Research Article

Improvement on Conformable Fractional Derivative and Its Applications in Fractional Differential Equations

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In this paper, we made improvement on the conformable fractional derivative. Compared to the original one, the improved conformable fractional derivative can be a better replacement of the classical Riemann-Liouville and Caputo fractional derivative in terms of physical meaning. We also gave the definition of the corresponding fractional integral and illustrated the applications of the improved conformable derivative to fractional differential equations by some examples.

1. Introduction

The fractional order derivative has always been an interesting research topic in the theory of functional space for many years [1–11]. Various types of fractional derivatives were introduced, among which the following Riemann-Liouville and Caputo are the most widely used ones.

- (1) *Riemann-Liouville Definition*. For $\alpha \in [n-1, n)$, the α derivative of f is

$${}^{RL}D^\alpha(f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}} dx. \quad (1)$$

- (2) *Caputo Definition*. For $\alpha \in [n-1, n)$, the α derivative of f is

$${}^CD^\alpha(f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}} dx. \quad (2)$$

Both Riemann-Liouville definition and Caputo definition are defined via fractional integrals. Therefore, these two fractional derivatives inherit some nonlocal behaviors including historical memory and future dependence. All definitions including (1) and (2) above satisfy the property that the frac-

tional derivative is linear. This is the only property inherited from the 1st derivative. However, the existing fractional derivatives do not satisfy the following properties which the integral derivatives have.

- (1) Most of the fractional derivatives except Caputo-type derivatives do not satisfy $D^\alpha(1) = 0$, if α is not a natural number
- (2) All fractional derivatives do not obey the familiar product rule for two functions:

$$D^\alpha(fg) = fD^\alpha(g) + gD^\alpha(f) \quad (3)$$

- (3) All fractional derivatives do not obey the familiar quotient rule for two functions:

$$D^\alpha(fg) = \frac{gD^\alpha(f) - fD^\alpha(g)}{g^2} \quad (4)$$

- (4) All fractional derivatives do not obey the chain rule:

$$D^\alpha f(g(t)) = f^\alpha(g(t))D^\alpha(g) \quad (5)$$

- (5) Fractional derivatives do not have corresponding Rolle's theorem

- (6) Fractional derivatives do not have a corresponding mean value theorem
- (7) In general, all fractional derivatives do not obey

$$D^\alpha D^\beta f = D^{\alpha+\beta} f \quad (6)$$

- (8) The Caputo definition assumes that the function f is differentiable

To overcome some of these difficulties, Khalil et al. [12] proposed a new interesting fractional derivative definition called conformable derivative that extends the familiar limit definition of the derivative of a function given by the following.

Definition 1. Given a function $f : [0, +\infty) \rightarrow \mathbb{R}$, then the conformable fractional derivative of f of order α is defined by

$$T_\alpha(f)(t) = \lim_{\varepsilon \rightarrow 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, \quad (7)$$

for $t > 0$ and $\alpha \in (0, 1]$. If f is α differentiable in some $(0, a)$, $a > 0$, and $\lim_{t \rightarrow 0+} T_\alpha(f)(t)$ exists, then define $T_\alpha(f)(0) = \lim_{t \rightarrow 0+} T_\alpha(f)(t)$. It is easy to see that if f is differentiable, then $T_\alpha(f)(t) = t^{1-\alpha} f'(t)$. One can find functions which are α -differentiable at a point but not differentiable at this point.

As a result of the above definition, the authors in [12] showed that the conformable derivative obeys the product rule and quotient rule and has results similar to Rolle's theorem and the mean value theorem in classical calculus.

The conformable fractional derivative has two advantages over the classical fractional derivatives. First, the conformable fractional derivative definition is natural and it satisfies most of the properties which the classical integral derivative has such as linearity, product rule, quotient rule, power rule, chain rule, vanishing derivatives for constant functions, Rolle's theorem, and mean value theorem. Second, the conformable derivative bring us a lot of convenience when it is applied for modelling many physical problems, because the differential equations with conformable fractional derivative are easier to solve numerically than those associated with the Riemann-Liouville or Caputo fractional derivative. In fact, many researchers have already applied conformable fractional derivative to many fields and a lot of corresponding techniques were developed [13–20].

However, there are still shortcomings or disadvantages for the conformable derivative. If we look at the Riemann-Liouville and Caputo fractional derivative definition, we have

$$\lim_{\alpha \rightarrow 0_a} {}^{RL}D^\alpha(f)(t) = \lim_{\alpha \rightarrow 0} \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{f(x)}{(t-x)^\alpha} dx = f(t),$$

$$\lim_{\alpha \rightarrow 0_a} {}^CD^\alpha(f)(t) = \lim_{\alpha \rightarrow 0} \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{f'(x)}{(t-x)^\alpha} dx = f(t) - f(a),$$

(8)

for $\alpha \in (0, 1]$, and

$$\lim_{\alpha \rightarrow n-1_a} {}^{RL}D^\alpha(f)(t) = \lim_{\alpha \rightarrow n-1} \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}} dx = f^{(n-1)}(t),$$

$$\lim_{\alpha \rightarrow n-1_a} {}^CD^\alpha(f)(t) = \lim_{\alpha \rightarrow n-1} \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}} dx = f^{(n-1)}(t) - f^{(n-1)}(a), \quad (9)$$

for $\alpha \in (n-1, n]$, $n = 2, 3, \dots$, whereas

$$\lim_{\alpha \rightarrow n-1} T_\alpha(f)(t) = \lim_{\alpha \rightarrow n-1} t^{(n-\alpha)} f^{(n)}(t) = t f^{(n)}(t), \quad (10)$$

for $\alpha \in (n-1, n]$, $n = 1, 2, \dots$.

That means the physical meaning of the classical Riemann-Liouville and Caputo fractional derivative is quite different from that of the conformable derivative, especially when $\alpha \in (n-1, n]$ and close to $n-1$, $n = 1, 2, \dots$. We graph the conformable derivatives of $y = \sin x$ and $y = e^x$ in Figures 1 and 2 and make comparison with the Riemann-Liouville and Caputo fractional derivatives. We can see if the Riemann-Liouville and Caputo fractional derivatives are replaced by conformable derivative, a large error will occur. To overcome this difficulty, we propose a kind of modified conformable fractional derivative in Section 2. This modified conformable fractional derivative is a local operator on the one hand and approximates the Riemann-Liouville and Caputo fractional derivative better on the other hand. So it is a better choice to replace the classical Riemann-Liouville and Caputo fractional derivative with the improved conformable fractional derivative, especially when $\alpha \in (n-1, n]$ and close to $n-1$, $n = 1, 2, \dots$.

2. Improvement on Conformable Fractional Derivative

First, we give the definition for the improved Caputo-type conformable fractional derivative ${}_a^C \tilde{T}_\alpha(t)$ and the improved Riemann-Liouville-type conformable fractional derivative ${}_a^{RL} \tilde{T}_\alpha(t)$ for $0 \leq \alpha \leq 1$.

Definition 2. Given a function $f : \mathbb{R} \rightarrow \mathbb{R}$, the improved Caputo-type conformable fractional derivative of f of order α is defined by

$${}_a^C \tilde{T}_\alpha(t) = \lim_{\varepsilon \rightarrow 0} \left[(1-\alpha)(f(t) - f(a)) + \alpha \frac{f(t + \varepsilon(t-a)^{1-\alpha}) - f(t)}{\varepsilon} \right], \quad (11)$$

where $-\infty < a < t < +\infty$, a is a given number.

FIGURE 1: 0.3th derivative of $y = \sin x$. FIGURE 2: 0.2th derivative of $y = e^x$.

The improved Riemann-Liouville-type conformable fractional derivative of f of order α is defined by

$${}^{RL}\tilde{T}_\alpha(t) = \lim_{\varepsilon \rightarrow 0} \left[(1 - \alpha)f(t) + \alpha \frac{f(t + \varepsilon(t - a)^{1-\alpha}) - f(t)}{\varepsilon} \right], \quad (12)$$

where $-\infty < a < t < +\infty$, a is a given number.

It is easy to see

$$\begin{aligned} \lim_{\alpha \rightarrow 0} {}^C\tilde{T}_\alpha(t) &= (f(t) - f(a)) = \lim_{\alpha \rightarrow 0} {}^C D^\alpha(t), \\ \lim_{\alpha \rightarrow 0} {}^{RL}\tilde{T}_\alpha(t) &= f(t) = \lim_{\alpha \rightarrow 0} {}^{RL} D^\alpha(t). \end{aligned} \quad (13)$$

If $\alpha = 1$, both ${}_a^C\tilde{T}_\alpha(t)$ and ${}_a^{RL}\tilde{T}_\alpha(t)$ coincide with $f'(t)$. In Definition 2, we introduce a to let ${}_a^{RL}\tilde{T}_\alpha(t)$ and ${}_a^{RL}\tilde{T}_\alpha(t)$ have a kind of historical memory as the Caputo and Riemann-Liouville fractional derivative have.

For $n < \alpha \leq n + 1$, $n = 1, 2, \dots$, we give the following.

Definition 3. Given a function $f : R \rightarrow R$, the improved Caputo-type conformable fractional derivative of f of order α is defined by

$$\begin{aligned} {}^C\tilde{T}_\alpha(t) &= \lim_{\varepsilon \rightarrow 0} \left[(n + 1 - \alpha) \left(f^{(n)}(t) - f^{(n)}(a) \right) \right. \\ &\quad \left. + (\alpha - n) \frac{f^{(n)}(t + \varepsilon(t - a)^{n+1-\alpha}) - f^{(n)}(t)}{\varepsilon} \right], \end{aligned} \quad (14)$$

where $-\infty < a < t < +\infty$.

The improved Riemann-Liouville-type conformable fractional derivative is defined by

$$\begin{aligned} {}^{RL}\tilde{T}_\alpha(t) &= \lim_{\varepsilon \rightarrow 0} \left[(n + 1 - \alpha) f^{(n)}(x) \right. \\ &\quad \left. + (\alpha - n) \frac{f^{(n)}(t + \varepsilon(t - a)^{n+1-\alpha}) - f^{(n)}(t)}{\varepsilon} \right], \end{aligned} \quad (15)$$

where $-\infty < a < t < +\infty$.

FIGURE 3: 0.2th fractional derivative of $y = x^2$.

It is easy to see

$$\begin{aligned} \lim_{\alpha \rightarrow n_a} {}^C \tilde{T}_\alpha(t) &= \left(f^{(n)}(t) - f^{(n)}(a) \right) = \lim_{\alpha \rightarrow n_a} {}^C D^\alpha(t), \\ \lim_{\alpha \rightarrow n_a} {}^{RL} \tilde{T}_\alpha(t) &= f^{(n)}(t) = \lim_{\alpha \rightarrow n_a} {}^{RL} D^\alpha(t). \end{aligned} \quad (16)$$

If $\alpha = n + 1$, both ${}_a \tilde{T}_\alpha(t)$ and ${}_a^{RL} \tilde{T}_\alpha(t)$ coincide with $f^{(n+1)}(t)$.

Remark 4. As a result of Definitions 2 and 3, we can easily show that if f is 1-differentiable at t , then

$$\begin{aligned} {}^C \tilde{T}_\alpha(f)(t) &= (1 - \alpha)(f(t) - f(a)) + \alpha(t - a)^{1-\alpha} f'(t), \\ {}^{RL} \tilde{T}_\alpha(f)(t) &= (1 - \alpha)f(t) + \alpha(t - a)^{1-\alpha} f'(t), \end{aligned} \quad (17)$$

for $\alpha \in (0, 1]$. If f is $(n + 1)$ -differentiable at t , then

$$\begin{aligned} {}^C \tilde{T}_\alpha(f)(t) &= (n + 1 - \alpha) \left(f^{(n)}(t) - f^{(n)}(a) \right) \\ &\quad + (\alpha - n)(t - a)^{n+1-\alpha} f^{(n+1)}(t), \end{aligned} \quad (18)$$

for $\alpha \in (n, n + 1]$, $n = 1, 2, \dots$.

In fact, compared with the conformable fractional derivative, the improved conformable fractional derivative can be a better replacement of the Caputo and Riemann-Liouville fractional derivative. One can see in Figures 3–9 that the improved conformable fractional derivative approximates the Caputo and Riemann-Liouville fractional derivative in a better way for most elementary functions especially when α is larger than and close to n and even when α is close to $n + 0.5$, $n = 0, 1, 2, \dots$.

Based on the results in [12], one can prove the following theorem.

Theorem 5. We can easily show that the improved conformable fractional derivative satisfies the following properties if $0 \leq \alpha \leq 1$.

- (1) ${}_a^C \tilde{T}_\alpha(mf + ng) = m {}_a^C \tilde{T}_\alpha(f) + n {}_a^C \tilde{T}_\alpha(g)$
- (2) ${}_a^{RL} \tilde{T}_\alpha(mf + ng) = m {}_a^{RL} \tilde{T}_\alpha(f) + n {}_a^{RL} \tilde{T}_\alpha(g)$
- (3) ${}_a^{RL} \tilde{T}_\alpha(fg) = (1 - \alpha) {}_a^{RL} \tilde{T}_\alpha(f)g + f {}_a^{RL} \tilde{T}_\alpha(g) - (1 - \alpha)fg$
- (4) ${}_a^{RL} \tilde{T}_\alpha(f(g(x))) = (1 - \alpha)f(g(x)) + \alpha f'(g(x))T_\alpha(g(x))$

Theorem 6. We get the following improved conformable fractional derivatives of certain functions for $0 \leq \alpha \leq 1$.

- (1) ${}_0^C \tilde{T}_\alpha(t^p) = {}_0^{RL} \tilde{T}_\alpha(t^p) = (1 - \alpha)(t^p) + \alpha p t^{p-\alpha}$
- (2) ${}_0^C \tilde{T}_\alpha(\lambda) = 0$, for any constant λ
- (3) ${}_0^{RL} \tilde{T}_\alpha(\lambda) = (1 - \alpha)$, for any constant λ
- (4) ${}_0^C \tilde{T}_\alpha(e^t) = (1 - \alpha)(e^t - 1) + \alpha t^{1-\alpha} e^t$
- (5) ${}_0^{RL} \tilde{T}_\alpha(e^t) = (1 - \alpha)(e^t) + \alpha t^{1-\alpha} e^t$
- (6) ${}_0^C \tilde{T}_\alpha(\sin t) = {}_0^{RL} \tilde{T}_\alpha(\sin t) = (1 - \alpha) \sin t + \alpha t^{1-\alpha} \cos t$
- (7) ${}_0^C \tilde{T}_\alpha(\cos t) = (1 - \alpha)(\cos t - 1) - \alpha t^{1-\alpha} \sin t$
- (8) ${}_0^{RL} \tilde{T}_\alpha(\cos t) = (1 - \alpha)(\cos t) - \alpha t^{1-\alpha} \sin t$
- (9) ${}_0^{RL} \tilde{T}_\alpha(e^{(1/a)t^a})$

Proof. It is very easy to verify properties (1)–(8) if we take into account the conclusions in [12]. We only prove property (9) here. From [12], we have

$$T_\alpha \left(e^{(1/a)t^a} \right) = e^{(1/a)t^a}. \quad (19)$$

FIGURE 4: 0.5th fractional derivative of $y = x^2$. FIGURE 5: 0.8th fractional derivative of $y = x^2$. FIGURE 6: 0.2th fractional derivative of $y = \sin x$.

FIGURE 7: 0.5th fractional derivative of $y = \sin x$. FIGURE 8: 0.8th fractional derivative of $y = \sin x$. FIGURE 9: 0.2th fractional derivative of $y = e^x$.

Therefore,

$$\begin{aligned} {}_0^{RL}\tilde{T}_\alpha\left(e^{(1/a)t^a}\right) &= (1-\alpha)e^{(1/a)t^a} + \alpha T_\alpha\left(e^{(1/a)t^a}\right) \\ &= (1-\alpha)e^{(1/a)t^a} + \alpha e^{(1/a)t^a} = e^{(1/a)t^a}. \end{aligned} \quad (20)$$

Now, we give the following definition for α fractional integral.

Definition 7. For $\alpha \in (0, 1]$ and continuous function f , let

$$I_\alpha(f) = \frac{1}{\alpha} \int_0^x \frac{f(t)}{t^{1-\alpha}} e^{(1-\alpha/\alpha^2)(t^\alpha-x^\alpha)} dt. \quad (21)$$

When $\alpha = 1$, $I_1(f) = \int_0^x f(t)dt$ and coincides with the usual Riemann integral.

Theorem 8. Suppose $y(0) = 0$, we have

$$\begin{aligned} (1) \quad I_{\alpha,0}^{RL}\tilde{T}_\alpha(y(t)) &= {}_0^{RL}\tilde{T}_\alpha \cdot I_\alpha(y(t)) = y(t) \\ (2) \quad I_{\alpha,0}^C\tilde{T}_\alpha(y(t)) &= {}_0^C\tilde{T}_\alpha \cdot I_\alpha(y(t)) = y(t) \end{aligned}$$

Proof. We only prove (1); (2) can be proved in the same way.

$$\begin{aligned} I_{\alpha,0}^{RL}\tilde{T}_\alpha(f(x)) &= \frac{1-\alpha}{\alpha} \int_0^x \frac{f(t)}{t^{1-\alpha}} e^{(1-\alpha/\alpha^2)(t^\alpha-x^\alpha)} dt \\ &+ \int_0^x f'(t) e^{(1-\alpha/\alpha^2)(t^\alpha-x^\alpha)} dt = \frac{1-\alpha}{\alpha} \int_0^x \frac{f(t)}{t^{1-\alpha}} e^{(1-\alpha/\alpha^2)(t^\alpha-x^\alpha)} dt \\ &+ \int_0^x e^{(1-\alpha/\alpha^2)(t^\alpha-x^\alpha)} df(t) = f(t) e^{(1-\alpha/\alpha^2)(t^\alpha-x^\alpha)} \Big|_0^x \\ &- \frac{1-\alpha}{\alpha} \int_0^x \frac{f(t)}{t^{1-\alpha}} e^{(1-\alpha/\alpha^2)(t^\alpha-x^\alpha)} dt \\ &+ \frac{1-\alpha}{\alpha} \int_0^x \frac{f(t)}{t^{1-\alpha}} e^{(1-\alpha/\alpha^2)(t^\alpha-x^\alpha)} dt = f(x). \end{aligned} \quad (22)$$

Let $(1-\alpha)y(x) + \alpha t^{1-\alpha}y'(x) = f(x)$; by using a variation of the constant method, we can get

$$y(x) = \frac{1}{\alpha} \int_0^x \frac{f(t)}{t^{1-\alpha}} e^{(1-\alpha/\alpha^2)(t^\alpha-x^\alpha)} dt. \quad (23)$$

Let

$$I_\alpha(f) = \frac{1}{\alpha} \int_0^x \frac{f(t)}{t^{1-\alpha}} e^{(1-\alpha/\alpha^2)(t^\alpha-x^\alpha)} dt, \quad (24)$$

we have ${}_0^{RL}\tilde{T}_\alpha \cdot I_\alpha(y(t)) = y(t)$.

3. Applications of Improved Conformable Fractional Derivative

We solve fractional differential equations by using the improved conformable fractional derivatives.

Example 1. Consider the following fractional differential equation:

$$y^{(0.5)} = \frac{2}{\Gamma(2.5)} x^{3/2}, \quad y(0) = 0. \quad (25)$$

If $y^{(0.5)}$ is understood as the Riemann-Liouville or Caputo fractional derivative, the solution to this problem is

$$y = x^2. \quad (26)$$

Now, we take $y^{(0.5)}$ as the improved conformable fractional derivative; the problem reduces to

$$\begin{cases} \frac{1}{2}y + \frac{1}{2}\sqrt{x}y' = \frac{2}{\Gamma(2.5)}x^{3/2}, \\ y(0) = 0. \end{cases} \quad (27)$$

We can solve this problem through the variation constant method and get the following solution:

$$y = \frac{4}{\Gamma(2.5)} \left(x^{3/2} - \frac{3}{2}x + \frac{3}{2}\sqrt{x} - \frac{3}{4} \right) + \frac{3}{\Gamma(2.5)} e^{-2\sqrt{x}}. \quad (28)$$

If $y^{(0.5)}$ is understood as the conformable fractional derivative, the problem reduces to

$$\begin{cases} \sqrt{x}y' = \frac{2}{\Gamma(2.5)}x^{3/2}, \\ y(0) = 0, \end{cases} \quad (29)$$

and the solution to this problem is

$$y = \frac{1}{\Gamma(2.5)} x^2. \quad (30)$$

We compare the three solutions in Figure 10. We can see that the improved conformable fractional derivative solution is much better than the solution obtained by using the conformable fractional derivative. That means if we prefer a replacement of the Riemann-Liouville or Caputo fractional derivative, the improved conformable fractional derivative is a better choice.

Example 2. In general, we consider the problem

$$y^{(\alpha)} = \frac{2}{\Gamma(3-\alpha)} x^{2-\alpha}, \quad y(0) = 0. \quad (31)$$

We can get the following three solutions:

FIGURE 10: Solution to Example 1.

- (i) If $y^{(\alpha)}$ is the Riemann-Liouville or Caputo fractional derivative, the solution is $y_1(x) = x^2$, and

$$\lim_{\alpha \rightarrow 0} y_1(x) = \lim_{\alpha \rightarrow 0} x^2 = x^2 \quad (32)$$

- (ii) If $y^{(\alpha)}$ is the conformable fractional derivative, the problem can be reduced to

$$\begin{cases} x^{1-\alpha} y' = \frac{2}{\Gamma(3-\alpha)} x^{2-\alpha}, \\ y(0) = 0. \end{cases} \quad (33)$$

The solution is

$$y_2(x) = \frac{1}{\Gamma(3-\alpha)} x^2. \quad (34)$$

We can see that

$$\lim_{\alpha \rightarrow 0} y_2(x) = \lim_{\alpha \rightarrow 0} \frac{1}{\Gamma(3-\alpha)} = \frac{1}{2} x^2 \quad (35)$$

- (iii) If $y^{(\alpha)}$ is the improved conformable fractional derivative, the problem can be reduced to

$$\begin{cases} (1-\alpha)y + \alpha x^{1-\alpha} y' = \frac{2}{\Gamma(3-\alpha)} x^{2-\alpha}, \\ y(0) = 0 \end{cases} \quad (36)$$

Its solution $y_3(x)$ can be obtained through the variation of the constant method.

Since

$$(1-\alpha)y_3 + \alpha x^{1-\alpha} y_3' = \frac{2}{\Gamma(3-\alpha)} x^{2-\alpha}, \quad (37)$$

we have

$$y_3 = \alpha y_3 - \alpha x^{1-\alpha} y_3' + \frac{2}{\Gamma(3-\alpha)} x^{2-\alpha}. \quad (38)$$

Therefore,

$$\begin{aligned} \lim_{\alpha \rightarrow 0} y_3 &= \lim_{\alpha \rightarrow 0} \left(\alpha y_3 - \alpha x^{1-\alpha} y_3' + \frac{2}{\Gamma(3-\alpha)} x^{2-\alpha} \right) \\ &= \lim_{\alpha \rightarrow 0} \frac{2}{\Gamma(3-\alpha)} x^{2-\alpha} = x^2 = \lim_{\alpha \rightarrow 0} y_1. \end{aligned} \quad (39)$$

The result for the general α equation shows that the improved conformable derivative has advantages over other fractional derivatives. First, it is a good approximation to the classical Riemann-Liouville or Caputo fractional derivative so it has a similar physical meaning with the Riemann-Liouville and Caputo fractional derivative. Second, it is a local derivative operator so it is easy for numerical computing.

Example 3. Consider the following fractional differential equation:

$$y^{(1.5)} = \frac{3}{\Gamma(2.5)} x^{3/2}, \quad y(0) = 0, y'(0) = 0. \quad (40)$$

If $y^{(1.5)}$ is the Caputo fractional derivative, the solution to this problem is

$$y = x^3. \quad (41)$$

If $y^{(1.5)}$ is the conformable fractional derivative, the problem reduces to

$$x^{0.5} y'' = \frac{3}{\Gamma(2.5)} x^{3/2}, \quad y(0) = 0, y'(0) = 0. \quad (42)$$

FIGURE 11: Solution of Example 3.

The solution to this problem is

$$y = \frac{1}{2\Gamma(2.5)} x^3. \quad (43)$$

If $y^{(1.5)'}$ is the improved conformable fractional derivative, the problem reduces to

$$\frac{1}{2}y' + \frac{1}{2}x^{0.5}y'' = \frac{3}{\Gamma(2.5)}x^{3/2}, \quad y(0) = 0, y'(0) = 0. \quad (44)$$

Let $y' = p(x)$, the problem further reduces to

$$\frac{1}{2}p + \frac{1}{2}x^{0.5}p' = \frac{3}{\Gamma(2.5)}x^{3/2}, \quad p(0) = 0. \quad (45)$$

Solve this problem by using the method of constant variation, we get

$$p = \frac{6}{\Gamma(2.5)} \left(x^{3/2} - \frac{3}{2}x + \frac{3}{2}\sqrt{x} - \frac{3}{4} + \frac{3}{4}e^{-2\sqrt{x}} \right), \quad (46)$$

$$y = \frac{6}{\Gamma(2.5)} \left(\frac{2}{5}x^{5/2} - \frac{3}{4}x^2 + x^{3/2} - \frac{3}{4}x - \frac{3}{8}e^{-2\sqrt{x}} - \frac{3}{4}\sqrt{x}e^{-2\sqrt{x}} \right) + \frac{9}{4\Gamma(2.5)}. \quad (47)$$

The solutions to Example 3 can be seen in Figure 11.

4. Conclusion

We propose a kind of improved conformable fractional derivative in this paper. This improved conformable fractional derivative is also local by its definition, and meanwhile, a kind of historical memory parameter is introduced to its definition. The advantage of the improved conformable derivative is that its physical behavior approximates the

Riemann-Liouville and Caputo fractional derivative better than the conformable fractional derivative. So this improved conformable fractional derivative has much potential in modelling many physical problems where the Riemann-Liouville and Caputo fractional derivative is usually used.

Data Availability

Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no competing interests.

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