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Research Article

Improved Fractional Subequation Method and Exact Solutions to Fractional Partial Differential Equations

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In this paper, the improved fractional subequation method is applied to establish the exact solutions for some nonlinear fractional partial differential equations. Solutions to the generalized time fractional biological population model, the generalized time fractional compound KdV-Burgers equation, the space-time fractional regularized long-wave equation, and the $(3+1)$-space-time fractional Zakharov-Kuznetsov equation are obtained, respectively.

1. Introduction

Fractional differential equations are widely used to describe lots of important phenomena and dynamic processes in physics, engineering, electromagnetics, acoustics, viscoelasticity electrochemistry, material science, stochastic dynamical system, plasma physics, controlled thermonuclear fusion, nonlinear control theory, image processing, nonlinear biological systems and astrophysics, etc. [1–7]. In order to find the solutions of fractional differential equations, many powerful and efficient methods have been introduced and developed, such as Darboux transformations [8], the hyperbolic function method [9], the variational iteration method [10, 11], the autofinite element method [12, 13], the auxiliary equation method [14], the finite difference method [15, 16], the Adomian decomposition method [17, 18], the homogenous balance method [19], Hirotas bilinear method [20], the homotopy analysis method [21, 22], (G'/G)-expansion method [23, 24], the subequation method [25, 26], the first integral method [27, 28], the improved fractional subequation method [29, 30], the extended Jacobi elliptic function expansion method [31], the generalized Kudryashov method [32], the exponential rational function method [33], the exp-function method [34, 35], the multiple exp-function method [36], the extended simple equation method [37].

In order to deal with nondifferentiable functions, Jumarie [38] has proposed a modification of the Riemann-Liouville definition which appears to provide a framework for a fractional calculus which is quite parallel to the classical calculus.

Jumarie's modified Riemann-Liouville derivative of order α for a function f is defined as follows:

$$D_x^\alpha f(x) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^x (x-\xi)^{-\alpha-1}[f(\xi)-f(0)]d\xi, & \alpha<0, \\ \dfrac{1}{\Gamma(1-\alpha)} \dfrac{d}{dx} \displaystyle\int_0^x (x-\xi)^{-\alpha}[f(\xi)-f(0)]d\xi, & 0<\alpha<1, \\ \left[f^{(n)}(x)\right]^{(\alpha-n)}, & n\le\alpha<n+1, n\ge 1. \end{cases} \tag{1}$$

Some useful properties of modified Riemann-Liouville derivative are given below:

$$D_x^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} x^{\gamma-\alpha}, \gamma > 0, \quad (2)$$

$$D_x^\alpha [f(x)g(x)] = g(x) D_x^\alpha f(x) + f(x) D_x^\alpha g(x), \quad (3)$$

$$D_x^\alpha f[g(x)] = f'_g[g(x)] D_x^\alpha g(x) = D_g^\alpha f[g(x)] \left[g'(x)\right]^\alpha, \quad (4)$$

which holds for nondifferentiable functions. Equations (2), (3), (4) which are important tools for fractional calculus. Based on these merits, the modified Riemann-Liouville derivative was successfully applied to the probability calculus, fractional Laplace problems, and fractional variational calculus.

In this paper, we aim to find new exact solutions of some important partial fractional differential equations under Jumarie's definition by improved fractional subequation method.

In what follows, we introduce the aforementioned fractional partial differential equations. They are the generalized time fractional biological population model, the generalized time fractional compound KdV-Burgers equation, the space-time fractional regularized long-wave equation, and the (3 + 1)-space-time fractional Zakharov-Kuznetsov equation.

Suppose time $t > 0$, $D_t^\alpha u$ is time modified Riemann-Liouville derivative of order α for a function u, $0 < \alpha \leq 1$, the parameters $k_1, k_2, \cdots, k_7$ are any real constants.

The generalized time fractional biological population model is given by

$$D_t^\alpha u + k_1 \left(u^2\right)_{xx} + k_2 \left(u^2\right)_{yy} + k_3 u_x + k_4 u_y + k_5 u^2 + k_6 u + k_7 = 0, \quad (5)$$

where $u = u(x, y, t)$ is an unknown function.

When $k_1 = -1, k_2 = -1, k_3 = k_4 = k_6 = 0, k_5 = -h \neq 0, k_7 = rh \neq 0$, Equation (5) is the time fractional biological population model [39]:

$$D_t^\alpha u = \left(u^2\right)_{xx} + \left(u^2\right)_{yy} + h\left(u^2 - r\right), \quad (6)$$

where $u = u(x, y, t)$ denotes the population density and h $(u^2 - r)$ represents the amount of population due to death and birth. Moreover, $h(u^2 - r)$ leads to Verhulst law. Equation (5) has an important role to understand the dynamic process of population changes, and it is also an assistant to achieve precision about it.

The generalized time fractional compound KdV-Burgers equation is given by

$$D_t^\alpha u + k_1 u u_x + k_2 u^2 u_x + k_3 u_{xx} + k_4 u_{xxx} = 0, \quad (7)$$

where $u = u(x, t)$ is an unknown function.

When $k_1 = k_3 = 0$, Equation (7) becomes the time fractional mKdV equation

$$D_t^\alpha u + k_2 u^2 u_x + k_4 u_{xxx} = 0\,; \quad (8)$$

when $k_1 = k_3 = 0$ Equation (7) becomes the time fractional KdV equation

$$D_t^\alpha u + k_2 u^2 u_x + k_3 u_{xx} + k_4 u_{xxx} = 0\,; \quad (9)$$

when $k_2 = k_4 = 0$, Equation (7) becomes the time fractional Burgers equation [40]

$$D_t^\alpha u + k_1 u u_x + k_3 u_{xx} = 0\,; \quad (10)$$

when $k_1 = 0$, Equation (7) becomes the time fractional mKdV-Burgers equation

$$D_t^\alpha u + k_2 u^2 u_x + k_3 u_{xx} + k_4 u_{xxx} = 0\,; \quad (11)$$

when $k_2 = 0$, Equation (7) becomes the time fractional KdV-Burgers equation

$$D_t^\alpha u + k_1 u u_x + k_3 u_{xx} + k_4 u_{xxx} = 0. \quad (12)$$

The space-time fractional regularized long-wave equation is given by [41]:

$$D_t^\alpha u + k_1 D_x^\alpha u + k_2 u D_x^\alpha u + k_3 D_t^\alpha D_x^2 \alpha u = 0, \quad (13)$$

where $u = u(x, t)$ is an unknown function, $D_x^\alpha u$ is the modified Riemann-Liouville derivative of order α for a function u, and $D_x^{2\alpha} u = D_x^\alpha (D_x^\alpha u)$.

The regularized long-wave equation, which describes approximately the unidirectional propagation of long waves in certain nonlinear dispersive systems, was proposed by Benjamin et al. in 1972. The regularized long-wave equation is considered an alternative to the KdV equation, which is modeled to govern a large number of physical phenomena such as shallow waters and plasma waves.

The (3 + 1)-space-time fractional Zakharov-Kuznetsov equation given by [41].

$$D_t^? u + k_1 u D_x^\alpha u + k_2 D_x^{3\alpha} u + k_3 D_x^\alpha D_y^{2\alpha} u + k_4 D_x^\alpha D_z^{2\alpha} u = 0, \quad (14)$$

where $u = u(x, y, z, t)$ is an unknown function; $D_x^\alpha u$, $D_y^\alpha u$, and $D_z^\alpha u$ are the modified Riemann-Liouville derivatives of the function u; $D_x^{2\alpha} u = D_x^\alpha (D_x^\alpha u)$; $D_y^{2\alpha} u = D_y^\alpha (D_y^\alpha u)$; $D_z^{2\alpha} u = D_z^\alpha$ $(D_z^\alpha u)$; and $D_x^{3\alpha} u = D_x^\alpha (D_x^\alpha (D_x^\alpha u))$.

The Zakharov-Kuznetsov equation was first derived for analysing weakly nonlinear ion acoustic waves in heavily magnetized lossless plasma and geophysical flows in two dimensions. The ZK equation is one of the two well-established canonical two-dimensional extensions of the KdV equation. The ZK equation governs the behavior of weakly nonlinear ionacoustic waves in a plasma comprising cold ions and hot isothermal electrons in the presence of a uniform magnetic field.

Motivated by the above results, in this paper, we use the improved subequation method to find new exact solutions of the generalized time fractional biological population model, the generalized time fractional compound KdV-Burgers equation, the space-time fractional regularized long-wave equation, and the $(3+1)$-space-time fractional Zakharov-Kuznetsov equation, respectively.

2. A Brief Description of the Improved Fractional Subequation Method

In this section, basic steps of the improved subequation method [42] are presented.

Consider the following nonlinear fractional differential equation,

$$P\left(u, D_t^\alpha u, D_{x_1}^\alpha u, D_{x_1}^\alpha u^2, \cdots, D_{x_m}^\alpha u, D_{x_1}^\alpha u^2, D_{x_2}^\alpha u^2, \cdots, D_{x_m}^\alpha u^2, \cdots, D_{x_1}^{2\alpha} u, D_{x_m}^{2\alpha} u, \cdots\right) = 0, \tag{15}$$

where $t, x_1, x_2, \cdots, x_m$ are independent variables, $u(t, x_1, x_2, \cdots, x_m)$ is an unknown function, and P is a polynomial of u, $u^2, \cdots$ and their partial fractional derivatives. Also, $D^\alpha(\cdot)$ symbolizes the modified Riemann-Liouville fractional derivative.

Step 1. First of all, using a suitable fractional complex transform,

$$u(t, x_1, x_2, \cdots, x_m) = u(\xi), \xi(t, x_1, x_2, \cdots, x_m), \tag{16}$$

Equation (15) converts into nonlinear ordinary differential equation given below:

$$Q\left(u, u', u'', u''', \cdots\right) = 0, \tag{17}$$

where $u', u'', u''', \cdots$ denotes the derivations with respect to ξ.

We used to utilize real transformation, but actually, complex transformations are more useful. They make some equations easier to simplify.

Step 2. Suppose that the solution of ordinary differential equation (17) is

$$u = \sum_{i=n}^{-1} a_i \varphi^i + a_0 + \sum_{i=n}^{n} a_i \varphi^i, \tag{18}$$

where constants $a_i (i = -n, \cdots, -1, 1, \cdots, n)$ are going to be determined. Here, n is a positive integer, and it is obtained using the homogeneous balance of the highest order derivative and the nonlinear term seen in Equation (17).

$\varphi = \varphi(\xi)$ is the solution of the Riccati equation

$$\varphi' = \sigma + \varphi^2, \tag{19}$$

where σ is a constant, and the solutions of Equation (19) are obtained by Zhang et al. [34] as follows:

$$\varphi(\xi) = \begin{cases} -\sqrt{-\sigma}\tanh\left(\sqrt{-\sigma}\xi\right), & \sigma < 0, \\ -\sqrt{-\sigma}\coth\left(\sqrt{-\sigma}\xi\right), & \sigma < 0, \\ \sqrt{\sigma}\tan\left(\sqrt{\sigma}\xi\right), & \sigma > 0, \\ -\sqrt{\sigma}\cot\left(\sqrt{\sigma}\xi\right), & \sigma > 0, \\ -\dfrac{\Gamma(1+\alpha)}{\xi^\alpha + \omega}, & \sigma = 0, \omega = \text{const}, \end{cases} \tag{20}$$

In the previous literatures, $u = a_0 + \sum_{i=1}^n a_i \varphi^i$ was considered. In this paper, we assume $u = \sum_{i=-n}^{-1} a_i \varphi^i + a_0 + \sum_{i=1}^n a_i \varphi^i$, and we can get a solution that has both hyperbolic tangent function and hyperbolic cotangent function or both tangent function and cotangent function.

Step 3. Putting Equation (18) along with Equation (19) into Equation (17), we obtain a new polynomial in terms of φ. Then, all the coefficients of powers of $\varphi^k (k = 0, 1, 2, \cdots, -1, -2, \cdots)$ are set equal to zero, we get a system of algebraic equations.

Step 4. Finally, the system of algebraic equations is obtained in the previous step for $a_i (-n \le i \le n)$, and σ is solved by the Maple package. By substituting the newly obtained values into Equation (20), we get the exact solutions for the nonlinear fractional differential equation (15).

Applying a suitable fractional complex transform of the improved fractional subequation method and the chain rule, nonlinear fractional differential equations with the modified Riemann-Liouville derivative can be converted into nonlinear ordinary differential equations. Then, using the solutions of a Riccati equation, we can find exact analytical solutions expressed by triangle functions, hyperbolic functions, or power functions.

3. Applications of the Improved Fractional Subequation Method

In this section, the improved fractional subequation method is utilized to solve some nonlinear fractional differential equations introduced in Section 1.

3.1. The Generalized Time Fractional Biological Population Model. The generalized time fractional biological population model is given by

$$D_t^\alpha u + k_1 \left(u^2\right)_{xx} + k_2 \left(u^2\right)_{yy} + k_3 u_x + k_4 u_y + k_5 u^2 + k_6 u + k_7 = 0, \tag{21}$$

where $u = u(x, y, t)$ is an unknown function.

We considered two cases.

Case 1. When $k_1 k_2 > 0$, let

$$u(x, y, t) = u(\xi), \xi = ax + ia\sqrt{\frac{k_1}{k_2}}y + \frac{bt^\alpha}{\Gamma(1+\alpha)}, a \neq 0, \tag{22}$$

then Equation (21) is reduced to the ordinary differential equation as follows:

$$\begin{cases} ak_4\sqrt{\dfrac{k_1}{k_2}}u' = 0, \\ (ak_3 + b)u' + k_5u^2 + k_6u + k_7 = 0. \end{cases} \tag{23}$$

When $k_4 \neq 0$, Equation (23) has only constant solutions.

When $k_4 = 0$, Equation (23) has solutions in the form of (18). u' is obtained from the homogeneous balance between the highest order derivative and the nonlinear term u^2. We obtain the solution of Equation (21) as follows:

$$u(\xi) = a - 1\varphi^{-1} + a_0 + a_1\varphi, \tag{24}$$

Substituting Equation (24) together with its necessary derivatives into Equation (21), the algebraic equation is arranged according to the powers of the function $\varphi^k(\xi)$. Then, the following coefficients are obtained:

$$\begin{aligned} \varphi^{-2} &: -k_3\sigma a_{-1} - b\sigma a_{-1} + k_5a^2{}_{-1}, \\ \varphi^{-1} &: 2k_5a_{-1}a_0 + a_{-1}k_6, \\ \varphi^{0} &: -ak_3a_{-1} - ba_{-1} + ak_3\sigma a_1 + b\sigma a_1 \\ &\quad + k_5a_0^2 + 2k_5a_{-1}a_1 + k_6a_0 + k_7, \\ \varphi^{1} &: 2k_5a_0a_1 + k_6a_1, \\ \varphi^{2} &: ak_3a_1 + ba_1 + k_5a_1^2. \end{aligned} \tag{25}$$

Let the coefficients be zero. By solving the set of equations given above for a_{-1}, a_0, a_1, a, b, and σ, we obtain solution sets as follows:

Set 1

$$\begin{aligned} a_{-1} &= 0, \\ a_0 &= -\frac{k_6}{2k_5}, \\ a_1 &= a_1, \\ a &= a, \\ b &= -ak_3 - a_1k_5, \\ \sigma &= \frac{\lambda}{4a_1^2k_5^2}. \end{aligned} \tag{26}$$

Set 2

$$\begin{aligned} a_{-1} &= a_{-1}, \\ a_0 &= -\frac{k_6}{2k_5}, \\ a_1 &= \frac{-\lambda}{16k_5^2a_{-1}}, \\ a &= a, \\ b &= \frac{\lambda - 16aa_{-1}k_3k_5}{16k_5a_{-1}}, \\ \sigma &= \frac{16k_5^2a_{-1}^2}{\lambda}. \end{aligned} \tag{27}$$

Set 3

$$\begin{aligned} a_{-1} &= a_{-1}, \\ a_0 &= -\frac{k_6}{2k_5}, \\ a_1 &= 0, \\ a &= a, \\ b &= \frac{\lambda - 4aa_{-1}k_3k_5}{4k_5a_{-1}}, \\ \sigma &= \frac{4k_5^2a_{-1}^2}{\lambda}. \end{aligned} \tag{28}$$

where $\lambda = 4k_5k_7 - k_6^2$.

Case 2. When $k_1k_2 < 0$, let

$$u(x, y, t) = u(\xi), \xi = ax + a\sqrt{-\frac{k_1}{k_2}}y + \frac{bt^\alpha}{\Gamma(1+\alpha)}, a \neq 0, \tag{29}$$

then Equation (21) is reduced to the ordinary differential equation as follows:

$$\left(b + k_3a + ak_4\sqrt{-\frac{k_1}{k_2}}\right)u' + k_5u^2 + k_6u + k_7 = 0. \tag{30}$$

The solution of Equation (30) is in the form of (18). $n = 1$ is taken from the homogeneous balance between the highest order derivative u' and the nonlinear term u^2. We obtain the solution of Equation (30) as Equation (24). Substituting Equation (24) together with its necessary derivatives into Equation (30), the algebraic equation is arranged according to the powers of the function $\varphi^k(\xi)$. Then, the following coefficients are obtained:

$$\begin{aligned} \varphi^{-2} &: \quad -a\sqrt{-\frac{k_1}{k_2}}k_4\sigma a_{-1} - ak_3\sigma a_{-1} - b\sigma a_{-1} + k_5a_{-1}^2, \\ \varphi^{-1} &: \quad 2k_5a_{-1}a_0 + a_{-1}k_6, \\ \varphi^{0} &: \quad a\sqrt{-\frac{k_1}{k_2}}k_4\sigma a_1 + ak_3\sigma a_1 + b\sigma a_1 - a\sqrt{-\frac{k_1}{k_2}}k_4a_{-1}, \\ \varphi^{1} &: \quad 2k_5a_0a_1 + k_6a_1, \end{aligned}$$

$$\varphi^2: \quad a\sqrt{-\frac{k_1}{k_2}}k_4a_1 + ak_3a_1 + ba_1 + k_5a_1^2. \tag{31}$$

Let the coefficients be zero. By solving the set of equations given above for a_{-1}, a_0, a_1, a, b, and σ, we obtain solution sets as follows:

Set 4

$$\begin{aligned} a_{-1} &= 0, \\ a_0 &= -\frac{k_6}{2k_5}, \\ a_1 &= a_1, \\ a &= a, \\ b &= -ak_3 - ak_4\sqrt{-\frac{k_1}{k_2}} - a_1k_5, \\ \sigma &= \frac{\lambda}{4a_1^2k_5^2}. \end{aligned} \tag{32}$$

Set 5

$$\begin{aligned} a_{-1} &= a_{-1}, \\ a_0 &= -\frac{k_6}{2k_5}, \\ a_1 &= \frac{-\lambda}{16k_5^2a_{-1}}, \\ a &= a, \\ b &= \frac{\lambda - 16\mu}{16k_5a_{-1}}, \\ \sigma &= \frac{16k_5^2a_{-1}^2}{\lambda}. \end{aligned} \tag{33}$$

Set 6

$$\begin{aligned} a_{-1} &= a_{-1}, \\ a_0 &= -\frac{k_6}{2k_5}, \\ a_1 &= 0, \\ a &= a, \\ b &= \frac{\lambda - 4\mu}{4k_5a_{-1}}, \\ \sigma &= \frac{4k_5^2a_{-1}^2}{\lambda}. \end{aligned} \tag{34}$$

where $\mu = aa_{-1}k_3k_5 + aa_{-1}k_4k_5\sqrt{-k_1/k_2}$.

We find that a_{-1}, a_0, a_1, and σ are equal in set 1 and set 4, set 2 and set 5, and set 3 and set 6, respectively. In this study, the solutions of differential equations are symbolized as $u_{(i,j)(x,y,t)}$, $(i, j \in Z^+)$, where i denotes obtained set number and j is the solution number of the Riccati equation, respectively. Thus, using set 1 to set 6, we obtain the solution of Equation (21) as $u_{i,j}(x, y, t)$, $(i = 1, 2, \cdots, 6; j = 1, 2, \cdots, 5)$. $u_{i,j}(x, y, t)$ is the following:

When $k_1k_2 > 0$, $\lambda = 4k_5k_7 - k_6^2 < 0$, we have $\sigma < 0$, then

$$u_{1,1}(x, y, t) = -\frac{k_6}{2k_5} - \frac{a_1\sqrt{-\lambda}}{2|a_1k_5|}\tanh\left[\frac{\sqrt{-\lambda}}{2|a_1k_5|}\left(ax + ia\sqrt{\frac{k_1}{k_2}}y - \frac{(ak_3 + a_1k_5)t^\alpha}{\Gamma(1+\alpha)}\right)\right],$$

$$u_{1,2}(x, y, t) = -\frac{k_6}{2k_5} - \frac{a_1\sqrt{-\lambda}}{2|a_1k_5|}\coth\left[\frac{\sqrt{-\lambda}}{2|a_1k_5|}\left(ax + ia\sqrt{\frac{k_1}{k_2}}y - \frac{(ak_3 + a_1k_5)t^\alpha}{\Gamma(1+\alpha)}\right)\right],$$

$$\begin{aligned} u_{2,1}(x, y, t) = &-\frac{a_{-1}\sqrt{-\lambda}}{4|k_5a_{-1}|}\coth\left[\frac{4|k_5a_{-1}|}{\sqrt{-\lambda}}\left(ax - ia\sqrt{\frac{k_1}{k_2}}y + \frac{(\lambda - 16aa_{-1}k_3k_5)t^\alpha}{16k_5a_{-1}\Gamma(1+\alpha)}\right)\right] - \frac{k_6}{2k_5} \\ &- \frac{|k_5a_{-1}|\sqrt{-\lambda}}{4k_5^2a_{-1}}\tanh\left[\frac{4|k_5a_{-1}|}{\sqrt{-\lambda}}\left(ax - ia\sqrt{\frac{k_1}{k_2}}y + \frac{(\lambda - 16aa_{-1}k_3k_5)t^\alpha}{16k_5a_{-1}\Gamma(1+\alpha)}\right)\right], \end{aligned}$$

$$\begin{aligned} u_{2,2}(x, y, t) = &-\frac{a_{-1}\sqrt{-\lambda}}{4|k_5a_{-1}|}\tanh\left[\frac{4|k_5a_{-1}|}{\sqrt{-\lambda}}\left(ax - ia\sqrt{\frac{k_1}{k_2}}y + \frac{(\lambda - 16aa_{-1}k_3k_5)t^\alpha}{16k_5a_{-1}\Gamma(1+\alpha)}\right)\right] - \frac{k_6}{2k_5} \\ &- \frac{|k_5a_{-1}|\sqrt{-\lambda}}{4k_5^2a_{-1}}\coth\left[\frac{4|k_5a_{-1}|}{\sqrt{-\lambda}}\left(ax - ia\sqrt{\frac{k_1}{k_2}}y + \frac{(\lambda - 16aa_{-1}k_3k_5)t^\alpha}{16k_5a_{-1}\Gamma(1+\alpha)}\right)\right], \end{aligned}$$

$$u_{3,1}(x, y, t) = \frac{a_1\sqrt{-\lambda}}{2|a_1k_5a_{-1}|}\coth\left[\frac{2|a_1k_5a_{-1}|}{\sqrt{-\lambda}}\left(ax + ia\sqrt{\frac{k_1}{k_2}}y - \frac{(\lambda - 4aa_{-1}k_3k_5)t^\alpha}{4k_5a_{-1}\Gamma(1+\alpha)}\right)\right] - \frac{k_6}{2k_5},$$

$$u_{3,2}(x,y,t)=\frac{a_{-1}\sqrt{-\lambda}}{2|a_1k_5a_{-1}|}\tanh\left[\frac{2|a_1k_5a_{-1}|}{\sqrt{-\lambda}}\left(ax+ia\sqrt{\frac{k_1}{k_2}}y-\frac{(\lambda-4aa_{-1}k_3k_5)t^{\alpha}}{4k_5a_{-1}\Gamma(1+\alpha)}\right)\right]-\frac{k_6}{2k_5}. \quad (35)$$

When $k_1k_2<0$, $\lambda=4k_5k_7-k_6^2<0$, we have $\sigma<0$, then

$$u_{4,1}(x,y,t)=-\frac{k_6}{2k_5}-\frac{a_1\sqrt{-\lambda}}{2|a_1k_5|}\tanh\left[\frac{\sqrt{-\lambda}}{2|a_1k_5|}\left(ax+a\sqrt{-\frac{k_1}{k_2}}y-\frac{\left(ak_3+ak_4\sqrt{-k_1/k_2}+a_1k_5\right)t^{\alpha}}{\Gamma(1+\alpha)}\right)\right],$$

$$u_{4,2}(x,y,t)=-\frac{k_6}{2k_5}-\frac{a_1\sqrt{-\lambda}}{2|a_1k_5|}\coth\left[\frac{\sqrt{-\lambda}}{2|a_1k_5|}\left(ax+a\sqrt{-\frac{k_1}{k_2}}y-\frac{\left(ak_3+ak_4\sqrt{-k_1/k_2}+a_1k_5\right)t^{\alpha}}{\Gamma(1+\alpha)}\right)\right],$$

$$u_{5,1}(x,y,t)=-\frac{a_{-1}\sqrt{-\lambda}}{4|k_5a_{-1}|}\coth\left[\frac{4|k_5a_{-1}|}{\sqrt{-\lambda}}\left(ax-a\sqrt{-\frac{k_1}{k_2}}y+\frac{(\lambda-16\mu)t^{\alpha}}{16k_5a_{-1}\Gamma(1+\alpha)}\right)\right]-\frac{k_6}{2k_5}-\frac{|k_5a_{-1}|\sqrt{-\lambda}}{4k_5^2a_{-1}}\tanh\left[\frac{4|k_5a_{-1}|}{\sqrt{-\lambda}}\left(ax-a\sqrt{-\frac{k_1}{k_2}}y+\frac{(\lambda-16\mu)t^{\alpha}}{16k_5a_{-1}\Gamma(1+\alpha)}\right)\right],$$

$$u_{5,2}(x,y,t)=-\frac{a_{-1}\sqrt{-\lambda}}{4|k_5a_{-1}|}\tanh\left[\frac{4|k_5a_{-1}|}{\sqrt{-\lambda}}\left(ax-a\sqrt{-\frac{k_1}{k_2}}y+\frac{(\lambda-16\mu)t^{\alpha}}{16k_5a_{-1}\Gamma(1+\alpha)}\right)\right]-\frac{k_6}{2k_5}-\frac{|k_5a_{-1}|\sqrt{-\lambda}}{4k_5^2a_{-1}}\coth\left[\frac{4|k_5a_{-1}|}{\sqrt{-\lambda}}\left(ax-a\sqrt{-\frac{k_1}{k_2}}y+\frac{(\lambda-16\mu)t^{\alpha}}{16k_5a_{-1}\Gamma(1+\alpha)}\right)\right],$$

$$u_{6,1}(x,y,t)=\frac{a_{-1}\sqrt{-\lambda}}{2|a_1k_5a_{-1}|}\coth\left[\frac{2|a_1k_5a_{-1}|}{\sqrt{-\lambda}}\left(ax+a\sqrt{-\frac{k_1}{k_2}}y+\frac{(\lambda-4\mu)t^{\alpha}}{4k_5a_{-1}\Gamma(1+\alpha)}\right)\right]-\frac{k_6}{2k_5},$$

$$u_{6,2}(x,y,t)=\frac{a_{-1}\sqrt{-\lambda}}{2|a_1k_5a_{-1}|}\tanh\left[\frac{2|a_1k_5a_{-1}|}{\sqrt{-\lambda}}\left(ax+a\sqrt{-\frac{k_1}{k_2}}y+\frac{(\lambda-4\mu)t^{\alpha}}{4k_5a_{-1}\Gamma(1+\alpha)}\right)\right]-\frac{k_6}{2k_5}. \quad (36)$$

When $k_1k_2<0$ and $\lambda=4k_5k_7-k_6^2<0$, we have $\sigma<0$, then

$$u_{1,3}(x,y,t)=-\frac{k_6}{2k_5}+\frac{a_1\sqrt{\lambda}}{2|a_1k_5|}\tan\left[\frac{\sqrt{\lambda}}{2|a_1k_5|}\left(ax+ia\sqrt{\frac{k_1}{k_2}}y+\frac{(ak_3+a_1k_5)t^{\alpha}}{\Gamma(1+\alpha)}\right)\right],$$

$$u_{1,4}(x,y,t)=-\frac{k_6}{2k_5}+\frac{a_1\sqrt{\lambda}}{2|a_1k_5|}\cot\left[\frac{\sqrt{\lambda}}{2|a_1k_5|}\left(ax+ia\sqrt{\frac{k_1}{k_2}}y+\frac{(ak_3+a_1k_5)t^{\alpha}}{\Gamma(1+\alpha)}\right)\right],$$

$$u_{2,3}(x,y,t)=-\frac{a_{-1}\sqrt{\lambda}}{4|k_5a_{-1}|}\cot\left[\frac{4|k_5a_{-1}|}{\sqrt{\lambda}}\left(ax-ia\sqrt{\frac{k_1}{k_2}}y+\frac{(\lambda-16aa_{-1}k_3k_5)t^{\alpha}}{16k_5a_{-1}\Gamma(1+\alpha)}\right)\right]-\frac{k_6}{2k_5}+\frac{|k_5a_{-1}|\sqrt{\lambda}}{4k_5^2a_{-1}}\tan\left[\frac{4|k_5a_{-1}|}{\sqrt{\lambda}}\left(ax-ia\sqrt{\frac{k_1}{k_2}}y+\frac{(\lambda-16aa_{-1}k_3k_5)t^{\alpha}}{16k_5a_{-1}\Gamma(1+\alpha)}\right)\right],$$

$$u_{2,4}(x,y,t)=-\frac{a_{-1}\sqrt{\lambda}}{4|k_5a_{-1}|}\tan\left[\frac{4|k_5a_{-1}|}{\sqrt{\lambda}}\left(ax-ia\sqrt{\frac{k_1}{k_2}}y+\frac{(\lambda-16aa_{-1}k_3k_5)t^{\alpha}}{16k_5a_{-1}\Gamma(1+\alpha)}\right)\right]-\frac{k_6}{2k_5}-\frac{|k_5a_{-1}|\sqrt{\lambda}}{4k_5^2a_{-1}}\cot\left[\frac{4|k_5a_{-1}|}{\sqrt{\lambda}}\left(ax-ia\sqrt{\frac{k_1}{k_2}}y+\frac{(\lambda-16aa_{-1}k_3k_5)t^{\alpha}}{16k_5a_{-1}\Gamma(1+\alpha)}\right)\right],$$

$$u_{3,3}(x,y,t)=-\frac{a_{-1}\sqrt{\lambda}}{2|a_1k_5a_{-1}|}\cot\left[\frac{2|a_1k_5a_{-1}|}{\sqrt{\lambda}}\left(ax+ia\sqrt{\frac{k_1}{k_2}}y+\frac{(\lambda-4aa_{-1}k_3k_5)t^{\alpha}}{4k_5a_{-1}(1+\alpha)}\right)\right]-\frac{k_6}{2k_5}, \quad (37)$$

$u_{3,4}(x,y,t)=-(a_{-1}\sqrt{\lambda}/2\ |a_1k_5a_{-1}|)\tan[(2|a_1k_5a_{-1}|/\sqrt{\lambda})(ax+ia\sqrt{k_1/k_2}y+((\lambda-4aa_{-1}k_3k_5)t^{\alpha}/4k_5a_{-1}(1+\alpha)))]-k_6/2k_5$, When $k_1k_2 < 0$ and $\lambda = 4k_5k_7 - k_6^2 < 0$, we have $\sigma < 0$, then

$$u_{4,3}(x,y,t) = -\frac{k_6}{2k_5} + \frac{a_1\sqrt{\lambda}}{2|a_1k_5|}\tan\left[\frac{\sqrt{\lambda}}{4|a_1k_5|}\left(ax + a\sqrt{-\frac{k_1}{k_2}}y - \frac{\left(ak_3 + ak_4\sqrt{-k_1/k_2} + a_1k_5\right)t^\alpha}{\Gamma(1+\alpha)}\right)\right],$$

$$u_{4,4}(x,y,t) = -\frac{k_6}{2k_5} - \frac{a_1\sqrt{\lambda}}{2|a_1k_5|}\cot\left[\frac{\sqrt{\lambda}}{2|a_1k_5|}\left(ax + a\sqrt{-\frac{k_1}{k_2}}y - \frac{\left(ak_3 + ak_4\sqrt{-k_1/k_2} + a_1k_5\right)t^\alpha}{\Gamma(1+\alpha)}\right)\right],$$

$$u_{5,3}(x,y,t) = -\frac{a_{-1}\sqrt{\lambda}}{4|k_5a_{-1}|}\cot\left[\frac{4|k_5a_{-1}|}{\sqrt{\lambda}}\left(ax + a\sqrt{-\frac{k_1}{k_2}}y + \frac{(\lambda-16\mu)t^\alpha}{16k_5a_{-1}\Gamma(1+\alpha)}\right)\right] - \frac{k_6}{2k_5} + \frac{|k_5a_{-1}|\sqrt{\lambda}}{4k_5^2a_{-1}}\tan\left[\frac{4|k_5a_{-1}|}{\sqrt{\lambda}}\cdot\left(ax + a\sqrt{-\frac{k_1}{k_2}}y + \frac{(\lambda-16\mu)t^\alpha}{16k_5a_{-1}\Gamma(1+\alpha)}\right)\right],$$

$$u_{5,4}(x,y,t) = -\frac{a_{-1}\sqrt{\lambda}}{4|k_5a_{-1}|}\tan\left[\frac{4|k_5a_{-1}|}{\sqrt{\lambda}}\left(ax + a\sqrt{-\frac{k_1}{k_2}}y + \frac{(\lambda-16\mu)t^\alpha}{16k_5a_{-1}\Gamma(1+\alpha)}\right)\right] - \frac{k_6}{2k_5} - \frac{|k_5a_{-1}|\sqrt{\lambda}}{4k_5^2a_{-1}}\cot\left[\frac{4|k_5a_{-1}|}{\sqrt{\lambda}}\cdot\left(ax + a\sqrt{-\frac{k_1}{k_2}}y + \frac{(\lambda-16\mu)t^\alpha}{16k_5a_{-1}\Gamma(1+\alpha)}\right)\right],$$

$$u_{6,3}(x,y,t) = \frac{a_{-1}\sqrt{\lambda}}{2|k_5a_{-1}|}\cot\left[\frac{2|k_5a_{-1}|}{\sqrt{\lambda}}\left(ax + a\sqrt{-\frac{k_1}{k_2}}y + \frac{(\lambda-4\mu)t^\alpha}{4k_5a_{-1}\Gamma(1+\alpha)}\right)\right] - \frac{k_6}{2k_5},$$

$$u_{6,4}(x,y,t) = \frac{a_{-1}\sqrt{\lambda}}{2|k_5a_{-1}|}\tan\left[\frac{2|k_5a_{-1}|}{\sqrt{\lambda}}\left(ax + a\sqrt{-\frac{k_1}{k_2}}y + \frac{(\lambda-4\mu)t^\alpha}{4k_5a_{-1}\Gamma(1+\alpha)}\right)\right] - \frac{k_6}{2k_5}. \tag{38}$$

When $\lambda = 4k_5k_7 - k_6^2 = 0$, we have $\sigma = 0$, then

$$\begin{aligned}
u_{1,5}(x,y,t) &= -\frac{k_6}{2k_5} - \frac{\Gamma(1+\alpha)a_1}{\left[ax + ia\sqrt{(k_1/k_2)}y - ((ak_3 + a_1k_5)t^\alpha/\Gamma(1+\alpha))\right]^\alpha + \omega},\\
u_{2,5}(x,y,t) &= -\frac{a_{-1}\left[ax + ia\sqrt{(k_1/k_2)}y - (ak_3t^\alpha/\Gamma(1+\alpha))\right]^\alpha + a_{-1}\omega}{\Gamma(1+\alpha)} - \frac{k_6}{2k_5} = u_{3,5}(x,y,t),\\
u_{4,5}(x,y,t) &= -\frac{k_6}{2k_5}\frac{a_1\Gamma(1+\alpha)}{\left[ax + a\sqrt{(-k_1/k_2)}y - \left(\left(ak_3 + ak_3\sqrt{-k_1/k_2} + a_1k_5\right)t^\alpha/\Gamma(1+\alpha)\right)\right]^\alpha + \omega},\\
u_{5,5}(x,y,t) &= \frac{-a_{-1}}{\Gamma(1+\alpha)}\left[a^\alpha\left(x + \sqrt{(-k_1/k_2)}y - \frac{\left(k_3 + k_4\sqrt{-k_1/k_2}\right)t^\alpha}{\Gamma(1+\alpha)}\right)^\alpha + \omega\right] - \frac{k_6}{2k_5}.
\end{aligned} \tag{39}$$

Solutions $u_{i,1}, u_{i,2} (i = 1, 2, \cdots, 6)$ describe the soliton. Solitons exist everywhere in the nature; they are special kinds of solitary waves. $u_{2,1}, u_{2,2}, u_{5,1}$, and $u_{5,2}$ describe the multiple soliton solutions. Solutions $u_{i,3}$ and $u_{i,4} (i = 1, 2, \cdots, 6)$ represent the exact periodic traveling wave solutions. Periodic solutions are traveling wave solutions.

Figures 1–6 present the solutions: $u_{1,1}, u_{2,1}, u_{1,3}, u_{2,3}, u_{1,5}$, and $u_{5,5}$ of the generalized time fractional biological population model with $-10 < x < 10, 0 < t < 10$. Solution $u_{1,1}$ is presented for values $\alpha = 0.5, k_1 = k_2 = 1, k_3 = k_4 = 0, k_5 = 2, k_6 = 3, k_7 = 1, a_1 = 1, a = 2$, and $y = 0$; solution $u_{2,1}$ is presented for values $\alpha = 0.5, k_1 = k_2 = 1, k_3 = k_4 = 0, k_5 = 2, k_6 = 3, k_7 = 1, a_{-1} = 1, a = 2$, and $y = 0$; solution $u_{1,3}$ is presented for values $\alpha = 0.5, k_1 = k_2 = 1, k_3 = k_4 = 0, k_5 = 2, k_6 = 2, k_7 = 1, a_1 = 1, a = 2$, and $y = 0$; solution $u_{2,3}$ is presented for values $\alpha = 0.5, k_1 = k_2 = 1, k_3 = k_4 = 0, k_5 = 2, k_6 = 3, k_7 =$

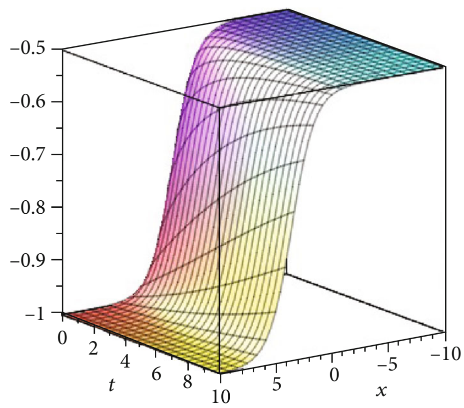

FIGURE 1: $u_{1,1}(x, 0, t)$ of Equation (21).

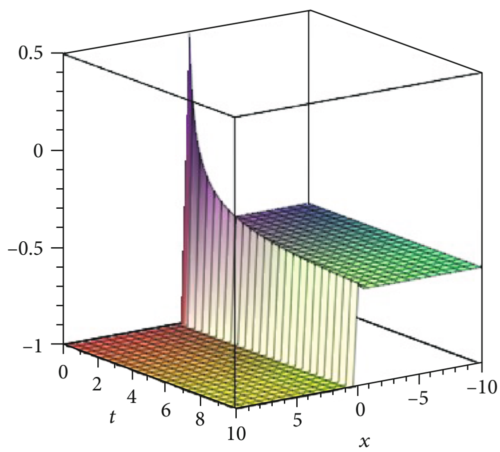

FIGURE 2: $u_{2,1}(x, 0, t)$ of Equation (21).

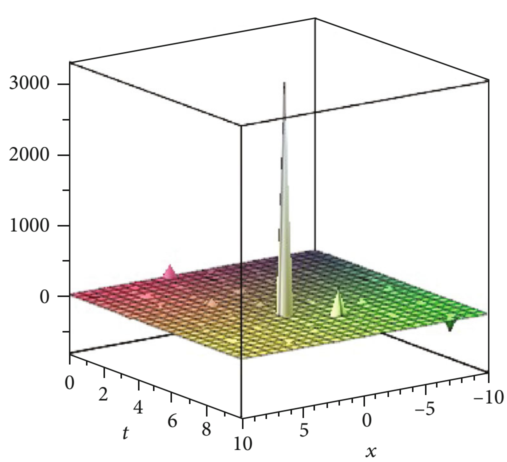

FIGURE 3: $u_{1,3}(x, 0, t)$ of Equation (21).

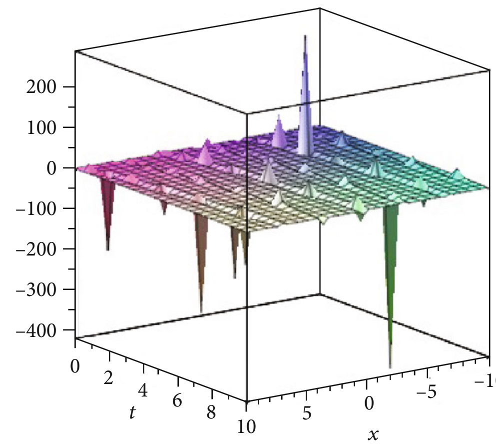

FIGURE 4: $u_{2,3}(x, 0, t)$ of Equation (21).

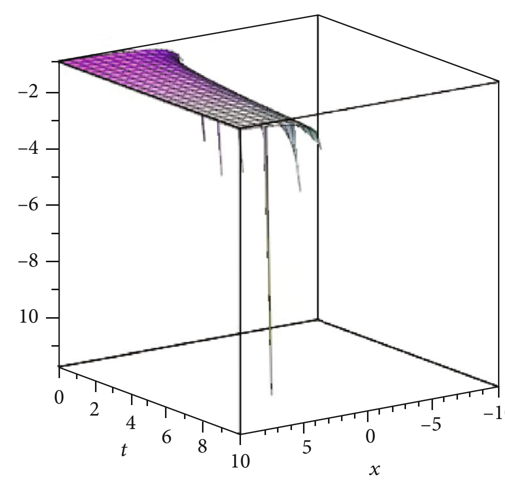

FIGURE 5: $u_{1,5}(x, 0, t)$ of Equation (21).

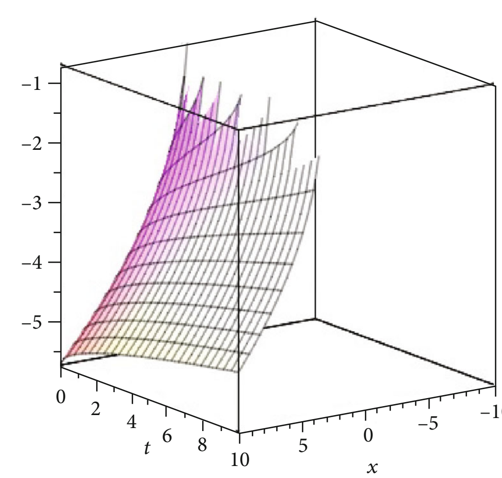

FIGURE 6: $u_{5,5}(x, 0, t)$ of Equation (21).

1, $a_{-1} = 1$, $a = 2$, and $y = 0$; solution $u_{1,5}$ is presented for values $\alpha = 0.5$, $k_1 = k_2 = 1$, $k_3 = k_4 = 0$, $k_5 = 2$, $k_6 = \sqrt{8}$, $k_7 = 1$, $a = 2$, $a_1 = 1$, $y = 0$, and $\omega = 0$; solution $u_{5,5}$ is presented for values $\alpha = 0.5$, $k_1 = 1$, $k_2 = -1$, $k_3 = 0$, $k_4 = 1$, $k_5 = 2$, $k_6 = \sqrt{8}$, $k_7 = 1$, $a_1 = 1$, $a = 2$, $y = 0$, and $\omega = 0$.

When $k_1 = -1$, $k_2 = -1$, $k_3 = k_4 = k_6 = 0$, $k_5 = -h \neq 0$, and $k_7 = rh \neq 0$, Equation (21) is the time fractional biological population model [39]

$$D_t^\alpha u = \left(u^2\right)_{xx} + \left(u^2\right)_{yy} + h\left(u^2 - r\right). \quad (40)$$

We denote

$$u(x, y, t) = u(\xi), \xi = ax + iay + \frac{bt^\alpha}{\Gamma(1+\alpha)}, a \neq 0, \quad (41)$$

and find the solution of Equation (40) in the form of $u(\xi) = a_{-1}\varphi^{-1} + a_0 + a_1\varphi$. Then, by solving the set of equations given above for a_{-1}, a_0, a_1, a, b, and σ, we obtain solution sets as follows:

Set 1

$$\begin{aligned} a_{-1} &= 0, \\ a_0 &= 0, \\ a_1 &\neq 0, \\ \sigma &= -\frac{r}{a_1^2}. \end{aligned} \quad (42)$$

Let $a_1 = c/h$, then $\sigma = -rh^2/c^2$, we can obtain set 1 in [42].

Set 2

$$\begin{aligned} a_{-1} &\neq 0, \\ a_0 &= 0, \\ a_1 &= \frac{r}{4a_{-1}}, \\ \sigma &= -\frac{4a_{-1}^2}{r}. \end{aligned} \quad (43)$$

Let $a_{-1} = rh/4c$, then $\sigma = -rh^2/4c^2$, we can obtain set 3 in [42].

Set 3

$$\begin{aligned} a_{-1} &\neq 0, \\ a_0 &= 0, \\ a_1 &= 0, \\ \sigma &= -\frac{a_{-1}^2}{r}. \end{aligned} \quad (44)$$

Let $a_{-1} = rh/c$, then $\sigma = -rh^2/c^2$, we can obtain set 2 in [42].

Clearly, we get more solutions of the time fractional biological population model (40) than the literature [42].

3.2. The Generalized Compound KdV-Burgers Equation. The generalized compound KdV-Burgers equation is given by

$$D_t^\alpha u + k_1 u u_x + k_2 u^2 u_x + k_3 u_{xx} + k_4 u_{xxx} = 0, \quad (45)$$

Let

$$u(x, t) = u(\xi), \xi = ax + \frac{bt^\alpha}{\Gamma(1+\alpha)}, a \neq 0, \quad (46)$$

then Equation (45) is reduced to ordinary differential equation as

$$bu' + k_1 a u u' + k_2 a u^2 u' + k_3 a^2 u'' + k_4 a^3 u''' = 0. \quad (47)$$

The solution of Equation (45) is in the form of (18), and here, $n = 1$ is taken from the homogeneous balance between the highest order derivative u''' and the nonlinear term $u^2 u'$. We obtain the solution of Equation (45) as follows:

$$u(\xi) = a_{-1}\varphi^{-1} + a_0 + a_1\varphi. \quad (48)$$

Substituting Equation (48) together with its necessary derivatives into Equation (47), the algebraic equation is arranged according to the powers of the function $\varphi^k(\xi)$. Then, the following coefficients are obtained:

$$\begin{aligned} \varphi^{-4} &: -6a^3k_4\sigma^3 a_{-1} - ak_2\sigma a_{-1}^3, \\ \varphi^{-3} &: 2a^2k_3\sigma^2 a_{-1} - 2ak_2\sigma a_{-1}^2 a_0 - ak_1\sigma a_{-1}^2, \\ \varphi^{-2} &: b\sigma a_{-1} - ak_1\sigma a_{-1}a_0 - ak_2 a_{-1}^3 - ak_2\sigma a_{-1}a_0^2 \\ &\quad - ak_2\sigma a_{-1}^2 a_1 - 8a^3k_4\sigma^2 a_{-1}, \\ \varphi^{-1} &: -ak_1 a_{-1}^2 - 2ak_2 a_{-1}^2 a_0 + 2a^2k_3\sigma a_{-1}, \end{aligned} \quad (49)$$

$$\begin{aligned} \varphi^0 &: ba_{-1} - b\sigma a_1 + ak_1\sigma a_0 a_1 - ak_1 a_{-1}a_0 \\ &\quad + ak_2\sigma a_0^2 a_1 + ak_2\sigma a_{-1}a_1^2 - ak_2 a_0^2 a_{-1} \\ &\quad - ak_2 a_{-1}^2 a_1 + 2a^3k_4\sigma^2 a_1 - 2a^3k_4\sigma a_{-1}, \\ \varphi^1 &: ak_1\sigma a_1^2 + 2ak_2\sigma a_0 a_1^2 + 2a^2k_3\sigma a_1, \\ \varphi^2 &: -ba_1 + ak_1 a_0 a_1 + ak_2 a_0^2 a_1 + ak_2 a_{-1}a_1^2 \\ &\quad + ak_2\sigma a_1^3 + 8a^3k_4\sigma a_1, \\ \varphi^3 &: 2ak_2 a_0 a_1^2 + 2a^2k_3 a_1 + ak_1 a_1^2, \\ \varphi^4 &: 6a^3k_4 a_1 + ak_2 a_1^3. \end{aligned} \quad (50)$$

Let the coefficients of $\varphi^k(\xi)$ to be zero. By solving the set of equations given above for a_{-1}, a_0, a_1, a, b, and σ, we obtain solution sets as follows:

When $k_2k_4 < 0$, it denotes $\mu = 3ak_1^2k_4 + 2ak_2k_3^2 - 12bk_2k_4$ and $\lambda_1 = \pm\sqrt{-6k_4/k_2}$, $\lambda_2 = -k_1/2 \pm ((|k_3 \mid \sqrt{-6k_2k_4})/6k_4)$, we have

Set 1

$$\begin{aligned} a_{-1} &= -\frac{\lambda_1\mu}{96a^2k_2k_4^2}, \\ a_0 &= -\frac{\lambda_1 k_1 + 2k_3}{2\lambda_1 k_2}, \\ a_1 &= a\lambda_1, \\ a &= a, \\ b &= b, \\ \sigma &= \frac{\mu}{96a^3k_2k_4^2}. \end{aligned} \quad (51)$$

Set 2

$$\begin{aligned} a_{-1} &= 0, \\ a_0 &= -\frac{\lambda_1 k_1 + 2k_3}{2\lambda_1 k_2}, \\ a_1 &= a\lambda_1, \\ a &= a, \\ b &= b, \\ \sigma &= \frac{\mu}{24a^3 k_2 k_4^2}. \end{aligned} \tag{52}$$

Set 3

$$\begin{aligned} a_{-1} &= -\frac{\mu k_3}{12a^2 k_2 k_4^2 (2\lambda_2 + k_1)}, \\ a_0 &= \frac{\lambda_2}{k_2}, \\ a_1 &= 0, \\ a &= a, \\ b &= b, \\ \sigma &= \frac{\mu}{24a^3 k_2 k_4^2}. \end{aligned} \tag{53}$$

Thus, we obtain the solution of Equation (45) as $u_{i,j}(x, t)$, $(i = 1, 2, 3 ; j = 1, 2, 3, 4, 5)$. Using set 1 to set 3, $u_{i,j}(x, t)$ is as follows:

When $k_2 k_4 < 0$ and $a\mu k_2 < 0$, we have $\sigma < 0$, then

$$\begin{aligned} u_{1,1}(x, t) = {} & \frac{\lambda_1 \mu |a|}{24a^2 k_4} \sqrt{\frac{-6a}{\mu k_2}} \coth \left[\frac{1}{4|ak_4|} \sqrt{\frac{-\mu}{6ak_2}} \cdot \left(ax + \frac{bt^\alpha}{\Gamma(1+\alpha)} \right) \right] - \frac{\lambda_1 k_1 + 2k_3}{2\lambda_1 k_2} \\ & - \frac{a\lambda_1}{4|ak_4|} \sqrt{\frac{-\mu}{6ak_2}} \tanh \left[\frac{1}{4|ak_4|} \cdot \sqrt{\frac{-\mu}{6ak_2}} \left(ax + \frac{bt^\alpha}{\Gamma(1+\alpha)} \right) \right] \end{aligned}$$

$$\begin{aligned} u_{1,2}(x, t) = {} & \frac{\lambda_1 \mu |a|}{24a^2 k_4} \sqrt{\frac{-6a}{\mu k_2}} \tanh \left[\frac{1}{4|ak_4|} \sqrt{\frac{-\mu}{6ak_2}} \cdot \left(ax + \frac{bt^\alpha}{\Gamma(1+\alpha)} \right) \right] - \frac{\lambda_1 k_1 + 2k_3}{2\lambda_1 k_2} \\ & - \frac{a\lambda_1}{4|ak_4|} \sqrt{\frac{-\mu}{6ak_2}} \coth \left[\frac{1}{4|ak_4|} \cdot \sqrt{\frac{-\mu}{6ak_2}} \left(ax + \frac{bt^\alpha}{\Gamma(1+\alpha)} \right) \right] \end{aligned}$$

$$\begin{aligned} u_{2,1}(\mathrm{x}, t) = {} & -\frac{\lambda_1 k_1 + 2k_3}{2\lambda_1 k_2} - \frac{a\lambda_1}{2|ak_4|} \sqrt{\frac{-\mu}{6ak_2}} \tanh \cdot \left[\frac{1}{2|ak_4|} \sqrt{\frac{-\mu}{6ak_2}} \left(ax + \frac{bt^\alpha}{\Gamma(1+\alpha)} \right) \right] \\ u_{2,2}(x, t) = {} & -\frac{\lambda_1 k_1 + 2k_3}{2\lambda_1 k_2} - \frac{a\lambda_1}{2|ak_4|} \sqrt{\frac{-\mu}{6ak_2}} \coth, \\ u_{3,1}(x, t) = {} & \frac{\mu k_3 |a|}{6a^2 k_4 (2\lambda_2 + k_1)} \sqrt{\frac{-6a}{\mu k_2}} \coth + \frac{\lambda_2}{k_2}, \\ u_{3,2}(x, t) = {} & \frac{\mu k_3 |a|}{6a^2 k_4 (2\lambda_2 + k_1)} \sqrt{\frac{-6a}{\mu k_2}} \tanh + \frac{\lambda_2}{k_2}. \end{aligned} \tag{54}$$

When $k_2 k_4 < 0$ and $a\mu k_2 > 0$, we have $\sigma > 0$, then

$$\begin{aligned} u_{1,3}(x, t) = {} & -\frac{\lambda_1 \mu |a|}{24a^2 k_4} \sqrt{\frac{6a}{\mu k_2}} \cot \left[\frac{1}{4|ak_4|} \sqrt{\frac{\mu}{6ak_2}} \cdot \left(ax + \frac{bt^\alpha}{\Gamma(1+\alpha)} \right) \right] - \frac{\lambda_1 k_1 + 2k_3}{2\lambda_1 k_2} \\ & + \frac{a\lambda_1}{4|ak_4|} \sqrt{\frac{\mu}{6ak_2}} \tan \left[\frac{1}{4|ak_4|} \sqrt{\frac{\mu}{6ak_2}} \cdot \left(ax + \frac{bt^\alpha}{\Gamma(1+\alpha)} \right) \right], \end{aligned}$$

$$\begin{aligned} u_{1,4}(x, t) = {} & \frac{\lambda_1 \mu |a|}{24a^2 k_4} \sqrt{\frac{6a}{\mu k_2}} \tan \left[\frac{1}{4|ak_4|} \sqrt{\frac{\mu}{6ak_2}} \cdot \left(ax + \frac{bt^\alpha}{\Gamma(1+\alpha)} \right) \right] - \frac{\lambda_1 k_1 + 2k_3}{2\lambda_1 k_2} \\ & - \frac{a\lambda_1}{4|ak_4|} \sqrt{\frac{\mu}{6ak_2}} \cot \left[\frac{1}{4|ak_4|} \sqrt{\frac{\mu}{6ak_2}} \cdot \left(ax + \frac{bt^\alpha}{\Gamma(1+\alpha)} \right) \right], \\ u_{2,3}(x, t) = {} & -\frac{\lambda_1 k_1 + 2k_3}{2\lambda_1 k_2} + \frac{a\lambda_1}{2|ak_4|} \sqrt{\frac{\mu}{6ak_2}} \tan \cdot \left[\frac{1}{2|ak_4|} \sqrt{\frac{\mu}{6ak_2}} \left(ax + \frac{bt^\alpha}{\Gamma(1+\alpha)} \right) \right], \\ u_{2,4}(x, t) = {} & -\frac{\lambda_1 k_1 + 2k_3}{2\lambda_1 k_2} - \frac{a\lambda_1}{2|ak_4|} \sqrt{\frac{\mu}{6ak_2}} \coth \cdot \left[\frac{1}{2|ak_4|} \sqrt{\frac{\mu}{6ak_2}} \left(ax + \frac{bt^\alpha}{\Gamma(1+\alpha)} \right) \right], \\ u_{3,3}(x, t) = {} & -\frac{\mu k_3}{6a^2 k_4 (2\lambda_2 + k_1)} \sqrt{\frac{6a^3}{\mu k_2}} \cot \cdot \left[\frac{1}{2|k_4|} \sqrt{\frac{\mu}{6a^3 k_2}} \left(ax + \frac{bt^\alpha}{\Gamma(1+\alpha)} \right) \right] + \frac{\lambda_2}{k_2}, \end{aligned}$$

$$u_{3,4}(x,t) = \frac{\mu k_3|a|}{6a^2k_4(2\lambda_2+k_1)}\sqrt{\frac{6a}{\mu k_2}}\tan \cdot\left[\frac{1}{2|ak_4|}\sqrt{\frac{\mu}{6ak_2}}\left(ax+\frac{bt^\alpha}{\Gamma(1+\alpha)}\right)\right]+\frac{\lambda_2}{k_2}. \quad (55)$$

When $k_2k_4 < 0$ and $\mu = 0$, we have $\sigma = 0$, then

$$u_{1,5}(x,t) = -\frac{\lambda_1k_1+2k_3}{2\lambda_1k_2}+\frac{a\lambda_1\Gamma(1+\alpha)}{(ax+(bt^\alpha/\Gamma(1+\alpha)))^\alpha+\omega} = u_{2,5}(x,t). \quad (56)$$

Solutions $u_{i,1}$ and $u_{i,2}$ $(i = 1, 2, 3)$ described the soliton. $u_{1,1}$ and $u_{1,2}$ describe the multiple soliton solutions. Solutions $u_{i,3}$ and $u_{i,4}$ $(i = 1, 2, 3)$ represent the exact periodic traveling wave solutions.

Figures 7–10 present the solutions: $u_{1,1}$, $u_{2,1}$, $u_{1,3}$, and $u_{2,3}$ of the generalized compound KdV-Burgers equation with $-10 < x < 10$, $0 < t < 1$. Solutions $u_{1,1}$ and $u_{2,1}$ are presented for values $\alpha = 0.5$, $k_1 = k_2 = 1$, $k_3 = k_4 = -1$, $a = -4$, $b = 1$, and $y = 0$; solutions $u_{1,3}$ and $u_{2,3}$ are presented for values $\alpha = 0.5$, $k_1 = k_2 = 1$, $k_3 = k_4 = -1$, $a = 4$, b = 1, and $y = 0$.

3.3. The Space-Time Fractional Regularized Long-Wave Equation. Consider the space-time fractional regularized long-wave equation as follows:

$$D_t^\alpha u + k_1 D_x^\alpha u + k_2 u D_x^\alpha u + k_3 D_t^\alpha D_x^{2\alpha} u = 0. \quad (57)$$

Let

$$u(x,t) = u(\xi), \xi = \frac{ax^\alpha}{\Gamma(1+\alpha)}+\frac{bt^\alpha}{\Gamma(1+\alpha)}, a \neq 0, \quad (58)$$

then Equation (57) is reduced to the following ordinary differential equation:

$$bu' + k_1au' + k_2auu' + k_3a^2bu''' = 0. \quad (59)$$

The solution of Equation (57) is in the form of (18), and here, $n = 2$ is taken from the homogeneous balance between the highest order derivative u''' and the nonlinear term uu'. We obtain the solution of Equation (57) as follows:

$$u(\xi) = a_{-2}\varphi^{-2} + a_{-1}\varphi^{-1} + a_0 + a_1\varphi + a_2\varphi^2. \quad (60)$$

Substituting Equation (60) together with its necessary derivatives into Equation (57), the algebraic equation is arranged according to the powers of the function $\varphi^k(\xi)$. Then, the following coefficients are obtained:

$$\begin{aligned}
&\varphi^{-5}: -24k_3a^2ba_{-2}\sigma^3 - 2k_2aa_{-2}^2\sigma,\\
&\varphi^{-4}: -6k_3a^2ba_{-1}\sigma^3 - 3k_2aa_{-1}a_{-2}\sigma,\\
&\varphi^{-3}: -2k_1aa_{-2}\sigma - 2ba_{-2}\sigma - 2k_2aa_0a_{-2}\sigma - k_2aa_{-1}^2\sigma - 2k_2aa_{-2}^2 - 40k_3a^2ba_{-2}\sigma^2,\\
&\varphi^{-2}: -k_1aa_{-1}\sigma - ba_{-1}\sigma - k_2aa_1a_{-2}\sigma - 3k_2aa_{-1}a_{-2} - k_2aa_0a_{-1}\sigma - 8k_3a^2ba_{-1}\sigma^2,\\
&\varphi^{-1}: -2ba_{-2} - 2k_1aa_{-2} - k_2aa_{-1}^2 - 2k_2aa_0a_{-2} - 16k_3a^2ba_{-2}\sigma,\\
&\varphi^{0}: k_1aa_1\sigma - k_1aa_{-1} + a_1b\sigma - a_{-1}b - k_2aa_1a_{-2} + k_2aa_{-1}a_2\sigma + k_2aa_0a_1\sigma - k_2aa_0a_{-1} + 2k_3a^2ba_1\sigma^2 - 2k_3a^2ba_{-1}\sigma,
\end{aligned} \quad (61)$$

$$\begin{aligned}
&\varphi^{1}: 2k_1aa_2\sigma + 2ba_2\sigma + 2k_2aa_0a_2\sigma + k_2aa_1^2\sigma + 16k_3a^2ba_2\sigma^2,\\
&\varphi^{2}: k_1aa_1 + ba_1 + k_2aa_2a_{-1} + k_2aa_1a_0 + 3k_2aa_1a_2\sigma + 8k_3a^2ba_1\sigma,\\
&\varphi^{3}: 2k_1aa_2 + 2ba_2 + 2k_2aa_2^2\sigma + 2k_2aa_0a_2 + k_2aa_1^2 + 40k_3a^2ba_2\sigma,\\
&\varphi^{4}: 6k_3a^2ba_1 + 3k_2aa_1a_2,\\
&\varphi^{5}: 24k_3a^2ba_2 + 2k_2aa_2^2.
\end{aligned} \quad (62)$$

Let the coefficients of $\varphi^k(\xi)$ tbe zero. By solving the set of equations given above for a_{-1}, a_0, a_1, a, b, and σ, we obtain solution sets as follows:

Set 1

$$\begin{aligned}
a_{-2} &= 0,\\
a_{-1} &= 0,\\
a_0 &= a_0,\\
a_1 &= 0,\\
a_2 &= -\frac{12k_3ab}{k_2},\\
a &= a,\\
b &= b,\\
\sigma &= -\frac{k_1a + k_2a_0a + b}{8k_3a^2b}.
\end{aligned} \quad (63)$$

Set 2

$$a_{-2} = -\frac{3(k_1a + k_2a_0a + b)^2}{16k_2k_3a^3b},$$

$$a_{-1} = 0,$$

$$a_0 = a_0,$$

$$a_1 = 0,$$
$$a_2 = 0,$$
$$a = a,$$
$$b = b,$$
$$\sigma = -\frac{k_1 a + k_2 a_0 a + b}{8k_3 a^2 b}. \tag{64}$$

Set 3

$$\begin{aligned} a_{-2} &= -\frac{3(k_1 a + k_2 a_0 a + b)^2}{16k_2 k_3 a^3 b},\\ a_{-1} &= 0,\\ a_0 &= a_0,\\ a_1 &= 0,\\ a_2 &= -\frac{12k_3 ab}{k_2},\\ a &= a,\\ b &= b,\\ \sigma &= -\frac{k_1 a + k_2 a_0 a + b}{8k_3 a^2 b}. \end{aligned} \tag{65}$$

Thus, we obtain the solution of Equation (57) as $u_{i,j}(\xi)$, $(i = 1, 2, 3 ; j = 1, 2, 3, 4, 5)$ and $\xi = (ax^\alpha/\Gamma(1+\alpha)) + (bt^\alpha/\Gamma(1+\alpha))$; $u_{i,j}(\xi)$ is as follows:

When $(k_1 a + k_2 a_0 a + b)/(8k_3 a^2 b) > 0$, we have $\sigma < 0$, then

$$\begin{aligned} u_{1,1}(\xi) &= a_0 - \frac{3(k_1 a + k_2 a_0 a + b)}{2k_2 a} \tanh^2 \cdot \left(\sqrt{\frac{k_1 a + k_2 a_0 a + b}{8k_3 a^2 b}}\xi\right) = u_{2,2}(\xi),\\ u_{2,1}(\xi) &= -\frac{3(k_1 a + k_2 a_0 a + b)}{2k_2 a} \coth^2 \cdot \left(\sqrt{\frac{k_1 a + k_2 a_0 a + b}{8k_3 a^2 b}}\xi\right) + a_0 = u_{1,2}(\xi),\\ u_{3,1}(\xi) &= -\frac{3(k_1 a + k_2 a_0 a + b)}{2k_2 a} \coth^2\left(\sqrt{\frac{k_1 a + k_2 a_0 a + b}{8k_3 a^2 b}}\xi\right) + a_0 - \frac{3(k_1 a + k_2 a_0 a + b)}{2k_2 a} \tanh^2 \cdot \left(\sqrt{\frac{k_1 a + k_2 a_0 a + b}{8k_3 a^2 b}}\xi\right) = u_{3,2}(\xi). \end{aligned} \tag{66}$$

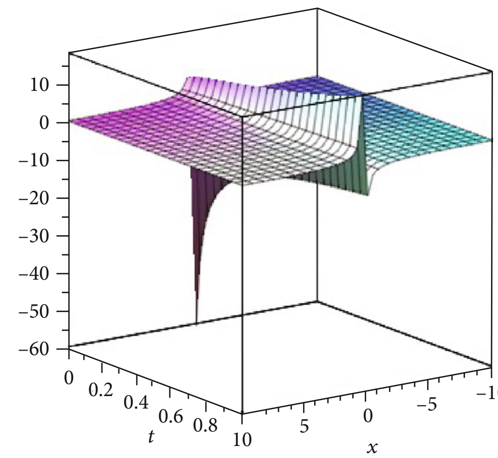

Figure 7: $u_{1,1}(x,t)$ of Equation (45).

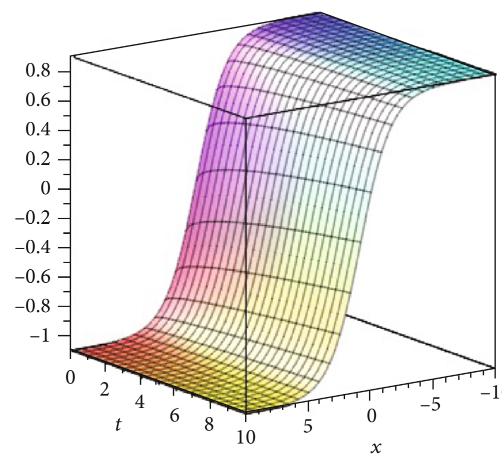

Figure 8: $u_{2,1}(x,t)$ of Equation (45).

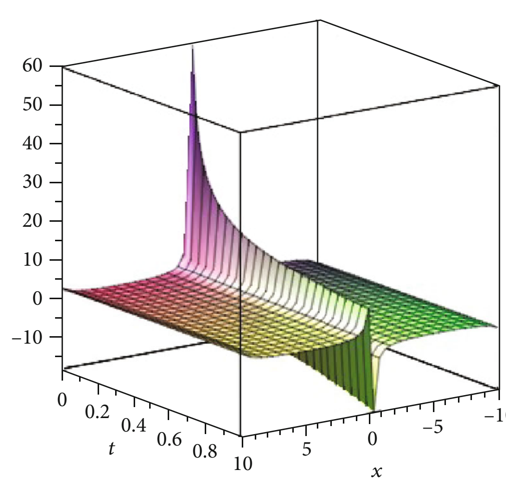

Figure 9: $u_{1,3}(x,t)$ of Equation (45).

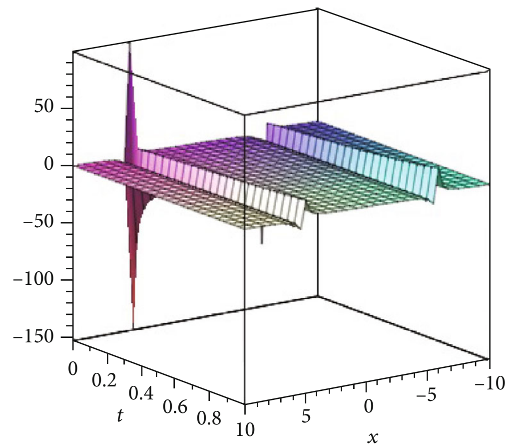

FIGURE 10: $u_{2,3}(x,t)$ of Equation (45).

When $(k_1a+k_2a_0a+b)/(8k_3a^2b)<0$, we have $\sigma>0$, then

$$u_{1,3}(\xi)=a_0+\frac{3(k_1a+k_2a_0a+b)}{2k_2a}\tan^2\cdot\left(\sqrt{-\frac{k_1a+k_2a_0a+b}{8k_3a^2b}}\xi\right)=u_{2,4}(\xi),$$
$$u_{2,3}(\xi)=\frac{3(k_1a+k_2a_0a+b)}{2k_2a}\cot^2\left(\sqrt{-\frac{k_1a+k_2a_0a+b}{8k_3a^2b}}\xi\right)+a_0=u_{1,4}(\xi),$$
$$u_{3,3}(\xi)=\frac{3(k_1a+k_2a_0a+b)}{2k_2a}\cot^2\left(\sqrt{-\frac{k_1a+k_2a_0a+b}{8k_3a^2b}}\xi\right)+a_0+\frac{3(k_1a+k_2a_0a+b)}{2k_2a}\tan^2\cdot\left(\sqrt{-\frac{k_1a+k_2a_0a+b}{8k_3a^2b}}\xi\right)=u_{3,4}(\xi).\tag{67}$$

When $(k_1a+k_2a_0a+b)/(8k_3a^2b)=0, a_{-2}=0$, we have $\sigma=0$, then

$$u_{1,5}(\xi)=a_0-\frac{12k_3ab\Gamma^2(1+\alpha)}{k_2\left(\xi^\alpha+\omega\right)^2}=u_{3,5}(\xi).\tag{68}$$

Solutions $u_{i,1}$ and $u_{i,2}(i=1,2,3)$ describe the multiple soliton. Solutions $u_{i,3}$ and $u_{i,4}(i=1,2,3)$ represent the exact periodic traveling wave solutions.

Figures 11–14 present the solutions: $u_{1,1}, u_{2,1}, u_{2,3}$, and $u_{1,5}$ of the generalized compound KdV-Burgers equation with $0<x<1, 0<t<2$. Solution $u_{1,1}$ and solution $u_{2,1}$ are presented for values $\alpha=0.5, k_1=k_2=1, k_3=k_4=-1, a=-4, b=1, a_0=0$, and $y=0$; $u_{2,3}$ is presented for values $\alpha=0.5, k_1=k_2=1, k_3=k_4=-1, a=4, b=1, a_0=0$, and $y=0$; solution $u_{1,5}$ is presented for values $\alpha=0.5, k_1=k_2=1, k_3=k_4=-1, a=4, b=-4, a_0=0, \omega=0$, and $y=0$.

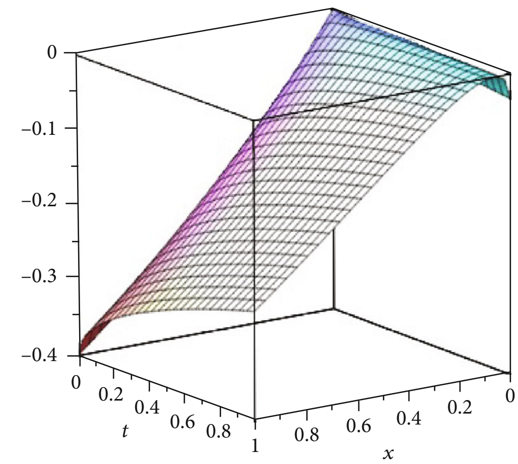

FIGURE 11: $u_{1,1}(x,t)$ of Equation (57).

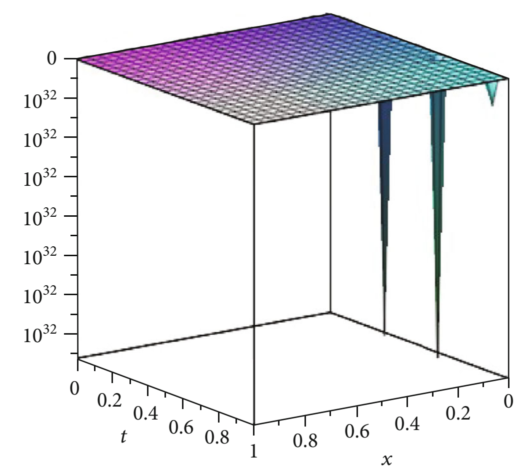

FIGURE 12: $u_{2,1}(x,t)$ of Equation (57).

3.4. The (3 + 1)-Space-Time Fractional Zakharov-Kuznetsov Equation.

The (3 + 1)-space-time fractional Zakharov-Kuznetsov Equation is given by

$$D_t^\alpha u+k_1uD_x^\alpha u+k_2D_x^{3\alpha}u+k_3D_x^\alpha D_y^{2\alpha}u+k_4D_x^\alpha D_z^{2\alpha}u=0,\tag{69}$$

Let

$$u(x,t)=u(\xi), \xi=\frac{at^\alpha}{\Gamma(1+\alpha)}+\frac{bx^\alpha}{\Gamma(1+\alpha)}+\frac{cy^\alpha}{\Gamma(1+\alpha)}+\frac{dz^\alpha}{\Gamma(1+\alpha)}, a\neq0,\tag{70}$$

then Equation (69) is reduced to the ordinary differential equation as

$$au'+k_1buu'+k_2b^3u'''+k_3bc^2u'''+k_4bd^2u'''=0.\tag{71}$$

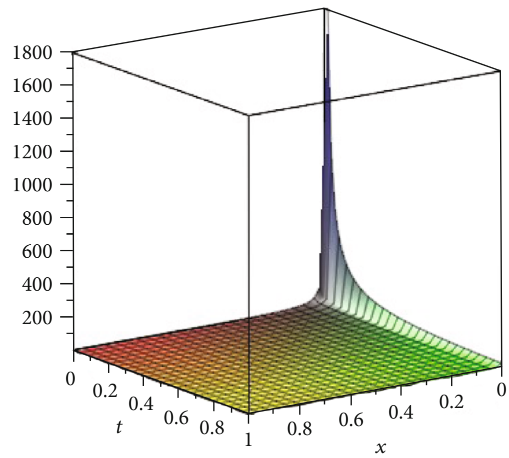

FIGURE 13: $u_{2,3}(x,t)$ of Equation (57).

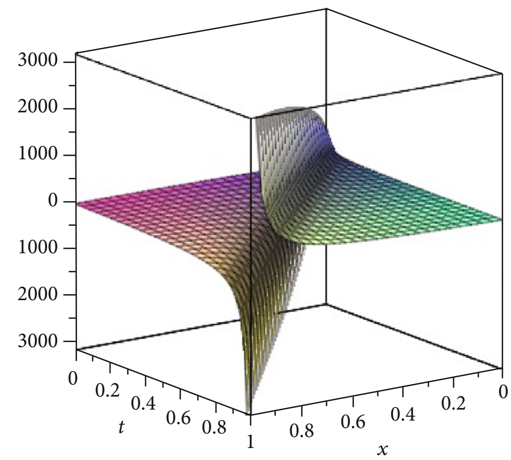

FIGURE 14: $u_{1,5}(x,t)$ of Equation (57).

The solution of Equation (71) is the form (18), and here, $n = 2$ is taken from the homogeneous balance between the highest order derivative u''' and the nonlinear term uu'. We obtain the solution of Equation (71) as

$$u(\xi) = a_{-2}\varphi^{-2} + a_{-1}\varphi^{-1} + a_0 + a_1\varphi + a_2\varphi^2. \tag{72}$$

Substituting Equation (72) together with its necessary derivatives into Equation (71), the algebraic equation is arranged according to the powers of the function $\varphi^k(\xi)$. Then, the following coefficients are obtained:

$$\begin{aligned}
\varphi^{-5} &: -2k_1ba_{-2}^2\sigma + 24\lambda a_{-2}\sigma^3,\\
\varphi^{-4} &: -3k_1ba_{-1}a_{-2}\sigma + 6\lambda a_{-1}\sigma^3,\\
\varphi^{-3} &: -2aa_{-2}\sigma - 2k_1ba_0a_{-2}\sigma - k_1ba_{-1}^2\sigma\\
&\quad - 2k_1ba_{-2}^2 + 40\lambda a_{-2}\sigma^2,\\
\varphi^{-2} &: -aa_{-1}\sigma - 3k_1ba_{-1}a_{-2} - k_1ba_0a_{-1}\sigma\\
&\quad - k_1ba_1a_{-2}\sigma + 8\lambda a_{-1}\sigma^2,\\
\varphi^{-1} &: -2aa_{-2} - k_1ba_{-1}^2 - 2k_1ba_0a_{-2} + 16\lambda a_{-2}\sigma,\\
\varphi^{0} &: aa_1\sigma - aa_{-1} - k_1ba_{-2}a_1 + k_1ba_{-1}a_2\sigma\\
&\quad + k_1ba_0a_1\sigma - k_1ba_0a_{-1} - 2\lambda a_1\sigma^2 + 2\lambda a_{-1}\sigma,\\
\varphi^{1} &: 2aa_2\sigma + 2k_1ba_0a_2\sigma + k_1ba_1^2\sigma - 16\lambda a_2\sigma^2,\\
\varphi^{2} &: aa_1 + k_1ba_{-1}a_2 + k_1ba_0a_1 + 3k_1ba_1a_2\sigma - 8\lambda a_1\sigma,\\
\varphi^{3} &: 2aa_2 + 2k_1ba_2^2\sigma + 2k_1ba_0a_2 + k_1ba_1^2 - 40\lambda a_2\sigma,\\
\varphi^{4} &: 3k_1ba_1a_2 - 6\lambda a_1,\\
\varphi^{5} &: 2k_1ba_2^2 - 24\lambda a_2.
\end{aligned} \tag{73}$$

where $\lambda = -k_2b^3 - k_3bc^2 - k_4bd^2$.

Let the coefficients of $\varphi^k(\xi)$ be zero. By solving the set of equations given above for a_{-1}, a_0, a_1, a, b, and σ, we obtain solution sets as follows:

Set 1

$$\begin{aligned}
a_{-2} &= 0,\\
a_{-1} &= 0,\\
a_0 &= a_0,\\
a_1 &= 0,\\
a_2 &= \frac{12\lambda}{k_1},\\
a &= a,\\
b &= b,\\
c &= c,\\
d &= d,\\
\sigma &= \frac{a + k_1a_0b}{8b\lambda}.
\end{aligned} \tag{74}$$

Set 2

$$\begin{aligned}
a_{-2} &= \frac{3(a + k_1a_0b)^2}{16k_1b^2\lambda},\\
a_{-1} &= 0,\\
a_0 &= a_0,\\
a_0 &= a_0,\\
a_1 &= 0,\\
a_2 &= 0,\\
a &= a,\\
b &= b,\\
c &= c,\\
d &= d,\\
\sigma &= \frac{a + k_1a_0b}{8b\lambda}.
\end{aligned} \tag{75}$$

Set 3

$$
\begin{aligned}
a_{-2} &= \frac{3(a+k_1a_0b)^2}{16k_1b^2\lambda},\\
a_{-1} &= 0,\\
a_0 &= a_0,\\
a_1 &= 0,\\
a_2 &= \frac{12\lambda}{k_1},\\
a &= a,\\
b &= b,\\
c &= c,\\
d &= d,\\
\sigma &= \frac{a+k_1a_0b}{8b\lambda}.
\end{aligned} \tag{76}
$$

Thus, we obtain the solution of Equation (59) as $u_{i,j}(\xi)$, $(i=1,2,3\,;j=1,2,3,4,5)$ and $\xi=(at^\alpha/\Gamma(1+\alpha))+(bx^\alpha/\Gamma(1+\alpha))+(cy^\alpha/\Gamma(1+\alpha))+(dz^\alpha/\Gamma(1+\alpha))$; $u_{i,j}(\xi)$ is tas follows:

When $(a+k_1a_0b)/(8b\lambda)<0$, we have $\sigma<0$, then

$$
\begin{aligned}
u_{1,1}(\xi) &= a_0 - \frac{3(a+k_1a_0b)}{2k_1b}\tanh^2\left(\sqrt{-\frac{a+k_1a_0b}{8b\lambda}}\xi\right),\\
u_{2,1}(\xi) &= -\frac{12\lambda(a+k_1a_0b)}{k_1}\coth^2\left(\sqrt{-\frac{a+k_1a_0b}{8b\lambda}}\xi\right)+a_0,\\
u_{3,1}(\xi) &= -\frac{12\lambda(a+k_1a_0b)}{k_1}\coth^2\left(\sqrt{-\frac{a+k_1a_0b}{8b\lambda}}\xi\right)\\
&\quad + a_0 - \frac{3(a+k_1a_0b)}{2k_1b}\tanh^2\left(\sqrt{-\frac{a+k_1a_0b}{8b\lambda}}\xi\right),\\
u_{1,2}(\xi) &= a_0 - \frac{3(a+k_1a_0b)}{2k_1b}\coth^2\left(\sqrt{-\frac{a+k_1a_0b}{8b\lambda}}\xi\right),\\
u_{2,2}(\xi) &= -\frac{12\lambda(a+k_1a_0b)}{k_1}\tanh^2\left(\sqrt{-\frac{a+k_1a_0b}{8b\lambda}}\xi\right)+a_0,\\
u_{3,2}(\xi) &= -\frac{12\lambda(a+k_1a_0b)}{k_1}\tanh^2\left(\sqrt{-\frac{a+k_1a_0b}{8b\lambda}}\xi\right)\\
&\quad + a_0 - \frac{3(a+k_1a_0b)}{2k_1b}\coth^2\left(\sqrt{-\frac{a+k_1a_0b}{8b\lambda}}\xi\right).
\end{aligned} \tag{77}
$$

When $(a+k_1a_0b)/(8b\lambda)>0$, we have $\sigma>0$, then

$$
\begin{aligned}
u_{1,3}(\xi) &= a_0 + \frac{3(a+k_1a_0b)}{2k_1b}\tan^2\left(\sqrt{\frac{a+k_1a_0b}{8b\lambda}}\xi\right),\\
u_{2,3}(\xi) &= \frac{12\lambda(a+k_1a_0b)}{k_1}\cot^2\left(\sqrt{\frac{a+k_1a_0b}{8b\lambda}}\xi\right)+a_0,\\
u_{3,3}(\xi) &= \frac{12\lambda(a+k_1a_0b)}{k_1}\cot^2\left(\sqrt{\frac{a+k_1a_0b}{8b\lambda}}\xi\right)\\
&\quad + a_0 + \frac{3(a+k_1a_0b)}{2k_1b}\tan^2\left(\sqrt{\frac{a+k_1a_0b}{8b\lambda}}\xi\right),\\
u_{1,4}(\xi) &= a_0 + \frac{3(a+k_1a_0b)}{2k_1b}\cot^2\left(\sqrt{\frac{a+k_1a_0b}{8b\lambda}}\xi\right),\\
u_{2,4}(\xi) &= \frac{12\lambda(a+k_1a_0b)}{k_1}\tan^2\left(\sqrt{\frac{a+k_1a_0b}{8b\lambda}}\xi\right)+a_0,\\
u_{3,4}(\xi) &= \frac{12\lambda(a+k_1a_0b)}{k_1}\tan^2\left(\sqrt{\frac{a+k_1a_0b}{8b\lambda}}\xi\right)\\
&\quad + a_0 + \frac{3(a+k_1a_0b)}{2k_1b}\cot^2\left(\sqrt{\frac{a+k_1a_0b}{8b\lambda}}\xi\right).
\end{aligned} \tag{78}
$$

When $(a+k_1a_0b)/(8b\lambda)=0$, we have $\sigma=0$, then

$$
u_{1,5}(\xi) = a_0 + \frac{12\lambda\Gamma^2(1+\alpha)}{k_1\left(\xi^\alpha+\omega\right)^2} = u_{3,5}(\xi). \tag{79}
$$

Solutions $u_{i,1}$ and $u_{i,2}\,(i=1,2,3)$ describe the multiple soliton. Solutions $u_{i,3}$ and $u_{i,4}$ $(i=1,2,3)$ represent the exact periodic traveling wave solutions.

Figures 15–18 present the solutions: $u_{1,1}$, $u_{1,3}$, $u_{2,3}$, and $u_{1,5}$ of the generalized compound KdV-Burgers equation with $0<x<1, 0<t<2$. Solution $u_{1,1}$ is presented for values $\alpha=0.5$, $k_1=k_2=1$, $k_3=k_4=-1$, $a=-4$, $b=1$, $c=1$, $d=1$, $a_0=0$, and $y=z=0$; solution $u_{1,3}$ and $u_{2,3}$ are presented for values $\alpha=0.5$, $k_1=k_2=1$, $k_3=k_4=-1$, $a=4$, $b=1$, $c=1$, $d=1$, $a_0=0$, and $y=z=0$; solution $u_{1,5}$ is presented for values $\alpha=0.5$, $k_1=k_2=1$, $k_3=k_4=-1$, $a=-4$, $b=1$, $c=1$, $d=1$, $a_0=4$, $\omega=0$, and $y=z=0$.

4. Results and Discussion

The generalized time fractional biological population model, the generalized time fractional compound KdV-Burgers equation, the space-time fractional regularized long-wave equation, and the $(3+1)$-space-time fractional Zakharov-Kuznetsov equation have gained the focus of many studies due to their frequent appearance in various applications.

The improved fractional subequation method has several advantages according to other traditional methods. Applying a suitable fractional complex transform $u(t,x_1,x_2,\cdots,x_m)=u(\xi)$, $\xi=\xi(t,x_1,x_2,\cdots,x_m)$ and chain rule, the nonlinear fractional differential equations with the modified Riemann-Liouville derivative are converted into the nonlinear ordinary differential equations. This is a significant impact because

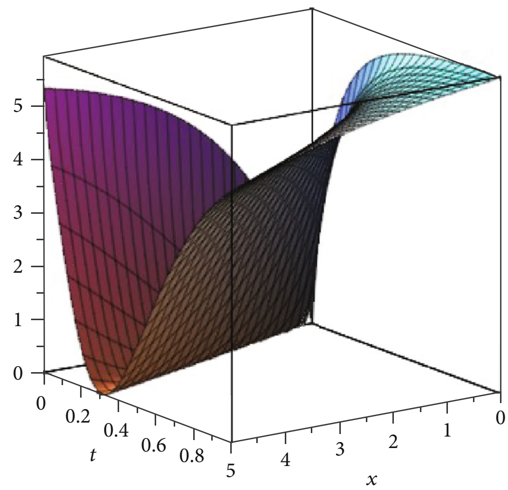

FIGURE 15: $u_{1,1}(x, 0, 0, t)$ of Equation (69).

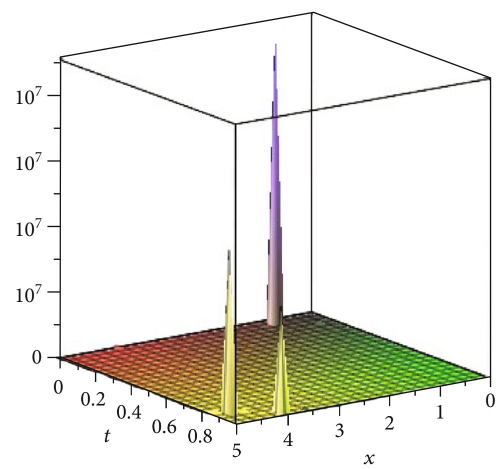

FIGURE 16: $u_{1,3}(x, 0, 0, t)$ of Equation (69).

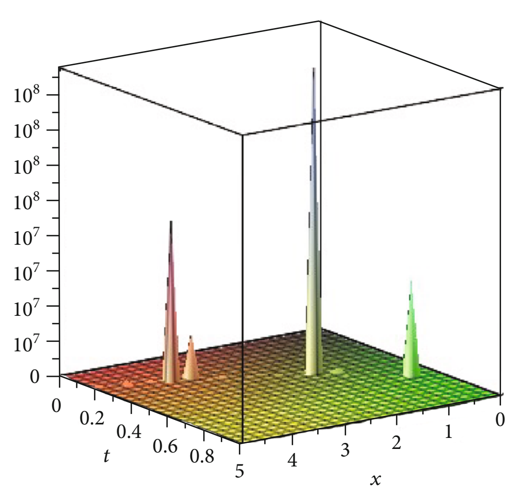

FIGURE 17: $u_{2,3}(x, 0, 0, t)$ of Equation (69).

neither Caputo definition nor Riemann-Liouville definition satisfies the chain rule. With the help of the Riccati equation, the method has been employed for finding the exact analytical solutions of these equations.

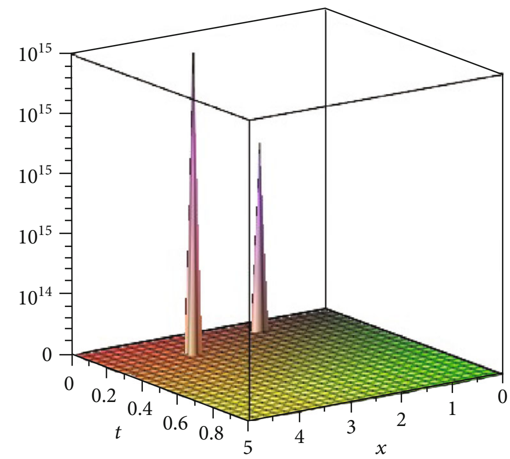

FIGURE 18: $u_{1,5}(x, 0, 0, t)$ of Equation (69).

These obtained solutions are traveling solutions. Furthermore, solutions $u_{i,1}(\xi)$ and $u_{i,2}(\xi)$ describe the solitons which are everywhere in nature. $u_{i,3}(\xi)$ and $u_{i,4}(\xi)$ represent the exact periodic traveling wave solutions.

The improved fractional subequation method is reliable and effective for finding more solutions for some space-time fractional nonlinear differential equations. We can substitute $\varphi' = r + p\varphi + q\varphi^2$ [39] for $\varphi' = \sigma + \varphi^2$, then, we will obtain more solutions.

Data Availability

All data included in this study are available upon request by contact with the corresponding author.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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