases} \quad (1)$$

where $(u, v) = (u(t, x), v(t, x))$, m and N are positive natural numbers, $p, q, r, s > 1$, σ and δ are nonnegative numbers that will be specified later, $0 < \alpha, \beta < 1$, and ${}^C D_{0,t}^\kappa$, $0 < \kappa < 1$, is the Caputo fractional derivative (with respect to t) of order κ . Namely, we are interested in obtaining sufficient conditions for which the considered system admits no global weak solution. Our approach is based on the nonlinear capacity method (see, e.g., [1]).

Before we state and prove our result, let us dwell on existence literature. Single wave equations or systems of wave equations have been studied in large; we may mention the books of Lions [2], Reed [3], Georgiev [4], and Strauss [5] and the papers of Aliev et al. [6], Said-Houari [7], Takeda [8], Georgiev and Todorova [9], Todorova and Yordanov [10], Zhang [11], and Kirane and Qafsaoui [12] for equations and systems with classical linear or nonlinear damping and Tatar [13], Kirane and Tatar [14], and Kirane and Laskri [15] for wave equations with fractional damping. In particular, in [13], the following problem was considered:

$$\begin{cases} u_{tt} + {}^C D_{0,t}^\alpha u - \Delta u = a|u|^{(p-1)}u, & t > 0, x \in \Omega, \\ u(t, x) = 0, & t > 0, x \in \partial\Omega, \\ (u(0, x), u_t(0, x)) = (u_0(x), u_1(x)), & x \in \Omega, \end{cases} \quad (2)$$

where Ω is a bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $a > 0$, $p > 1$, and $0 < \alpha < 1$. It was shown that, if u is a solution to (2), then there exist $T^* \leq \infty$ and sufficiently large

initial data so that u blows up at T^* . Problem (2) was also considered in [14]. Namely, it was shown that the solution of (2) is unbounded and grows up exponentially in the L^{p+1} -norm for sufficiently large initial data. In [15], the following problem was studied:

$$\begin{cases} u_{tt} - \Delta u + (-\Delta)^{\beta/2} D_{(0,t)}^\alpha u = |u|^p, & t > 0, x \in \mathbb{R}^N, \\ (u(0, x), u_t(0, x)) = (u_0(x), u_1(x)), & x \in \mathbb{R}^N, \end{cases} \quad (3)$$

where $p > 1$, $1 \leq \beta \leq 2$, $(-\Delta)^{\beta/2}$ is the fractional Laplacian of order $\beta/2$, $0 < \alpha < 1$, and $D_{0,t}^\alpha$ is the Riemann-Liouville fractional derivative of order α . It was shown that for all $p > 1$, if $\lim_{|x| \rightarrow +\infty} \inf u_1(x) = +\infty$, then (3) does not admit a local weak solution for any $T > 0$.

In the next section, we recall some notions on fractional calculus. In Section 3, we define global weak solutions of system (1) and state our main result. Moreover, as a consequence, we deduce a nonexistence result in the case of a single equation. Finally, the proof of the main result is given in Section 4.

2. Preliminaries

Some preliminaries on fractional calculus (see, e.g., [16–21]) are provided in this section. Given $\varsigma > 0$, the fractional integrals $I_0^\varsigma f$ and $I_T^\varsigma f$ of $f \in L^1[0, T]$ are defined by

$$\begin{aligned} (I_0^\varsigma f)(t) &= \frac{1}{\Gamma(\varsigma)} \int_0^t (t - \xi)^{\varsigma-1} f(\xi) d\xi, & t \in [0, T] \text{ a.e.}, \\ (I_T^\varsigma f)(t) &= \frac{1}{\Gamma(\varsigma)} \int_t^T (\xi - t)^{\varsigma-1} f(\xi) d\xi, & t \in [0, T] \text{ a.e.}, \end{aligned} \quad (4)$$

where Γ denotes the gamma function.

Lemma 1. Let $\varsigma > 0$, $f \in L^1[0, T]$, and $g \in L^\infty[0, T]$. Then,

$$\int_0^T (I_0^\varsigma f)(t) g(t) dt = \int_0^T f(t) (I_T^\varsigma g)(t) dt. \quad (5)$$

Given $0 < \varsigma < 1$, the Caputo fractional derivative ${}^C D_0^\varsigma f$ of $f \in AC[0, T]$ is defined by

$$({}^C D_0^\varsigma f)(t) = (I_0^1 - \varsigma) f'(t), \quad t \in [0, T] \text{ a.e.} \quad (6)$$

The following lemma will be used later in the proof of our main result.

Lemma 2. Let $0 < \varsigma < 1$ and $A : [0, T] \rightarrow \mathbb{R}$ be the function defined by

$$A(t) = T^{-\gamma}(T - t)^\gamma, \quad (7)$$

where $\gamma > 0$. Then,

$$(I_T^{1-\varsigma} A)(t) = \frac{\Gamma(\gamma + 1)}{\Gamma(2 - \varsigma + \gamma)} T^{-\gamma} (T - t)^{\gamma+1-\varsigma}, \quad (8)$$

$$(I_T^{1-\varsigma} A)(t) = \frac{-\Gamma(\gamma + 1)}{\Gamma(1 - \varsigma + \gamma)} T^{-\gamma} (T - t)^{\gamma-\varsigma}, \quad (9)$$

for all $t \in (0, T)$.

Proof. For $0 < t < T$, one has

$$\begin{aligned} (I_T^{1-\varsigma} A)(t) &= \frac{1}{\Gamma(1 - \varsigma)} \int_t^T (\xi - t)^{-\varsigma} A(\xi) d\xi \\ &= \frac{T^{-\gamma}}{\Gamma(1 - \varsigma)} \int_t^T (\xi - t)^{-\varsigma} (T - \xi)^\gamma d\xi \\ &= \frac{T^{-\gamma} (T - t)^\gamma}{\Gamma(1 - \varsigma)} \int_t^T (\xi - t)^{-\varsigma} \left(1 - \frac{\xi - t}{T - t}\right)^\gamma d\xi. \end{aligned} \quad (10)$$

Taking $\rho = (\xi - t)/(T - t)$, one obtains (8). Next, (9) follows immediately from (8).

3. Main Result

We begin with the definition of the intended solutions of system (1). Given $0 < T < \infty$, let $\mathcal{Q}_T = [0, T] \times \mathbb{R}^N$ and Φ_T be the set of functions $\varphi = \varphi(t, x) \in C_{t,x}^{2,2m}(\mathcal{Q}_T)$ satisfying the following conditions:

- (a) $\varphi(T, \cdot) = \varphi_t(T, \cdot) \equiv 0$
- (b) $\text{supp}_x \varphi \subset \subset \mathbb{R}^N$, i.e., $\text{supp}_x \varphi(t, x) \subset K \subset \mathbb{R}^N$ uniformly in $t \in [0, T]$, where K is a compact
- (c) $(I_T^{1-\kappa} \varphi)_t \in L^\infty(\mathcal{Q}_T^-)$, for all $0 < \kappa < 1$

Definition 3. Let $u_i, v_i \in L_{\text{loc}}^1(\mathbb{R}^N)$, $i = 0, 1$. A pair of functions

$$(u, v) \in L_{\text{loc}}^{\max\{q,s\}}([0, \infty) \times \mathbb{R}^N) \times L_{\text{loc}}^{\max\{p,r\}}([0, \infty) \times \mathbb{R}^N) \quad (11)$$

is said to be a global weak solution of system (1), if for all $0 < T < \infty$ and $\varphi \in \Phi_T$,

$$\begin{aligned} &\int_{\mathcal{Q}_T} |v|^p \varphi dx dt + \int_{\mathbb{R}^N} \varphi(0, x) u_1 dx - \int_{\mathbb{R}^N} \varphi_t(0, x) u_0 dx \\ &= \int_{\mathcal{Q}_T} u \varphi_{tt} dx dt + \int_{\mathcal{Q}_T} u (-\Delta)^m \varphi dx dt \\ &\quad - \int_{\mathcal{Q}_T} t^\sigma |u|^q (I_T^{1-\alpha} \varphi)_t dx dt, \end{aligned} \quad (12)$$

$$\begin{aligned}
& \int_{\mathcal{Q}_T} |u|^s \varphi \, dx \, dt + \int_{\mathbb{R}^N} \varphi(0, x) v_1 \, dx - \int_{\mathbb{R}^N} \varphi_t(0, x) v_0 \, dx \\
&= \int_{\mathcal{Q}_T} v \varphi_{tt} \, dx \, dt + \int_{\mathcal{Q}_T} v(-\Delta)^m \varphi \, dx \, dt \\
&\quad - \int_{\mathcal{Q}_T} t^\delta |v|^r \left(I_T^{1-\beta} \varphi \right)_t \, dx \, dt.
\end{aligned} \tag{13}$$

Next, we introduce the parameters

$$\begin{aligned}
\rho_1 &= \left(\frac{N}{m} + 1 \right) \left(\frac{ps-1}{p} \right) - 2(s+1), \\
\rho_2 &= \left(\frac{N}{m} + 1 \right) \left(\frac{ps-r}{p} \right) - 2s + \delta - \beta, \\
\rho_3 &= \left(\frac{N}{m} + 1 \right) \left(\frac{ps-q}{p} \right) + s(\sigma - \alpha) - 2q, \\
\rho_4 &= \left(\frac{N}{m} + 1 \right) \left(\frac{ps-rq}{p} \right) + s(\sigma - \alpha) + q(\delta - \beta) \\
\bar{\rho}_1 &= \left(\frac{N}{m} + 1 \right) \left(\frac{ps-1}{s} \right) - 2(p+1), \\
\bar{\rho}_2 &= \left(\frac{N}{m} + 1 \right) \left(\frac{ps-q}{s} \right) - 2p + \sigma - \alpha, \\
\bar{\rho}_3 &= \left(\frac{N}{m} + 1 \right) \left(\frac{ps-r}{s} \right) + p(\delta - \beta) - 2r, \\
\bar{\rho}_4 &= \left(\frac{N}{m} + 1 \right) \left(\frac{ps-rq}{s} \right) + p(\delta - \beta) + r(\sigma - \alpha).
\end{aligned} \tag{14}$$

Here, m and N are positive natural numbers, $p, q, r, s > 1$, σ and δ are positive numbers that will be specified later, $0 < \alpha, \beta < 1$, and ${}^C D_{0,t}^\kappa, 0 < \kappa < 1$.

Theorem 4. Let $0 < \alpha, \beta < 1$, $\sigma, \delta \geq 0$, $s > q > 1$, and $p > r > 1$. Suppose that $u_i, v_i \in L^1(\mathbb{R}^N)$, $i = 0, 1$, and

$$\min \left\{ \int_{\mathbb{R}^N} u_1(x) \, dx, \int_{\mathbb{R}^N} v_1(x) \, dx \right\} > 0. \tag{16}$$

If

$$\min \{ \max \{ \rho_i : i = 1, 2, 3, 4 \}, \max \{ \bar{\rho}_i : i = 1, 2, 3, 4 \} \} < 0, \tag{17}$$

then there exists no global weak solution of system (1).

Consider now the case of a single equation. Namely,

$$\begin{cases} u_{tt} + (-\Delta)^m u + {}^C D_{0,t}^\alpha (t^\sigma |u|^q) = |u|^p, & t > 0, x \in \mathbb{R}^N, \\ (u(0, x), u_t(0, x)) = (u_0(x), u_1(x)), & x \in \mathbb{R}^N, \end{cases} \tag{18}$$

where $0 < \alpha < 1$, $\sigma \geq 0$, and $p, q > 1$. A global weak solution of (18) can be defined in a similar way as in Definition 3. Taking in Theorem 4 $u = v$, $\alpha = \beta$, $\sigma = \delta$, $q = r$, $p = s$, and $(u_0, u_1) = (v_0, v_1)$, one deduces the following corollary.

Corollary 5. Let $p > q > 1$ and $u_i \in L^1(\mathbb{R}^N)$, $i = 0, 1$. Suppose that

$$\int_{\mathbb{R}^N} u_1(x) \, dx > 0. \tag{19}$$

If

$$\frac{N}{m} < \min \left\{ \frac{p+1}{p-1}, \frac{p^2 + (\alpha - \sigma)p + q}{p^2 - q}, \frac{(\alpha - \sigma - 1)p^2 + 2pq + q}{p^2 - q}, \frac{(\alpha - \sigma - 1)p + q}{p - q} \right\}, \tag{20}$$

then there exists no global weak solution of problem (18).

Example 6. Consider the equation

$$\begin{cases} u_{tt} + (-\Delta)^4 u + {}^C D_{0,t}^{2/3} (|u|^2) = |u|^3, & t > 0, x \in \mathbb{R}^2, \\ (u(0, x), u_t(0, x)) = (e^{-|x|}, |x|^2 e^{-|x|}), & x \in \mathbb{R}^2. \end{cases} \tag{21}$$

Observe that (21) is a special case of (18) with $m = 4$, $N = 2$, $\alpha = 2/3$, $\sigma = 0$, $q = 2$, $p = 3$, $u_0(x) = e^{-|x|}$, and $u_1(x) = |x|^2 e^{-|x|}$. One has $p > q > 1$, $u_0, u_1 \in L^1(\mathbb{R}^N)$, and

$$\int_{\mathbb{R}^N} u_1(x) \, dx = \int_{\mathbb{R}^2} |x|^2 e^{-|x|} \, dx > 0. \tag{22}$$

On the other hand, one can check easily that

$$\min \left\{ \frac{p+1}{p-1}, \frac{p^2 + (\alpha - \sigma)p + q}{p^2 - q}, \frac{(\alpha - \sigma - 1)p^2 + 2pq + q}{p^2 - q}, \frac{(\alpha - \sigma - 1)p + q}{p - q} \right\} = 1. \tag{23}$$

Since

$$\frac{N}{m} = \frac{2}{4} < 1, \tag{24}$$

it follows from Corollary 5 where (21) admits no global weak solution.

4. Proof of Theorem 4

Before proving Theorem 4, we need some preliminary results.

Given $0 < T < \infty$, let

$$\varphi(t, x) = A(t)B(x), \quad (t, x) \in \mathcal{Q}_T, \tag{25}$$

where A is defined by (7) with $\gamma \gg 1$ and

$$B(x) = \Xi \left(\frac{|x|^{2m}}{T^{2\ell m}} \right)^\mu, \quad x \in \mathbb{R}^N, \quad (26)$$

where $\mu \gg 1$, $\ell > 0$ is a certain parameter that will be specified later, and $\Xi \in C_0^\infty([0, \infty))$ is a decreasing function satisfying

$$\Xi(\rho) = \begin{cases} 1 & \text{if } 0 \leq \rho \leq 1, \\ 0 & \text{if } \rho \geq 2. \end{cases} \quad (27)$$

Lemma 7. *For all $0 < T < \infty$, the function φ defined by (25) belongs to Φ_T .*

Proof. One can check easily that $\varphi \in C_{t,x}^{2,2m}(\mathcal{Q}_T)$ and satisfies conditions (a) and (b). On the other hand, by (9), for all $0 < \kappa < 1$, one obtains

$$\begin{aligned} (I_T^{1-\kappa} \varphi)_t(t, x) &= B(x) (I_T^{1-\kappa} A)_t(t) \\ &= \frac{-\Gamma(\gamma + 1)}{\Gamma(1 - \kappa + \gamma)} T^{-\gamma} (T - t)^{\gamma - \kappa} B(x) \\ &:= -C(\gamma, \kappa) T^{-\gamma} (T - t)^{\gamma - \kappa} B(x), \quad (t, x) \in \mathcal{Q}_T, \end{aligned} \quad (28)$$

which yields

$$|(I_T^{1-\kappa} \varphi)_t(t, x)| \leq C(\gamma, \kappa) T^{-\kappa}, \quad (t, x) \in \mathcal{Q}_T. \quad (29)$$

This shows that φ satisfies condition (c).

The following estimate follows from elementary calculations.

Lemma 8. *There exists a constant $C > 0$ such that*

$$|\Delta^m [\Xi(|y|^{2m})]^\mu| \leq C \Xi(|y|^{2m})^{\mu-2m}, \quad y \in \mathbb{R}^N, \quad (30)$$

for any positive natural number m .

Proof of Theorem 4. Suppose that

$$(u, v) \in L_{\text{loc}}^{\max\{q, s\}}([0, \infty) \times \mathbb{R}^N) \times L_{\text{loc}}^{\max\{p, r\}}([0, \infty) \times \mathbb{R}^N) \quad (31)$$

is a global weak solution of system (1). For $T \gg 1$, using (12) and Lemma 7, one obtains

$$\begin{aligned} &\int_{\mathcal{Q}_T} |v|^p \varphi \, dx \, dt + \int_{\mathbb{R}^N} \varphi(0, x) u_1 \, dx - \int_{\mathbb{R}^N} \varphi_t(0, x) u_0 \, dx \\ &\leq \int_{\mathcal{Q}_T} |u| |\varphi_{tt}| \, dx \, dt + \int_{\mathcal{Q}_T} |u| |\Delta^m \varphi| \, dx \, dt \\ &\quad + \int_{\mathcal{Q}_T} t^\sigma |u|^q |(I_T^{1-\alpha} \varphi)_t| \, dx \, dt, \end{aligned} \quad (32)$$

where φ is defined by (25). Similarly, using (13), one obtains

$$\begin{aligned} &\int_{\mathcal{Q}_T} |u|^s \varphi \, dx \, dt + \int_{\mathbb{R}^N} \varphi(0, x) v_1 \, dx - \int_{\mathbb{R}^N} \varphi_t(0, x) v_0 \, dx \\ &\leq \int_{\mathcal{Q}_T} |v| |\varphi_{tt}| \, dx \, dt + \int_{\mathcal{Q}_T} |v| |\Delta^m \varphi| \, dx \, dt \\ &\quad + \int_{\mathcal{Q}_T} t^\delta |v|^r |(I_T^{1-\beta} \varphi)_t| \, dx \, dt. \end{aligned} \quad (33)$$

Using Hölder's inequality, the following holds:

$$\begin{aligned} &\int_{\mathcal{Q}_T} |u| |\varphi_{tt}| \, dx \, dt \leq \left(\int_{\mathcal{Q}_T} |u|^s \varphi \, dx \, dt \right)^{1/s} \\ &\quad \cdot \left(\int_{\mathcal{Q}_T} \varphi^{-1/(s-1)} |\varphi_{tt}|^{s/(s-1)} \, dx \, dt \right)^{(s-1)/s}. \end{aligned} \quad (34)$$

Similarly, one has

$$\begin{aligned} &\int_{\mathcal{Q}_T} |u| |\Delta^m \varphi| \, dx \, dt \leq \left(\int_{\mathcal{Q}_T} |u|^s \varphi \, dx \, dt \right)^{1/s} \\ &\quad \cdot \left(\int_{\mathcal{Q}_T} \varphi^{-1/(s-1)} |\Delta^m \varphi|^{s/(s-1)} \, dx \, dt \right)^{(s-1)/s}, \end{aligned} \quad (35)$$

$$\begin{aligned} &\int_{\mathcal{Q}_T} t^\sigma |u|^q |(I_T^{1-\alpha} \varphi)_t| \, dx \, dt \leq \left(\int_{\mathcal{Q}_T} |u|^s \varphi \, dx \, dt \right)^{q/s} \\ &\quad \cdot \left(\int_{\mathcal{Q}_T} t^{\sigma s/(s-q)} \varphi^{-q/(s-q)} |(I_T^{1-\alpha} \varphi)_t|^{s/(s-q)} \, dx \, dt \right)^{(s-q)/s}. \end{aligned} \quad (36)$$

It follows from (32), (34), (35), and (36) that

$$\begin{aligned} &\int_{\mathcal{Q}_T} |v|^p \varphi \, dx \, dt + \int_{\mathbb{R}^N} \varphi(0, x) u_1 \, dx - \int_{\mathbb{R}^N} \varphi_t(0, x) u_0 \, dx \\ &\leq \left(\int_{\mathcal{Q}_T} |u|^s \varphi \, dx \, dt \right)^{1/s} \left(\int_{\mathcal{Q}_T} \varphi^{-1/(s-1)} |\varphi_{tt}|^{s/(s-1)} \, dx \, dt \right)^{(s-1)/s} \\ &\quad + \left(\int_{\mathcal{Q}_T} |u|^s \varphi \, dx \, dt \right)^{1/s} \\ &\quad \cdot \left(\int_{\mathcal{Q}_T} \varphi^{-1/(s-1)} |\Delta^m \varphi|^{s/(s-1)} \, dx \, dt \right)^{(s-1)/s} \\ &\quad + \left(\int_{\mathcal{Q}_T} |u|^s \varphi \, dx \, dt \right)^{q/s} \\ &\quad \cdot \left(\int_{\mathcal{Q}_T} t^{\sigma s/(s-q)} \varphi^{-q/(s-q)} |(I_T^{1-\alpha} \varphi)_t|^{s/(s-q)} \, dx \, dt \right)^{(s-q)/s}. \end{aligned} \quad (37)$$

Similarly, using (33) and Hölder's inequality, one obtains

$$\begin{aligned} & \int_{\mathcal{Q}_T} |u|^s \varphi \, dx \, dt + \int_{\mathbb{R}^N} \varphi(0, x) v_1 \, dx - \int_{\mathbb{R}^N} \varphi_t(0, x) v_0 \, dx \\ & \leq \left(\int_{\mathcal{Q}_T} |v|^p \varphi \, dx \, dt \right)^{1/p} \left(\int_{\mathcal{Q}_T} \varphi^{-1/(p-1)} |\varphi_{tt}|^{p/(p-1)} \, dx \, dt \right)^{(p-1)/p} \\ & \quad + \left(\int_{\mathcal{Q}_T} |v|^p \varphi \, dx \, dt \right)^{1/p} \left(\int_{\mathcal{Q}_T} \varphi^{-1/(p-1)} |\Delta^m \varphi|^{p/(p-1)} \, dx \, dt \right)^{(p-1)/p} \\ & \quad + \left(\int_{\mathcal{Q}_T} |v|^p \varphi \, dx \, dt \right)^{r/p} \left(\int_{\mathcal{Q}_T} t^{\delta p/(p-r)} \varphi^{-r/(p-r)} \left| (I_T^{1-\beta} \varphi)_t \right|^{p/(p-r)} \, dx \, dt \right)^{(p-r)/p}. \end{aligned} \quad (38)$$

Now, set

$$\begin{aligned} I &= \left(\int_{\mathcal{Q}_T} |u|^s \varphi \, dx \, dt \right)^{1/s}, \\ J &= \left(\int_{\mathcal{Q}_T} |v|^p \varphi \, dx \, dt \right)^{1/p}. \end{aligned} \quad (39)$$

For $\iota > 1$, let

$$\begin{aligned} K_t &= \left(\int_{\mathcal{Q}_T} \varphi^{-1/(\iota-1)} |\varphi_{tt}|^{\iota/(\iota-1)} \, dx \, dt \right)^{(\iota-1)/\iota}, \\ L_t &= \left(\int_{\mathcal{Q}_T} \varphi^{-\iota/(\iota-1)} |\Delta^m \varphi|^{\iota/(\iota-1)} \, dx \, dt \right)^{(\iota-1)/\iota}. \end{aligned} \quad (40)$$

For $\zeta \geq 0$, $0 < \kappa < 1$, and $a > b > 1$, let

$$M_{\zeta, \kappa, a, b} = \left(\int_{\mathcal{Q}_T} t^{\zeta a/(a-b)} \varphi^{-b/(a-b)} \left| (I_T^{1-\kappa} \varphi)_t \right|^{a/(a-b)} \, dx \, dt \right)^{(a-b)/a}. \quad (41)$$

From (37) and (38), we may write

$$\begin{aligned} J^p &+ \int_{\mathbb{R}^N} \varphi(0, x) u_1 \, dx - \int_{\mathbb{R}^N} \varphi_t(0, x) u_0 \, dx \\ &\leq I(K_s + L_s) + I^q M_{\sigma, \alpha, s, q}, \end{aligned} \quad (42)$$

$$\begin{aligned} I^s &+ \int_{\mathbb{R}^N} \varphi(0, x) v_1 \, dx - \int_{\mathbb{R}^N} \varphi_t(0, x) v_0 \, dx \\ &\leq J(K_p + L_p) + J^r M_{\delta, \beta, p, r}. \end{aligned} \quad (43)$$

On the other hand, from (25), for $i = 0, 1$, one has

$$\int_{\mathbb{R}^N} \varphi(0, x) u_i \, dx = \int_{\mathbb{R}^N} \Xi \left(\frac{|x|^{2m}}{T^{2\ell m}} \right)^{\iota} u_i \, dx. \quad (44)$$

Since $u_i \in L^1(\mathbb{R}^N)$, $i = 0, 1$, by the dominated convergence

theorem, one obtains

$$\lim_{T \rightarrow \infty} \int_{\mathbb{R}^N} \varphi(0, x) u_i \, dx = \int_{\mathbb{R}^N} u_i \, dx < \infty. \quad (45)$$

Similarly, since $v_i \in L^1(\mathbb{R}^N)$, $i = 0, 1$, the following holds:

$$\lim_{T \rightarrow \infty} \int_{\mathbb{R}^N} \varphi(0, x) v_i \, dx = \int_{\mathbb{R}^N} v_i \, dx < \infty. \quad (46)$$

Again, from (25), one has

$$-\int_{\mathbb{R}^N} \varphi_t(0, x) u_0 \, dx = T^{-1} \int_{\mathbb{R}^N} \varphi(0, x) u_0 \, dx, \quad (47)$$

$$-\int_{\mathbb{R}^N} \varphi_t(0, x) v_0 \, dx = T^{-1} \int_{\mathbb{R}^N} \varphi(0, x) v_0 \, dx. \quad (48)$$

Using (16), (45), and (47), one obtains

$$\begin{aligned} & \lim_{T \rightarrow \infty} \int_{\mathbb{R}^N} \varphi(0, x) u_1 \, dx - \int_{\mathbb{R}^N} \varphi_t(0, x) u_0 \, dx \\ &= \int_{\mathbb{R}^N} u_1 \, dx > 0. \end{aligned} \quad (49)$$

Similarly, using (16), (46), and (48), one obtains

$$\lim_{T \rightarrow \infty} \int_{\mathbb{R}^N} \varphi(0, x) v_1 \, dx - \int_{\mathbb{R}^N} \varphi_t(0, x) v_0 \, dx = \int_{\mathbb{R}^N} v_1 \, dx > 0. \quad (50)$$

Consequently, one has

$$\int_{\mathbb{R}^N} \varphi(0, x) u_1 \, dx - \int_{\mathbb{R}^N} \varphi_t(0, x) u_0 \, dx > 0, \quad T \gg 1, \quad (51)$$

$$\int_{\mathbb{R}^N} \varphi(0, x) v_1 \, dx - \int_{\mathbb{R}^N} \varphi_t(0, x) v_0 \, dx > 0, \quad T \gg 1. \quad (52)$$

Next, it follows from (42) and (51) that

$$J^p \leq I(K_s + L_s) + I^q M_{\sigma, \alpha, s, q}. \quad (53)$$

Similarly, (43) and (52) yield

$$I^s \leq J(K_p + L_p) + J^r M_{\delta, \beta, p, r}. \quad (54)$$

Using the inequality

$$(w + z)^\iota \leq 2^{\iota-1} (w^\iota + z^\iota), \quad w, z \geq 0, \iota > 1, \quad (55)$$

it follows from (53) that

$$\begin{aligned} J^{ps} &\leq 2^{s-1} \left[I^s (K_s + L_s)^s + I^{qs} M_{\sigma, \alpha, s, q}^s \right] \\ &\leq 2^{s-1} \left[2^{s-1} I^s (K_s^s + L_s^s) + I^{qs} M_{\sigma, \alpha, s, q}^s \right], \end{aligned} \quad (56)$$

where upon

$$J^{ps} \leq 2^{2s-2} \left[(K_s^s + L_s^s) I^s + I^{qs} M_{\sigma, \alpha, s, q}^s \right]. \quad (57)$$

Similarly, it follows from (54) that

$$I^{qs} \leq 2^{2q-2} \left[(K_p^q + L_p^q) J^q + J^{rq} M_{\delta, \beta, p, r}^q \right]. \quad (58)$$

Using (54), (57), and (58), the following holds:

$$J^{ps} \leq C \times \left[(K_p + L_p) (K_s^s + L_s^s) J + (K_s^s + L_s^s) M_{\delta, \beta, p, r} J^r + M_{\sigma, \alpha, s, q}^s (K_p^q + L_p^q) J^q + M_{\sigma, \alpha, s, q}^s M_{\delta, \beta, p, r}^q J^{rq} \right], \quad (59)$$

where $C > 0$ is a constant that depends only on s and q . Here and below, any positive constant independent of T is denoted by C . Next, using the ε -Young inequality with $\varepsilon > 0$ which is small enough, one deduces from (59) that

$$J^{ps} \leq C \times \left[(K_p + L_p)^{ps/(ps-1)} (K_s^s + L_s^s)^{ps/(ps-1)} + (K_s^s + L_s^s)^{ps/(ps-r)} M_{\delta, \beta, p, r}^{ps/(ps-r)} + \left(M_{\sigma, \alpha, s, q}^s (K_p^q + L_p^q) \right)^{ps/(ps-q)} + \left(M_{\sigma, \alpha, s, q}^s M_{\delta, \beta, p, r}^q \right)^{ps/(ps-rq)} \right]. \quad (60)$$

Similarly, one obtains

$$I^{ps} \leq C \times \left[(K_s + L_s)^{ps/(ps-1)} (K_p^p + L_p^p)^{ps/(ps-1)} + (K_p^p + L_p^p)^{ps/(ps-q)} M_{\sigma, \alpha, s, q}^{ps/(ps-q)} + \left(M_{\delta, \beta, p, r}^p (K_s^r + L_s^r) \right)^{ps/(ps-r)} + \left(M_{\delta, \beta, p, r}^p M_{\sigma, \alpha, s, q}^r \right)^{ps/(ps-rq)} \right]. \quad (61)$$

Further, we shall estimate K_l and L_l for $l > 1$ and $M_{\zeta, \kappa, a, b}$ for $\zeta > 0$, $0 < \kappa < 1$, and $a > b > 1$. From (25), one has

$$\begin{aligned} K_l &= \left(\int_{\mathcal{Q}_T} \varphi^{-1/(l-1)} |\varphi_{tt}|^{l/(l-1)} dx dt \right)^{(l-1)/l} \\ &= \left(\int_0^T A(t)^{-1/(l-1)} |A_{tt}(t)|^{l/(l-1)} dt \right)^{(l-1)/l} \\ &\quad \cdot \left(\int_{\mathbb{R}^N} B(x) dx \right)^{(l-1)/l} \\ &= C \left(T^{-\gamma} \int_0^T (T-t)^{\gamma-(2l/(l-1))} dt \right)^{(l-1)/l} \end{aligned}$$

$$\begin{aligned} &\cdot \left(\int_{\mathbb{R}^N} \Xi \left(\frac{|x|^{2m}}{T^{2\ell m}} \right)^\mu dx \right)^{(l-1)/l} \\ &= CT^{-(i+1)/i} \left(T^{\ell N} \int_{0 < |y|^{2m} < 2} \Xi(|y|^{2m}) dy \right)^{(l-1)/l}, \quad (62) \end{aligned}$$

which yields

$$K_l = CT^{\ell N - 1 - ((\ell N + 1)/i)}, \quad l > 1. \quad (63)$$

Similarly, using (25) and Lemma 8, one obtains

$$\begin{aligned} L_l &= \left(\int_{\mathcal{Q}_T} \varphi^{-1/(l-1)} |\Delta^m \varphi|^{l/(l-1)} dx dt \right)^{(l-1)/l} \\ &= \left(\int_0^T A(t) dt \right)^{(l-1)/l} \left(\int_{\mathbb{R}^N} B(x)^{-1/(l-1)} |\Delta^m B(x)|^{l/(l-1)} dx \right)^{(l-1)/l} \\ &= \left(\int_0^T T^{-\gamma} (T-t)^\gamma dt \right)^{(l-1)/l} \\ &\quad \cdot \left(\int_{\mathbb{R}^N} \Xi \left(\frac{|x|^{2m}}{T^{2\ell m}} \right)^{-\mu/(l-1)} \left| \Delta^m \Xi \left(\frac{|x|^{2m}}{T^{2\ell m}} \right)^\mu \right|^{l/(l-1)} dx \right)^{(l-1)/l} \\ &= CT^{-(l-1)/l} \left(T^{\ell N} \int_{\mathbb{R}^N} E(|y|^{2m})^{-\mu/(l-1)} |T^{-2\ell m} \Delta^m E(|y|^{2m})^\mu|^{l/(l-1)} dy \right)^{(l-1)/l} \\ &= CT^{-(l-1)/l} T^{((l-1)/i)(\ell N - (2\ell m)/(l-1))} \left(\int_{1 < |y|^{2m} < 2} E(|y|^{2m})^{-\mu/(l-1)} \right. \\ &\quad \cdot |\Delta^m E(|y|^{2m})^\mu|^{l/(l-1)} dy \left. \right)^{(l-1)/l} \\ &\leq CT^{\ell N - 2\ell m + 1 - ((\ell N + 1)/i)} \left(\int_{1 < |y|^{2m} < 2} E(|y|^{2m})^{\mu - (2m)/(l-1)} dy \right)^{(l-1)/l}, \quad (64) \end{aligned}$$

so

$$L_l \leq CT^{\ell N - 2\ell m + 1 - ((\ell N + 1)/i)}, \quad l > 1. \quad (65)$$

Next, using (9) and (25), for $\zeta \geq 0$, $0 < \kappa < 1$, and $a > b > 1$, one obtains

$$\begin{aligned} M_{\zeta, \kappa, a, b} &= \left(\int_{\mathcal{Q}_T} t^{\zeta a/(a-b)} \varphi^{-b/(a-b)} |(I_T^{1-\kappa} \varphi)_t|^{a/(a-b)} dx dt \right)^{(a-b)/a} \\ &= \left(\int_0^T t^{\zeta a/(a-b)} A(t)^{-b/(a-b)} |(I_T^{1-\kappa} A)(t)|^{a/(a-b)} dt \right)^{(a-b)/a} \\ &\quad \cdot \left(\int_{\mathbb{R}^N} \Xi \left(\frac{|x|^{2m}}{T^{2\ell m}} \right)^\mu dx \right)^{(a-b)/a} \\ &= C \left(T^{-\gamma} \int_0^T t^{\zeta a/(a-b)} (T-t)^{\gamma-(\kappa a/(a-b))} dt \right)^{(a-b)/a} \\ &\quad \cdot \left(T^{\ell N} \int_{0 < |y|^{2m} < 2} \Xi(|y|^{2m})^\mu dy \right)^{(a-b)/a}, \quad (66) \end{aligned}$$

whereupon

$$M_{\zeta, \kappa, a, b} = CT^{((a-b)/a)(\ell N+1)+\zeta-\kappa}, \quad \zeta \geq 0, 0 < \kappa < 1, a > b > 1. \quad (67)$$

Next, taking $\ell = 1/m$, it follows from (63), (65), and (67) that

$$K_i = L_i = O\left(T^{((N/m)+1)((i-1)/i)-2}\right), \quad i > 1, T \gg 1, \quad (68)$$

$$M_{\zeta, \kappa, a, b} = CT^{((a-b)/a)((N/m)+1)+\zeta-\kappa}, \quad \zeta \geq 0, 0 < \kappa < 1, a > b > 1. \quad (69)$$

Using (60), (68), and (69), the following holds:

$$J^p \leq C\left(T^{\rho_1(ps/(ps-1))} + T^{\rho_2(ps/(ps-r))} + T^{\bar{\rho}_3(ps/(ps-q))} + T^{\rho_4(ps/(ps-rq))}\right)^{1/s}, \quad (70)$$

where ρ_i , $i = 1, 2, 3, 4$, are defined by (14). Similarly, using (61), (68), and (69), one obtains

$$I^s \leq C\left(T^{\bar{\rho}_1(ps/(ps-1))} + T^{\bar{\rho}_2(ps/(ps-q))} + T^{\bar{\rho}_3(ps/(ps-r))} + T^{\bar{\rho}_4(ps/(ps-rq))}\right)^{1/p}, \quad (71)$$

where $\bar{\rho}_i$, $i = 1, 2, 3, 4$, are defined by (15).

Now, from condition (17), we have two cases.

Case 1. If $\max \{\rho_i : i = 1, 2, 3, 4\} < 0$, in this case, taking the infimum limit as $T \rightarrow \infty$ in (70) and using Fatou's lemma, the following holds:

$$v(t, x) = 0, \quad (t, x) \in [0, \infty) \times \mathbb{R}^N \text{ a.e.} \quad (72)$$

Hence, by (33), the following holds:

$$\int_{\mathbb{R}^N} \varphi(0, x) v_1 dx - \int_{\mathbb{R}^N} \varphi_t(0, x) v_0 dx \leq 0, \quad T \gg 1, \quad (73)$$

which contradicts (52).

Case 2. If $\max \{\bar{\rho}_i : i = 1, 2, 3, 4\} < 0$, as in the previous case, taking the infimum limit as $T \rightarrow \infty$ in (71), one obtains

$$u(t, x) = 0, \quad (t, x) \in [0, \infty) \times \mathbb{R}^N \text{ a.e.} \quad (74)$$

Hence, by (32), the following holds:

$$\int_{\mathbb{R}^N} \varphi(0, x) u_1 dx - \int_{\mathbb{R}^N} \varphi_t(0, x) u_0 dx \leq 0, \quad T \gg 1, \quad (75)$$

which contradicts (51).

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

All authors contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

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